THE MOTIVIC NEARBY FIBER (after Denef-Loeser)

k - field of characteristic zero

Ref's:
- Denef-Loeser, "Geometry of arc spaces", ECM 2000
- Looijenga, "Motivic measures", Bourbaki, 1999/2000
- Denef-Loeser, "Lefschetz numbers", Topology (2002)
- Nicaise-Shinder, "Motivic nearby fiber", Inv. Math (2019)

1. Goal [NS, §3.1]

$K := k((t))$, $R := k[[t]]$

Thm $\exists!$ ring morph.

$$\mathrm{Vol}_K : K(\mathrm{Var}_K) \longrightarrow K^{\hat{\mu}}(\mathrm{Var}_k)$$

st.

(i) $\forall$ X smooth & proper / K

$\forall$ $\mathcal{X} \to \mathrm{Spec}\, R$ flat, proper/R, regular st.

$$\mathcal{X}_k = \sum_{i \in I} N_i \cdot E_i \quad \text{snc divisor}$$

$$[*] \qquad \mathrm{Vol}_K(X) = \sum_{\phi \neq J \subseteq I} (1-\mathbb{L})^{|J|-1} \cdot [\widetilde{E}_J^o]$$

we'll see later...

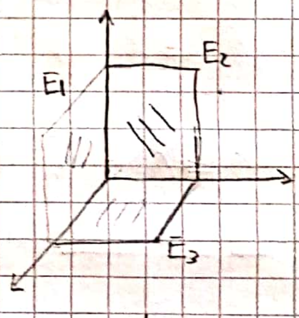
unramif. Galois cover of $E_J^o := \left(\bigcap_{j \in J} E_j\right) \setminus \left(\bigcup_{i \in I \setminus J} E_i\right)$

(Anne's talk)

(ii) If $\mathcal{X}$ smooth/R, then

$$\mathrm{Vol}_K(X) = [\mathcal{X}_k]$$

(iii) $\mathrm{Vol}_K(\mathbb{L}) = \mathbb{L}$



$E^o_{\{12\}}$:

Today: Where does [*] come from?

(⇝ Denef-Loeser work on "motivic nearby fiber")

Next:
- Intro to log-geometry (Joël)
- Pf of Thm (Benjamin)

2. Topological intuition

$\mathbb{K} = \mathbb{C}$

- (local) $f : (\mathbb{C}^{n+1}, 0) \longrightarrow (\mathbb{C}, 0)$, $n \geq 1$

  w/ a critical pt. at 0

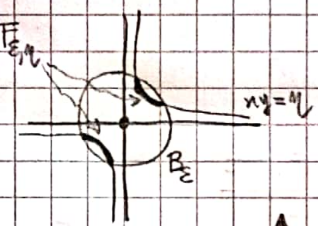
  ball at 0 of radius ε

  Milnor fibre: $F_{\varepsilon,\eta} := f^{-1}(\eta) \cap B_\varepsilon$, $\forall\, 0 < \eta \ll \varepsilon$

  [†]
  - $F_{\varepsilon,\eta}$ connected ($\Leftrightarrow H^0(F_{\varepsilon,\eta}, \mathbb{C})$ has dim $= 1$)
  - $H^k(F_{\varepsilon,\eta}, \mathbb{C})$ "vanishing cycles" + monodromy action... (we'll see)

    $1 \leq k \leq n$, oth. $= 0$

  Ex $f : \mathbb{C}^2 \to \mathbb{C}$
  $(x,y) \longmapsto xy$



- (global) X complex manifold
  $f : X \to \Delta$
  $X_0 := f^{-1}(0)$

$$\begin{tikzcd} X_0 \arrow[r, hook, "i_0"] \arrow[d] & X \arrow[d, "f"] & X^* \arrow[l, hook', "j"'] \arrow[d] & \tilde{X}^* \arrow[l, "p"'] \arrow[d] \\ 0 \arrow[r, hook] & \Delta & \Delta^* \arrow[l, hook'] & \tilde{\Delta}^* \arrow[l] \end{tikzcd}$$

univ. cover
gp $\simeq \mathbb{Z}$

$\psi_f \underline{\mathbb{C}}_X := i_0^{-1} R(j \circ p)_* (j \circ p)^{-1} \underline{\mathbb{C}}_X \in D^b_c(X)$ + monodromy action... (we'll see)

complex of nearby cycles — constant sheaf — derived category of constructible sheaves

