The Cohomology Ring of the Dolbeault Moduli Space and Tautological Classes

(after E. Markman) $/\mathbb{C}$

1. Review: the Dolbeault moduli space

Σ : smooth irred. proj. curve, $g := g(\Sigma) \geq 2$

$\mathcal{L}$: effective line bdle on Σ (e.g., $\mathcal{L} \cong \mathcal{O}_\Sigma$)

$n, d \in \mathbb{Z}$, $n \geq 1$, $\gcd(n,d) = 1$.

wrt $\mu := \frac{\deg}{\mathrm{rk}}$ on ϕ-inv. subs. of E

Def $M^{\mathcal{L}}_{Dol}$:= moduli space of stable $\mathcal{L}$-twisted Higgs bundles (E,ϕ) on Σ w/ $\mathrm{rk}(E) = n$, $\deg(E) = d$

$\phi : E \to \underbrace{E \otimes \Omega^1_\Sigma \otimes \mathcal{L}}_{=: \Omega_{\mathcal{L}}}$

it depends on n and d, but suppressed from notation

Thm [Hitchin, Nitsure, Simpson]

$M^{\mathcal{L}}_{Dol}$ smooth irreduc. quasi-proj. variety w/ universal $\mathcal{L}$-twisted Higgs bdle $(\mathcal{E}, \Phi)$ on $\Sigma \times M^{\mathcal{L}}_{Dol}$;

$$\Phi : \mathcal{E} \to \mathcal{E} \otimes p_\Sigma^* \Omega_{\mathcal{L}} .$$

$\dim = \begin{cases} 2 \cdot (n^2(g-1)+1), & \text{if } \mathcal{L} \cong \mathcal{O}_\Sigma \\ n^2 \cdot (2(g-1) + \deg \mathcal{L}), & \text{oth.} \end{cases}$

Moreover: • $M^{\mathcal{L}}_{Dol}$ Poisson, and sympl. iff $\mathcal{L} \cong \mathcal{O}_\Sigma$

• $h : M^{\mathcal{L}}_{Dol} \longrightarrow A^{\mathcal{L}} := \bigoplus_{i=1}^{n} H^0(\Sigma, \Omega_{\mathcal{L}}^{\otimes i})$ Hitchin map

$(E,\phi) \longmapsto$ char. poly of ϕ

$\dim \text{fiber} = \begin{cases} n^2(g-1)+1, & \mathcal{L} \cong \mathcal{O}_\Sigma \\ n^2(g-1) + \frac{n(n-1)}{2} \deg \mathcal{L}, & \text{oth} \end{cases}$

proper, surjective, and (algebraic) integrable system

in the Poisson sense (i.e., on each sympl. leaf) [Bottacin]

<u>Recall</u> : Poisson variety :

- X smooth variety
- $\eta \in H^0(X, \Lambda^2 T_X) \rightsquigarrow \{f, g\} := \langle \eta, df \wedge dg \rangle$

s.t. Jacobi identity holds :

$$\{f, \{g, h\}\} + \{g, \{h, f\}\} + \{h, \{f, g\}\} = 0$$

$(\Longleftrightarrow [\eta, \eta] = 0$, where $[-,-]$ Shouten-Nijenhuis bracket

$\Longleftrightarrow \tilde{\partial}\eta = 0$, $\tilde{\partial} : H^0(X, \Lambda^2 T_X) \to H^0(X, \Lambda^3 T_X)$)

$\cdot\cdot$

2. Tautological classes and Main Thm

$(\mathcal{E}, \Phi)$ univ. Higgs bdle / $\Sigma \times M_{Dol}^{\chi}$

$\rightsquigarrow$ Chern classes $c_k(\mathcal{E}) \in H^{2k}(\Sigma \times M_{Dol}^{\chi}, \mathbb{Z})$, $\forall k \geq 1$.

