Character Variety.

§ Intro.

X: Riemann surface of genus g.

$$\{\pi_1(X)\} = \langle (a_i, b_i)_{i=1}^g \mid \prod_{i=1}^g [a_i, b_i] = 1 \rangle$$

$$\downarrow \rho$$

$$GL_n.$$

$\mathcal{M}_n$: twisted character variety.

$$\begin{array}{ccc} \mathcal{M}_n & \overset{\text{diffeo}}{\simeq} & \mathcal{M}_n^{Dol} \\ \text{Mixed Hodge} \rightsquigarrow & & \rightsquigarrow \text{Hitchin fib} \\ W_\bullet H^\bullet(\mathcal{M}_n) & & P_\bullet H^\bullet(\mathcal{M}_n^{Dol}). \end{array}$$

§. Twisted character variety.

Σ_g. $g \geq 0$ $n > 0$. $\zeta_n \in \mathbb{C}$ n-th root of unity.

Action: $\sigma : GL_n \curvearrowright GL_n^{2g}$ diagonal conjugation action.

induces $\bar{\sigma} : PGL_n \curvearrowright GL_n^{2g}$

$$\mathcal{U}_n := \{ (A_i, B_i)^g \in GL_n^{2g} \mid \prod_{i=1}^g [A_i, B_i] = \zeta_n I_n \}.$$

$$\mu_n : GL_n^{2g} \longrightarrow SL_n \quad (A_i, B_i)_{i=1}^g \longmapsto \prod_{i=1}^g [A_i, B_i]$$

$\rightsquigarrow$ $\mathcal{U}_n = \mu_n^{-1}(\zeta_n I_n)$ is $\bar{\sigma}$-stable.

<u>Def</u>: The twisted GL_n-character variety of Σ_g is defined as $\mathcal{M}_n := \mathcal{U}_n /\!/ PGL_n = \operatorname{Spec}(\mathbb{C}[\mathcal{U}_n]^{PGL_n})$

<u>Thm</u>: $\mathcal{M}_n$: non-singular affine variety.

§. Cohomology and tautological class.

$\hat{\mathcal{M}}_n := \mathcal{M}_n /\!/ (\mathbb{C}^\times)^{2g}$

$\mathcal{M}_n' := \mathcal{U}_n' /\!/ PGL_n$, $\mathcal{U}_n' = \{ (A_i, B_i)_{i=1}^{g} \in SL_n^{2g} \mid \prod_{i=1}^{g} [A_i, B_i] = \zeta_n I_n \}$

$\rightsquigarrow \hat{\mathcal{M}}_n = \mathcal{M}_n' /\!/ (\mu_n)^{2g}$

<u>Thm</u>: $H^\bullet(\mathcal{M}_n) \cong H^\bullet(\hat{\mathcal{M}}_n) \otimes H^\bullet((\mathbb{C}^\times)^{2g})$.

<u>Thm</u>: $\mathbb{U} \longrightarrow \mathcal{M}_n' \times \Sigma$ which is a PGL_n-principal bundle and $(\mu_n)^{2g}$-equivariant (universal bundle).

for $j = 2, \cdots, r$ $\quad c_j(\mathbb{U}) \in H^{2j}(\mathcal{M}_n' \times \Sigma)^{(\mu_n)^{2g}}$

$$= \bigoplus_{a=0}^{2} H^{2j-a}(\mathcal{M}_n') \otimes H^a(\Sigma).$$

$$c_j(\mathbb{U}) = \beta_j \otimes 1 + \sum_{k=1}^{2g} \psi_{j,k} \otimes c_k + \alpha_j \otimes [\Sigma]$$

<u>Thm</u>: $\{\varepsilon_i\}$, $\{\beta_j\}$, $\{\psi_{j,k}\}$, $\{\alpha_j\}$ form a set of $\mathbb{Q}$-alg. generators of $H^\bullet(\mathcal{M}_n)$.