We have:
- $\forall x \in X_0$, $\mathcal{H}^k(\psi_f \underline{\mathbb{C}}_X)_x \cong H^k(F_{\varepsilon,x}, \mathbb{C})$, $\forall k \geq 0$
- [†] $\Leftrightarrow$ $\exists$ ex triangle $\underline{\mathbb{C}}_{X_0} \xrightarrow{} \psi_f \underline{\mathbb{C}}_X \longrightarrow \phi_f \underline{\mathbb{C}}_X$

  $\underline{\mathbb{C}}_{X_0} = i_0^{-1}\underline{\mathbb{C}}_X$, adjointness, vanishing cycles
- compatible w/ resolutions $\tilde{X} \xrightarrow{\pi} X$, $\tilde{f} : \tilde{X} \to \Delta$, $f : X \to \Delta$: $R\pi_* \psi_{\tilde{f}} = \psi_f$

- if f proper, then in $D^b_c(X_0)$ we have:
  $\psi_f \underline{\mathbb{C}}_X \cong R\, sp_* (\underline{\mathbb{C}}_{X_t})$, $t \neq 0$
  sp specialization map $X_t \to X_0$ $\rightsquigarrow$ $\mathbb{H}^k(X_0, \psi_f \underline{\mathbb{C}}_X) \cong H^k(X_t, \mathbb{C})$ — compatible w/ monodromy

<u>Idea</u>: X : smooth alg. var. $/\mathbb{C}$

$f : X \to \mathbb{C}$, dominant, alg.

Hodge str + monodromy (quasi-unip endom.)

- $\exists\ \psi_f^{mot} \in K^{\hat{\mu}}(Var_{X_0})[\mathbb{L}^{-1}]$ st. $\chi_x(\_) := \sum(-1)^k [H^k(\_, \mathbb{C})]$

$\forall x \in X_0$, $\chi_x(\psi_{f,x}^{mot}) = \chi_x(F_{\varepsilon,x})$

- $Vol_K(X_{X_0}, K) \text{ "=" } \psi_f^{mot}$

<u>issue</u>: need to invert $\mathbb{L}$ ... not good ...

<u>But</u> [*] holds for ψ_f^{mot} and well-def. w/o inverting $\mathbb{L}$

$\rightsquigarrow$ we'll need to check (later, in Benjamin's talk!) that it is well-def. (indep. of snc-model + good properties ...)

$\bullet\ \bullet$

3. Equivariant Grothendieck groups

$\mu_n := Spec(\mathbb{Z}[x]/(x^n - 1))$ n-th roots of unity

$\hat{\mu} := \varprojlim \mu_n$

$\mu_{nd} \to \mu_n$

$x \mapsto x^d$

S base scheme $\rightsquigarrow Var_S$

e.g., $\mathbb{L} = [\mathbb{A}_S^1, \hat{\mu}]$ trivial action

μ_n-action on X, some n "good" ($\leftrightarrow X/\mu_n$ exists)

$K_0^{\hat{\mu}}(Var_S) \ni [X/S, \hat{\mu}]$

+ usual relations "$\hat{\mu}$-equiv." (Anne's talk) $\rightsquigarrow$ ring (w/ diagonal action on product)

$M_S^{\hat{\mu}} := K_0^{\hat{\mu}}(Var_S)[\mathbb{L}^{-1}]$

$\bullet\ \bullet$

4. The motivic zeta function [DL'00, § 3.2]

X smooth variety $/\kappa$ ⟵ we will secretely assuming $\kappa = \bar{\kappa}$ (or even $\mathbb{C}$) but it is not necessary

$f : X \to \mathbb{A}^1_\kappa$ dominant

$d = \dim X$

$\rightsquigarrow$ (Anne) $f_m : \mathcal{J}_m(X) \to \mathcal{J}_m(\mathbb{A}^1)$

$f_m(\varphi) \in \kappa[[t]]/t^{m+1}$,

$\mathcal{X}_n := \{ \varphi \in \mathcal{J}_m(X) : \mathrm{ord}_t f_m(\varphi) = n \}$