$$\rightsquigarrow \quad c_k(\mathcal{E}) = \sum_i p_\Sigma^* \gamma_i \otimes p_M^* \mu_i$$

<u>Künneth</u>: $H^{2k}(\Sigma \times M) \cong \bigoplus_{p+q=2k} H^p(\Sigma) \otimes H^q(M)$

$/\mathbb{Z}$, since $H^*(\Sigma)$ torsion-free

<u>Def</u> $\mu_i :=$ <u>tautological classes</u> $\in H^*(M_{Dol}^{\chi}, \mathbb{Z})$

<u>Rmk</u> In [Maulik-Shen], they use notation $\leftarrow$ equiv., over $\mathbb{Q}$.

$$c_k(\gamma) := p_{M*}(p_\Sigma^* \gamma \cup ch_k(\mathcal{E})) \in H^*(M_{Dol}^{\chi}, \mathbb{Q})$$

$\forall k \geq 1$, $\forall \gamma \in H^*(\Sigma, \mathbb{Q})$.

<u>Main Thm</u> [Markman, Hausel-Thaddeus]

The cohomology ring $H^*(M^{\mathcal{L}}_{Dol}, \mathbb{Z})$ generated by tautological classes.

$\blacksquare$

<u>Plan</u>:
- to simplify, consider the case $/\mathbb{Q}$ $\leftsquigarrow$ main idea of pf not too different (keep track of coeff's) and [MS] only need result $/\mathbb{Q}$
- the case $\mathcal{L} \not\cong \mathcal{O}_\Sigma$ easier $\leftsquigarrow$ use simplif. by Beauville
- sketch $\mathcal{L} \cong \mathcal{O}_\Sigma$ along way.

<u>Idea Pf</u>: $\delta \in H^*(M \times M)$ Poincaré dual of class of diagonal

Then, $\forall \lambda \in H^*(M)$,

$$\lambda = p_{2*}(\delta \cup p_1^* \lambda)$$

$\rightsquigarrow$ $H^*(M)$ generated by Künneth factors of δ

<u>Need</u>:
- worry that M <u>not</u> compact $\leftsquigarrow$ we'll need 2 cptf'ns: modular / not loose any cohomology
- express δ as function of Chern classes of $\mathcal{E}$

$\blacktriangle$

3. Moduli spaces of torsion sheaves

$$\begin{array}{l} S_{\mathcal{L}} := \mathbb{P}_\Sigma(\omega_{\mathcal{L}}^\vee \oplus \mathcal{O}_\Sigma) \quad \left(:= \underline{\mathrm{Proj}}_\Sigma\left(\mathrm{Sym}^\bullet(\omega_{\mathcal{L}}^\vee \oplus \mathcal{O}_\Sigma)\right)\right) \\ \big\downarrow f \\ \Sigma \end{array}$$

Then :
- $f_*\mathcal{O}_f(\ell) = \mathrm{Sym}^\ell(\omega_{\mathcal{L}}^\vee \oplus \mathcal{O}_\Sigma)$, $\forall \ell \geq 0$
- $\omega_{S_{\mathcal{L}}} \cong \mathcal{O}_f(-2) \otimes f^*\mathcal{L}^\vee$ (⇝ $S_{\mathcal{L}}$ Poisson surface) $\mathcal{L}$ eff.
- $D_\infty \in |\mathcal{O}_f(1)|$ unique section

  $D_\infty^2 = -(2g-2+\deg(\mathcal{L}))$, $\mathrm{tot}(\omega_{\mathcal{L}}) = S_{\mathcal{L}} \setminus D_\infty$.