$\mathrm{ord}_t :=$ largest e s.t. t^e divides $f_m(\varphi)$

$\subseteq \mathcal{J}_m(X)$ loc. closed subvariety

$\rightsquigarrow f_m(\varphi) = \underset{\neq 0}{\underbrace{a(\varphi)}}\, t^m$

$m \geq 1 \rightsquigarrow \pi^m_{m,0}(\mathcal{X}_m) \subseteq X_0$, where $\pi_{m,0} : \mathcal{J}_m(X) \to X$

$\rightsquigarrow \mathcal{X}_m \in \mathrm{Var}_{X_0}$

Also, $\bar{f}_m : \mathcal{X}_m \longrightarrow \mathbb{G}_{m,\kappa}$

$\varphi \longmapsto a(\varphi)$

$$\boxed{\mathcal{X}_{m,1} := \bar{f}_m^{-1}(1)} \in \mathrm{Var}_{X_0}$$

We have a $\mathbb{G}_m$-action on $\mathcal{X}_m$ $\rightsquigarrow$ good action of μ_m (and so $\hat{\mu}$) on $\mathcal{X}_{m,1}$

$(a \cdot \varphi(t) = \varphi(at))$

<u>Def</u> Motivic zeta function of $f : X \to \mathbb{A}^1$ is

$$Z(T) := \sum_{m \geq 1} [\mathcal{X}_{m,1}/X_0, \hat{\mu}] \cdot \mathbb{L}^{-md} \cdot T^m \in \mathcal{M}^{\hat{\mu}}_{X_0}[[T]].$$

The goal is to find formula for $Z(T)$ by using resolution of f.

Let $h : Y \to X$ proper, birational s.t.

- Y smooth $/ \mathbb{K}$
- $h : Y \setminus h^{-1}(X_0) \to X \setminus X_0$ isom.
- $h^{-1}(X_0) = \sum_{i \in I} N_i \cdot E_i$ snc divisor

Main Thm (DL, Looijenga)

In $\mathcal{M}_{X_0}^{\hat{\mu}}[[T]]$ we have:

$$Z(T) = \sum_{\emptyset \neq J \subseteq I} (\mathbb{L}-1)^{|J|-1} \cdot [\tilde{E}_J^\circ / X_0, \hat{\mu}] \cdot \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{-\nu_j} T^{N_j}}$$

Explanation on notation:

- ν_j : $K_Y - f^* K_X = \sum_{i \in I} (\nu_i - 1) E_i$

- $\tilde{E}_J^\circ$: $m = \operatorname{lcm}(N_i)_{i \in I}$

$$\begin{tikzcd} \tilde{Y} \arrow[r, "\text{normaliz.}"] \arrow[rr, bend left, "u"] & Y' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & Y \arrow[d, "f \circ h"] \\ & \mathbb{A}^1 \arrow[r] & \mathbb{A}^1 \\ & z \arrow[r, mapsto] & z^m \end{tikzcd}$$

$N_J := \gcd(N_j)_{j \in J}$

$\tilde{E}_J^\circ :=$ connect. cmpt. of $u^{-1}(E_J^\circ)$

Then: $\tilde{E}_J^\circ \to E_J^\circ$ unramif. Galois covering w/ Galois gp. μ_{N_J}

$\rightsquigarrow [\tilde{E}_J^\circ / X_0, \hat{\mu}]$

Def $\psi_f^{mot} := -\lim_{T\to\infty} Z(T) \underset{\text{Main Thm.}}{=} \sum_{\emptyset\neq J\subseteq I} (1-\mathbb{L})^{|J|-1}\cdot[\tilde{E}_J^o/X_0, \hat{\mu}]$

<u>motivic nearby fiber</u>

ψ_f^{mot} "virtual" motivic incarnation of Milnor fiber

<u>Ex.</u>

$e :=$ topological Euler characteristic

$f: X \to \Delta$ family of curves in $\mathbb{P}^2$ of degree d s.t.

$$X_0 = \sum_{i=1}^m E_i \quad , \quad \text{snc divisor} \quad , \quad d_i = \deg E_i$$

$(N_i = 1, \forall i \rightsquigarrow$ can "forget" $\hat{\mu})$

$$e(\psi_f) = e(X_t) = -d^2 + 3d$$

$$e(\psi_f^{mot}) = \sum_{i=1}^m e(E_i^o) = \sum_{i=1}^m \left(-d_i^2 + 3d_i - \sum_{j\neq i} d_i d_j\right)$$