Let $\xi := n \cdot c_1(\mathcal{O}_f(1) \otimes f^*\omega_{\mathcal{L}})$ ⇝ $\xi . D_\infty = 0$, $\xi . \omega_S^\vee = \deg(\mathcal{L})$

$\chi := d + n\cdot(1-g)$

⇝ we look at torsion sheaves F on $S_{\mathcal{L}}$ w/

$\mathrm{rk}(F)=0$

$c_1(F) = \xi$

$\chi(S,F) = \chi$

Lem [Simpson]

$= \mathcal{O}_f(1) \otimes f^*\mathcal{O}_\Sigma(a)$, $a \gg 0$

For a general polarization H on $S_{\mathcal{L}}$

if $C \in |\xi|$ general smooth curve these are line bdles of rk=1 on C

$\overline{M}^{\mathcal{L}}_{\xi} := M_{S,H}(\xi,\chi) :=$ moduli space of H-Gieseker stable torsion sheaves on $S_{\mathcal{L}}$ w/ c_1, χ as above

[Mukai]

irred., proj. variety w/ universal family $\mathbb{F}$ on $S \times \overline{M}$.

Moreover :
- $\overline{M}^{\mathcal{L}}$ Poisson (on its smooth locus) ⟵ [Mukai, Beauville, Bottacin]
- $\overline{M}^{\mathcal{L}}$ smooth iff $\mathcal{L} \not\cong \mathcal{O}_\Sigma$

- $\bar{h} : \bar{M}^{\mathcal{L}} \longrightarrow \mathbb{P}(H^0(S, \mathcal{O}_S(\xi))) =: \mathbb{P}^{\mathcal{L}}$ — <u>integrable system</u>

  $F \longmapsto$ Fitting supp. of F

- $\exists$ open embedding

$$\begin{tikzcd} M^{\mathcal{L}}_{Dol} \arrow[r, hook, "i", "open"'] \arrow[d, "h"'] \arrow[dr, phantom, "\circlearrowleft"] & \bar{M}^{\mathcal{L}} \arrow[d, "\bar{h}"] \\ A^{\mathcal{L}} \arrow[r, hook, "open"'] & \mathbb{P}^{\mathcal{L}} \end{tikzcd}$$

$\longleftarrow$ the complement is hyperplane

whose image consists of those torsion sheaves whose support does <u>not</u> intersect D_∞

Pf:

<u>Recall</u>: <u>H-Gieseker stability</u>

$P_H(-)(m) := \chi(S, - \otimes \mathcal{O}_S(mH))$, $\forall m \in \mathbb{Z}$ ... <u>Hilbert poly</u>

$p_H := \frac{P_H}{\text{lead. coeff.}}$ ... <u>reduced Hilb. poly</u>

Say F <u>H-Gies. (semi)stable</u> if <u>pure</u> and

$\forall 0 \neq F' \subsetneq F$, $\quad p(F')(m) \underset{(\leq)}{<} p(F)(m)$, $\forall m \gg 0$.

In our case, this is just a slope $\dfrac{\chi}{H.c_1}$

In our case, since $\gcd(n, d) = 1$, can choose H st.

- semist. $\equiv$ st.
- $\exists$ univ. family

$\rightsquigarrow$ first statement (except irred., which we prove later).

Tg. space : $\mathrm{Ext}^1_S(F,F)$ ($\rightsquigarrow$ cotg. space $\mathrm{Ext}^1_S(F, F\otimes\omega_S)$)

Obstr. space : $\mathrm{Ext}^2_S(F,F) \underset{SD}{\cong} \mathrm{Hom}(F, F\otimes\omega_S)^\vee$.

$\deg(\mathcal{L}) > 0 \Rightarrow \xi.\omega_S^\vee > 0 \Rightarrow \mathrm{slope}(F) > \mathrm{slope}(F\otimes\omega_S) \Rightarrow {}^\vee = 0$

$\Rightarrow \overline{M}^{\mathcal{L}}$ smooth of dim $\mathrm{ext}^1_S(F,F)$.