$e((1-\mathbb{L})^{|J|-1}\ \ldots) = 0$ if $|J|>1$

$$= -\left(\sum_{i=1}^m d_i^2 + \sum_{i<j} 2 d_i d_j\right) + 3\sum_{i=1}^m d_i = -d^2 + 3d \quad \checkmark$$

▲

<u>Rmk</u> RHS of [*] indep. on the resolution Y in $\mathcal{M}_{X_0}^{\hat{\mu}}$

$\rightsquigarrow$ as mentioned at the beginning, for application to stable rationality need well-defined element in $K_0^{\hat{\mu}}(\mathrm{Var}_{X_0})$ (or $K_0^{\hat{\mu}}(\mathrm{Var}_{\mathbb{C}})$)

$\rightsquigarrow$ this is part of the content of Nicaise-Shinder result.

▲

5. Idea of proof of Main Thm [DL'00, §4 ; DL'02, §2 ; Loo, §5]

We need to compute $[\mathcal{X}_{m,1}/X_0, \hat{\mu}]$.
To do this, we use motivic integration and the base-change formula (we saw in Anne's talk).

The precise result is the following

Lem In $\mathcal{M}^{\hat{\mu}}_{X_0}$, we have $\forall m \geq 1$,

$$[\mathcal{X}_{m,1}/X_0, \hat{\mu}] \cdot \mathbb{L}^{-md} = \sum_{\emptyset \neq J \subseteq I} (\mathbb{L}-1)^{|J|-1} \cdot [\tilde{E}^{\circ}_J / X_0, \hat{\mu}] \cdot \Bigg(\sum_{\substack{k_j \geq 1, j \in J \\ \sum_{j\in J} k_j N_j = m}} \mathbb{L}^{-\sum_{j \in J} k_j \nu_j} \Bigg)$$

Pf (Lem $\Rightarrow$ Main Thm):

$$Z(T) = \sum_{m \geq 1} [\mathcal{X}_{m,1}/X_0, \hat{\mu}] \cdot \mathbb{L}^{-md} \cdot T^m$$

$$\underset{\text{Lem}}{=} \sum_{\emptyset \neq J \subseteq I} (\mathbb{L}-1)^{|J|-1} \cdot [\tilde{E}^{\circ}_J / X_0, \hat{\mu}] \cdot \underbrace{\sum_{m \geq 1} \Bigg(\sum_{\substack{k_j \geq 1, j \in J \\ \sum_{j\in J} k_j N_j = m}} \mathbb{L}^{-\sum_{j \in J} k_j \nu_j} \Bigg) \cdot T^m}$$

$$= \sum_{k \in \mathbb{Z}^J_{\geq 1}} \mathbb{L}^{-\sum k_j \nu_j} \cdot T^{\sum k_j N_j}$$

$$= \sum_{k \in \mathbb{Z}^J_{\geq 1}} \prod_{j \in J} (\mathbb{L}^{-\nu_j} T^{N_j})^{k_j}$$

$$= \prod_{j \in J} \sum_{k \geq 1} (\mathbb{L}^{-\nu_j} T^{N_j})^k = \prod_{j \in J} \frac{\mathbb{L}^{-\nu_j} T^{N_j}}{1 - \mathbb{L}^{\nu_j} T^{N_j}}.$$

recall that X smooth ($\rightsquigarrow$ stable)

Pf (Lem): $\pi_m : J_\infty(X) \to J_m(X)$

let's forget $\hat{\mu}$-action! (and let's also do everything $/\mathcal{X}_0$ for simplicity!)