[$\deg(\mathcal{L}) = 0 \Rightarrow \mathrm{Ext}^2_S(F,F) \cong \mathbb{C}$

If $\overline{M}^{\mathcal{L}}$ smooth, then sympl. $\overset{\xi}{\Leftarrow}$ since contains $M^{\mathcal{L}}_{Dol}$ as Zariski open subset]

Poisson str : $\eta \in H^0(S, \omega_S^\vee)$ Poisson str.

$\mathrm{Ext}^1(F, F\otimes\omega_S) \otimes \mathrm{Ext}^1(F, F\otimes\omega_S) \xrightarrow{\circ} \mathrm{Ext}^2(F, F\otimes\omega_S^{\otimes 2}) \xrightarrow{\eta} \mathrm{Ext}^2(F, F\otimes\omega_S) \xrightarrow[\sim]{\mathrm{Tr}} \mathbb{C}$

Open embedd. : $F \in \overline{M}^{\mathcal{L}}$ st. $\mathrm{supp}(F) \cap D_\infty = \emptyset$

$\rightsquigarrow \mathrm{Supp}(F) \subseteq \mathrm{tot}(\omega_X) \rightsquigarrow E := f_* F$ has $\mathrm{rk} = n$, $\deg = d$

and action $E \xrightarrow{\phi} E \otimes \omega_X$

$\rightsquigarrow$ Higgs bdle

def. by char. poly of ϕ

Stability is compatible & constr. can be inverted via spectral curve.

$\rightsquigarrow$ done ✓

4. Beauville's Thm

Thm [Beauville]

Assume $\mathcal{L} \not\cong \mathcal{O}_\Sigma$.   same def.

Then the cohomology ring $H^*(\overline{M}^{\mathcal{L}}, \mathbb{Q})$ gener. by tautological classes.

Pf: We prove thm under following asspt'ns:

any stability cond. w/ "good" properties
v fixed numerical class

- X sm. proj. variety
- $M := M^{st}_{X,\sigma}(v)$ smooth proj., $m := \dim M$
- $\exists$ univ. family $\mathcal{F}$ on $X \times M$
- $\forall F_1, F_2 \in M$, $\mathrm{Ext}^{\geq 2}_X(F_1, F_2) = 0$

this is very strong cond., true for $\overline{M}^{\mathcal{L}}$, if $\mathcal{L} \not\cong \mathcal{O}_\Sigma$.

Consider $M \times X \times M$ and

$$\mathcal{H} := R\mathcal{H}om_{p_{13}}(p_{12}^* \mathcal{F}, p_{23}^* \mathcal{F}) \in D^b(M \times M)$$

Asspt'n $\underset{\text{Coh. \& Base Ch.}}{\Longrightarrow}$ $\exists V_0, V_1$ v.bdles on $M \times M$
$\exists u : V_0 \to V_1$ st.

- $\mathcal{H} \underset{qis}{\cong} \{V_0 \xrightarrow{u} V_1\}$
- $\forall x = (F_1, F_2) \in M \times M$, $\exists$ ex. seq.

$$0 \to \mathrm{Hom}(F_1, F_2) \to V_0|_x \xrightarrow{u_x} V_1|_x \to \mathrm{Ext}^1(F_1, F_2) \to 0$$

Since F_1, F_2 stable of same slope, then

$$\mathrm{Hom}(F_1, F_2) = \begin{cases} \mathbb{C}, & \text{if } F_1 \cong F_2 \\ 0, & \text{oth.} \end{cases}$$

Hence, if we let D_u be the degeneracy locus of u

$$D_u := \{x \in M : \mathrm{rk}(u_x) \le \mathrm{rk}(V_0) - 1\} .$$

w/ scheme str. defined by vanishing of all minors of max rk

then (as sets) $D_u = \Delta$ ~ diagonal

$$\rightsquigarrow \quad [D_u] = (\text{pos. const.}) \cdot \delta \in H^{2m}(M \times M, \mathbb{Z})$$

actually, $= 1$, but /$\mathbb{Q}$ irrelevant.

Recall: Thom-Porteous Formula ← see [Fulton, Thm. 14.4]

Y CM scheme,

V_0, V_1 v.bdles on Y, $u : V_0 \to V_1$

$r_0 := \mathrm{rk}(V_0)$, $r_1 := \mathrm{rk}(V_1)$

$k \le \min\{r_0, r_1\}$

$$D_k(u) := \{y \in Y : \mathrm{rk}(u_y) \le k\} = Z(\Lambda^{k+1} u)$$

Then:

- $D_k(u) \neq \emptyset \Rightarrow \operatorname{codim} D_k(u) \le (r_0 - k) \cdot (r_1 - k) =:$ exp. codim.
- if all cpts of $D_k(u)$ have exp. codim., then

$$[D_k(u)] = \det(a_{ij})_{i,j = 1, \dots, r_0 - k} \in H^{2 \cdot \text{exp. codim}}(Y, \mathbb{Z})$$

where $\quad a_{ij} := c_{r_1 - k + j - i}(V_1 - V_0)$

$$c(V_1 - V_0) = c(V_1) / c(V_0)$$

since c mult., this is same as c of complex $V_0 \xrightarrow{u} V_1$ up to sign.

■

For us, $K = r_0 - 1$ $\quad F \in M$

$$\text{exp. codim} = r_1 - r_0 + 1 = -\underbrace{\hom(F,F)}_{=1} + \operatorname{ext}^1(F,F) + 1 = \operatorname{ext}^1(F,F)$$

$$= \text{codim} = m$$

$$\underset{\text{Thom-Porteous formula}}{\rightsquigarrow} [D_u] = c_m(V_1 - V_0) = c_m(-\mathcal{H}) \in H^{2m}(M \times M, \mathbb{Z})$$

$\rightsquigarrow$ δ is (rational positive multiple of) $c_m(-\mathcal{H})$.

Grothendieck-Riemann-Roch Thm:

$$ch(\mathcal{H}) = p_{13*}\big(p_{12}^* ch(\mathcal{U}^\vee) \cup p_{23}^* ch(\mathcal{U}) \cup p_2^* td_X\big)$$

dual in derived categ.

$\rightsquigarrow$ done ✓. ■

for holy Poisson surface (!)

Furthermore:
- $H^i(M, \mathbb{Z})$ torsion-free, $\forall i$
- $H^1(S, \mathbb{Z}) = 0 \Rightarrow H^{odd}(M, \mathbb{Z}) = 0$

<u>Rmks</u> • Pf. over $\mathbb{Z}$ is same, just need to better control coefficients (see [Markman 2, Lem 21, 22, 23])

- If $\nexists$ univ. family $\rightsquigarrow$ use twisted sheaves & similar pf. [Markman 1 & 2]

- [Markman] Pf can be generalized to the case where $\operatorname{Ext}^2(E,E) \cong \mathbb{C}$, $X = K3$, abelian surface or S_{Q_2} $\longleftarrow$ then it works on $M_{Dol}^{Q} \times \overline{M}^{Q}$

In this case $\mathcal{H} \underset{qis}{\cong} (V_{-1} \xrightarrow{u} V_0 \xrightarrow{v} V_1)$ where:

here $m = \dim M$ even

- V_{-1}, V_0, V_1 v.bdles on $M \times M$ — as scheme: they are line bdles
- $\ker(u) = 0$, $\operatorname{supp}(\operatorname{coker}(v)) = \Delta$, $\operatorname{supp}(\operatorname{coker}(u^\vee)) = \Delta$
- $r_0 - r_{-1} - r_1 = m - 2$

use Coh. & Base Ch.

<u>Then</u>: $c_m(-\mathcal{H}) = \delta$ $\quad\rightsquigarrow$ conclude as before.