$\mathcal{X}^\infty_{m,1} := \pi_m^{-1}(\mathcal{X}_{m,1})$ cylindrical subset, since X smooth

$\rightsquigarrow \tilde{\mu}(\mathcal{X}^\infty_{m,1}) = [\mathcal{X}_{m,1}]\cdot \mathbb{L}^{-md}$ $\longleftarrow$ in $\mathcal{M}_{\mathcal{X}}$ (no need $\hat{\mathcal{M}}_{\mathcal{X}}$ here)

recall: $h : Y \to X$ proper birat'l, Y smooth, $h^{-1}(X_0) = \sum_{i\in I} N_i E_i$, h isom outside $h^{-1}(X_0)$

Base-change formula $\Rightarrow$

$$\tilde{\mu}(\mathcal{X}^\infty_{m,1}) = \int_{h^{-1}(\mathcal{X}^\infty_{m,1})} \mathbb{L}^{-\mathrm{ord}\, K_{Y/X}}\, d\mu$$

To evaluate RHS, $\forall \emptyset \neq J \subseteq I$, $\forall k = (k_i)_{i\in J} \in \mathbb{Z}^J_{\geq 1}$, we consider the subset

$$\mathcal{Y}_m(k) := \left\{ \varphi \in J_m(Y) : \begin{array}{l} \bullet\ \mathrm{ord}_{E_i}\varphi = k_i,\ \forall i \in I \\ \bullet\ \mathrm{ord}_t(\underbrace{(f\circ h)(\varphi)}_{\in J_m(A_{\mathbb{K}})}) = \sum_{j\in J} k_j \cdot N_j = m \end{array} \right\}$$

Notice that: $\mathrm{ord}_{K_{Y/X}}(\varphi) = \sum_{i\in I} k_i(\nu_i - 1)$, $\forall \varphi \in \mathcal{Y}_m(k)$

We also let $U_J :=$ fiber product of $(N_{E_j/Y} \setminus \{0\text{-section}\})|_{E_J^\circ}$, $j \in J$.

it is a $\mathbb{G}_m^J$-bdle over E_J°.

We have

$\pi_{m,0}(\mathcal{Y}_m(k)) = E_J^\circ$

$$[\mathcal{Y}_m(k)] = [U_J]\cdot \mathbb{L}^{md - \sum_{j\in J} k_j}$$

everything is smooth

condition of $\mathrm{ord}_{E_j}\varphi = k_j$

$$\left(= (\mathbb{L}-1)^{|J|}\cdot [E_J^\circ]\cdot \mathbb{L}^{md - \sum_{j\in J} k_j} \right)$$

If we restrict to $\mathcal{Y}_{m,1}(\underline{k}) := \{\varphi \in \mathcal{Y}_m(\underline{k}) : h_m(\varphi) \in \mathcal{X}_{m,1}\}$

we have

$$[\mathcal{Y}_{m,1}(\underline{k})] = (\mathbb{L}-1)^{|J|-1} \cdot [\tilde{E}_J^\circ] \cdot \mathbb{L}^{md - \sum_{j\in J} k_j}$$

this is compatible w/ $\hat{\mu}$-action...

Now we are done:

$$\int_{\underbrace{h^{-1}(\mathcal{X}^\infty_{m,1})}_{\mathcal{Y}^\infty_{m,1}}} \mathbb{L}^{-\mathrm{ord}_{K_{Y/X}}} d\mu = \sum_{e\geq 1} \mu(\mathcal{Y}^\infty_{m,1,e}) \mathbb{L}^{-e}$$

$\mathrm{ord}_{K_{Y/X}}(\varphi) = e$

$$= \sum_{\emptyset \neq J \subseteq I} \sum_{\substack{\underline{k} \in \mathbb{Z}^J_{\geq 1} \\ \sum_{j\in J} k_j N_j = m}} \mu(\mathcal{Y}^\infty_{m,1}(\underline{k})) \cdot \mathbb{L}^{-\sum_{j\in J} k_j(\nu_j - 1)}$$

$$= \sum_{\emptyset \neq J \subseteq I} \sum_{\substack{\underline{k} \in \mathbb{Z}^J_{\geq 1} \\ \sum_{j\in J} k_j N_j = m}} [\mathcal{Y}_{m,1}(\underline{k})] \cdot \mathbb{L}^{-md} \cdot \mathbb{L}^{-\sum_{j\in J} k_j(\nu_j - 1)}$$

$$= \sum_{\emptyset \neq J \subseteq I} (\mathbb{L}-1)^{|J|-1} \cdot [\tilde{E}_J^\circ] \cdot \sum_{\substack{\underline{k} \in \mathbb{Z}^J_{\geq 1} \\ \sum_{j\in J} k_j N_j = m}} \mathbb{L}^{-\sum_{j\in J} k_j \nu_j}$$

$\boxtimes$