$c_{m-1}(-\mathcal{H}) = 0$ ▲

Excess Porteous Formula [Ful, Ex. 14.4.7]

Coroll [Mukai]

true $\forall$ Poisson surface

Same ass'pt'ns $\rightsquigarrow$ M <u>connected</u>.

Pf: M_1 conn. cpt. of M.

By the Thom-Porteous Formula, $\forall\, t_i = [F_i] \in M$

$$[t_i] = c_m\big(\underline{R\mathcal{H}om}_{p_2}(p_1^* F_i, p_2^* \mathcal{F})\big) \neq 0 \in H^{2m}(M, \mathbb{Z})$$

on $X \times M$

By the GRR Thm, this class does depend only on $ch(F_i) = v$

Assume $M \neq M_1 \rightsquigarrow$ pick $F_2 \in M \setminus M_1$.

$\underset{\substack{\text{Coh \& Base Ch}\\ + \,\mathrm{Ext}^{\geq 2} = 0}}{\rightsquigarrow}$ $\underline{R\mathcal{H}om}_{p_2}(p_1^* F_2, p_2^* \mathcal{F})[1]$ <u>vector bdle</u>

of $rk = -\chi(v, v) = m - 1$

$\rightsquigarrow c_m = 0$ $\quad \underline{\underline{\mathcal{E}}}$ $\quad \blacksquare$

5. Proof of Main Thm

Consider open inclusion

$$M^{\mathcal{L}}_{Dol} \xhookrightarrow[\text{open}]{i} \overline{M}^{\mathcal{L}}$$

Key Lem ["compactif. of $M^{\mathcal{L}}_{Dol}$ by symplectic cuts", Hausel, Simpson]

$\exists\ \widetilde{M}^{\mathcal{L}}$ smooth proj

$$\begin{tikzcd} & \widetilde{M}^{\mathcal{L}} \arrow[d, "\gamma"] \\ M^{\mathcal{L}}_{Dol} \arrow[ur, hook, "\tilde{i}"] \arrow[r, hook, "i"'] \arrow[ur, phantom, "\circlearrowleft", shift right=3] & \overline{M}^{\mathcal{L}} \end{tikzcd}$$

γ birational, isom on $M^{\mathcal{L}}_{Dol}$

st. $\tilde{i}^*: H^*(\widetilde{M}^{\mathcal{L}}, \mathbb{Q}) \longrightarrow H^*(M^{\mathcal{L}}_{Dol}, \mathbb{Q})$ surjective

over $\mathbb{Z}$?
I could not find pf...
(maybe via Morse theory)

Pf (Main Thm, by assuming Key Lem):

Consider class of diagonal $\tilde{\delta}$ on $\widetilde{M} \times M$
$\bar{\delta}$ on $\overline{M} \times M$

need to use Borel-Moore homology: for X complex ~~manifold~~, $\dim X = m$
$H^{BM}_i(X, \mathbb{Z}) \cong H^{2m-i}(X, \mathbb{Z})$

Beauville/Markman Thm $\Rightarrow$ $\bar{\delta} = c_m(-(id \times i)^* \mathcal{H}) \ \left(= c_m\left(-p_{13*}\left(p_{12}^* \mathcal{E}^\vee \otimes p_{23}^* i^* \mathcal{E}\right)\right)\right)$

$\rightsquigarrow$ $\tilde{\delta} = c_m(-(\gamma \times i)^* \mathcal{H}) \ \left(= c_m\left(-p_{13*}\left(p_{12}^* \gamma^* \mathcal{E}^\vee \otimes p_{23}^* i^* \mathcal{E}\right)\right)\right)$

Then: • $\forall \lambda \in H^*(M, \mathbb{Q})$, pick $\tilde{\lambda}$ lift to $\widetilde{M}$

$\rightsquigarrow$ $\lambda = p_{2*}(\tilde{\delta} \cup p_1^* \tilde{\lambda})$

smooth & proper (!)

$\rightsquigarrow$ $H^*(M, \mathbb{Q})$ gen. by Künneth factors of $\tilde{\delta}$.

- By formula for $\tilde{\gamma}$, Kunneth factors of $\mathbb{F}|_{S_\Sigma \times M}$ generate $H^*(M,\mathbb{Q})$

- $(f\times id)_* \mathbb{F}' = \mathcal{E}$ & $H^*(S,\mathbb{Z}) = H^*(\Sigma,\mathbb{Z}) \oplus \underbrace{[\mathcal{O}_E(-1)]\cdot H^*(\Sigma,\mathbb{Z})}$

  doesn't matter, since supported on D_∞

$\rightsquigarrow$ Künneth factors of $\mathcal{E}$ gen. $\rightsquigarrow$ done ✓

$\boxtimes$

<u>Idea Pf</u> (Key Lem):

<u>Recall</u>: <u>Def</u> X quasi proj., $\mathbb{C}^* \circlearrowright X$

Say X <u>semiprojective</u> if

[Hausel, Rodriguez-Villegas, §1]

- fixed set $X^{\mathbb{C}^*}$ <u>proper</u>
- $\forall x \in X$, $\lim_{\lambda\to 0} \lambda\cdot x$ exists $\in X$

<u>Białynicki-Birula Decomposition</u>: X smooth, semiproj.

Write $X^{\mathbb{C}^*} = \coprod_{i\in I} F_i$, $|I|<\infty$, F_i smooth proj

Then $\exists$ $\mathbb{C}^*$-equiv. very ample line bdle L on X

$\rightsquigarrow$ $\mathbb{C}^* \circlearrowright L|_{F_i}$ as $\mathbb{C}^* \to \mathbb{C}^*$ w/ weight $d_i \in \mathbb{Z}$.

Choose L st. $d_i \geq 0$, $\forall i \in I$.

Partial ordering : $i < j$ iff $\alpha_i > \alpha_j$

Set

$$U_i := \{x \in X : \lim_{\lambda \to 0} \lambda \cdot x \in F_i\}$$

$$D_i := \{x \in X : \lim_{\lambda \to \infty} \lambda \cdot x \in F_i\}$$

Then $D_i, U_i \subseteq X$ locally-closed subschemes, isom. to affine bdles over F_i

We have :
- $X = \coprod_{i \in I} U_i$ BB decomp.
- $\overline{U_i} \subseteq \bigcup_{j \geq i} U_j$

Compactification : $\overline{X}^{HS}$

Hilbert-Mumford crit. $\rightsquigarrow$ $X^{ss} = X \setminus (\coprod D_i)$

Thm [Simpson] $Z := X^{ss}/\mathbb{C}^*$ proj. orbifold

$\square$

Def $\overline{X}^{HS} := (X \times \mathbb{C})^{ss}/\mathbb{C}^* = X \sqcup Z$

usual action

Then $\overline{X}^{HS}$ proj. orbifold   "symplectic cutting".

this also have $\mathbb{C}^*$-action, which is trivial on Z

Key pt. $C := \coprod_{i \in I} D_i \overset{\iota}{\hookrightarrow} X$ cone

Then C proj. and $\iota^*: H^*(X, \mathbb{Z}) \to H^*(C, \mathbb{Z})$ isom.

uses $\bar{X}^{HS}$ uses BB decomp.

(!)
$\Rightarrow$ X has pure cohomology $\leftarrow$ X smooth $\Rightarrow$ weights $\geq *$, C proj. $\Rightarrow$ weights $\leq *$ $\rightsquigarrow$ "=".

Hence: $H^*(\bar{X}^{HS}, \mathbb{Q}) \to H^*(X, \mathbb{Q})$ surjective

Indeed, by long ex. seq. $(m = \dim X)$

$$\begin{array}{ccccc} \rightsquigarrow H^p(\bar{X}^{HS}, \mathbb{Q}) & \to & H^p(X, \mathbb{Q}) & \overset{\delta}{\to} & H^{p+1}(Z, \mathbb{Q}) \\ \| & & \| & & \| \\ H_{2m-p}(\bar{X}^{HS}) & \to & H_{2m-p}(X) & \to & H_{2m-p-1}(Z) \end{array}$$

$\dim Z = m-1$

But orbifolds have pure cohom. & connecting morph. δ is morph. of Hodge str. $\rightsquigarrow$ must be trivial $\lrcorner$

[Simpson]

Apply this to $X = M^{\chi}_{Dol}$, which is semiproj by action $(E, \phi) \mapsto (E, \lambda \cdot \phi)$

Let $\tilde{M}$: resolution of closure of diagonal of $M \times M$ in $\bar{M}^{HS} \times \bar{M}$

$\rightsquigarrow$ done$^{\checkmark}$. $\blacktriangle$

Rmk $\bar{M}^{HS} \neq \bar{M}$

since Hitchin map for $\bar{M}^{HS}$ compactifies w/ $\mathbb{P}(1, 1, 2, \dots, m-1)$ on the base (action of $\mathbb{C}^*$ on A^{χ} is: t on $H^0(\Omega_\Sigma)$, t^2 on $H^0(\Omega_\Sigma^{\otimes 2})$...)

$\blacktriangle$

Easier pfs (thanks to Olivier!)

(1) $M^{\mathcal{L}}_{Dol}$ has pure cohom.:

Consider Hitchin map $h : M^{\mathcal{L}}_{Dol} \to A^{\mathcal{L}}$, which is proper and surj.

The action of $\mathbb{C}^*$ by $\lambda \cdot (E, \phi) = (E, \lambda\phi)$ on the base $A^{\mathcal{L}}$ is given by

$$\lambda \cdot (a_1, a_2, \ldots, a_m) = (\lambda a_1, \ldots, \lambda^m a_m) \quad , \quad a_i \in H^0(\Sigma, \Omega_{\mathcal{L}}^{\otimes i})$$

$$\det(t \cdot I - \lambda\phi) = \lambda^m \cdot \det\left(\frac{t}{\lambda} I - \phi\right) = \lambda^m \cdot \left(\frac{t^m}{\lambda^m} + a_1 \cdot \frac{t^{m-1}}{\lambda^{m-1}} + \ldots + a_m\right)$$

$\rightsquigarrow$ it gives a retraction of $M^{\mathcal{L}}_{Dol}$ to $h^{-1}(0) \leftarrow$ proper (!)

Now, use weights $\rightsquigarrow$ ✓

(2) $H^*(\overline{M}^{\mathcal{L}}, \mathbb{Q}) \to H^*(M^{\mathcal{L}}, \mathbb{Q})$ surj, if $\mathcal{L} \not\cong \mathcal{O}_\Sigma$.:

By [Deligne, Hodge II, Cor. 3.2.17], since both $\overline{M}^{\mathcal{L}}$ and $M^{\mathcal{L}}$ are smooth and $\overline{M}^{\mathcal{L}}$ proj., the image of $i^*|_{H^r(\overline{M}^{\mathcal{L}})}$ is exactly $W_r H^r(M^{\mathcal{L}}) \underset{\uparrow \text{ pure}}{=} H^r(M^{\mathcal{L}})$

$\rightsquigarrow$ ✓

Rmk This follows essentially from [Grothendieck, Brauer gps III, (9.3)]:

the image of $i^*|_{H^r(\overline{M}^{\mathcal{L}}, \mathbb{Z})}$ indep. on smooth compactif.

$\rightsquigarrow$ reduce to a compactif. w/ snc div. $\leftarrow$ I don't know if $\overline{M}^{\mathcal{L}} \setminus M^{\mathcal{L}}_{Dol}$ snc. ?

$\rightsquigarrow$ done by Deligne's constr. ✓

▲

• •