Strong perversity and vanishing cycles

fix $(r,d)=1$

Recall we have a Dolbeault moduli space $M^{st}_{r,d} = M^{Dol}_{d}(GL_r)$: parametrizing stable rank r, deg d Higgs bundle, with Hitchin map.

$$M^{Dol}_{r,d} \xrightarrow{h} \bigoplus_{i=1}^{n} H^0(X, \Omega_X^{\otimes i}) =: A$$

It ~~has a~~ induces a perverse filtration on $H^\cdot(M^{st}_{r,d}, \mathbb{Q})$ by A

$$P_i H^\cdot(M, \mathbb{Q}) = \mathrm{Im}\Big(H^{\cdot - \cancel{\dim M^{st}_{r,d}} + \underbrace{\dim M^{st}_{r,d} \times_A M^{st}_{r,d} - \dim M^{st}_{r,d}}_{\overset{\parallel}{-a+r}}}(A, \tau_{\le i} Rh_* \mathbb{Q}[\dim M^{st}_{r,d} - r]) \to H^\cdot(M^{st}_{r,d}, \mathbb{Q}) \Big)$$

defect of semismallness

which gives information on the non-semismallness of h.

Another is the Betti moduli space $M^B_{r,d}$ parametrizing repn $\rho : \pi_1(X \setminus \{x\}) \to GL_r(\mathbb{C})$ with monodromy $\prod X_i Y_i X_i^{-1} Y_i^{-1} = e^{\frac{2\pi i d}{r}} \mathrm{Id}$. It's a smooth affine variety. MHS gives

$$W_0 H^*(M^B_{r,d}, \mathbb{Q}) \subset \cdots \subset W_n H^*(M^B_{r,d}, \mathbb{Q}) \subset \cdots$$

Conj (P=W) Under nonabelian Hodge theorem,

$$P_k H^m(M^{Dol}_{r,d}, \mathbb{Q}) = W_{2k} H^m(M^B_{r,d}, \mathbb{Q}).$$

Variant: $\widehat{M}^{Dol}_{r,d} := \widetilde{M}^{Dol}_{r,d} / \Gamma$, where $\widetilde{M}^{Dol}_{r,d}$ parametrizing $(\mathcal{E}, \theta)$ with $\det(\mathcal{E}) \simeq \mathcal{L} \in \mathrm{Pic}^d$, $\mathcal{L}$ fixed and $\mathrm{tr}\, \theta = 0$, $\Gamma = \mathrm{Pic}^0 X [n]$: can be seen as PGL_n-Higgs with deg d. it's a $\boxed{\text{nonsingular DM stack}}$, with a Hitchin map

$$\widehat{M}^{Dol}_{r,d} \xrightarrow{\hat{h}} \bigoplus_{i \ge 2} H^0(X, \Omega_X^{\otimes i}).$$

We can also define $\widehat{M}^B_{r,d}$, under Simpson correspondence,

Conj (P=W for PGL_n, which is equiv. to GL_n)

$$P_k H^m(\widehat{M}^{Dol}_{r,d}, \mathbb{Q}) = W_{2k} H^m(\widehat{M}^B_{r,d}, \mathbb{Q})$$

Thm (Markman, Shende)

1) $c_k(\gamma) := \int_\gamma c_k(\mathcal{U})$ for $\mathcal{U} \to X \times \widehat{M}^{Dol}_{r,d}$, $\gamma \in H^*(X, \mathbb{Q})$, ~~generates~~ generates $H^*(\widehat{M}^{Dol}_{r,d}, \mathbb{Q})$.

2) $c_k(\gamma)$ lies in $W_{2k}H^{**} \cap F^kH^*(\hat{M}^B_{r,d}, \mathbb{Q})$ under Simpson correspondence

<u>Lem</u> If $W_{2k} \subset P_k$ for all k, then $W_{2k} = P_k$.

<u>Pf</u> Both filtrations satisfies curious Q hard Lefschetz.

$$\dim Gr^{P}_{\frac{\dim \hat{M}}{2}-m} H^j_{\bullet}(\hat{M}, \mathbb{Q}) \cong Gr^{P}_{\frac{\dim \hat{M}}{2}+m} H^{j+2m}(\hat{M}, \mathbb{Q})$$

and

$$\dim Gr^{W}_{\dim \hat{M}-2m} H^j(\hat{M}, \mathbb{Q}) \cong Gr^{W}_{\dim \hat{M}+2m} H^{j+2m}(\hat{M}, \mathbb{Q})$$ ∎

Hence, to show $P=W$, it suffices to show ~~for each~~

$$\prod_{i=1}^{s} c_{k_i}(\gamma_i) \in P_{\sum_i k_i} H^*(\hat{M}^{Dol}_{r,d}, \mathbb{Q}) \qquad (*)$$

But even if we know each $c_k(\gamma) \in P_k H^*$, we cannot conclude ~~the~~ $\prod c_k(\gamma)$ $\in P_{\sum k}$, we need the right notion: strong perversity.

<u>Def</u> For a $\gamma \in H^{\ell}(X, \mathbb{Q})$, $f: X \to Y$ proper, we say γ ~~is~~ has strong perversity c ~~of~~ wrt to f if (under $\gamma: \mathbb{Q} \to \mathbb{Q}[\ell]$)

$$(c<\ell) \qquad \gamma({}^p\tau_{\le i} Rf_*\mathbb{Q}) \subset {}^p\tau_{\le i+(c-\ell)}(Rf_*\mathbb{Q}[\ell])$$

<u>Properties</u>

- If γ has ... c, then

$$\gamma \cup - : P_i H^m(X, \mathbb{Q}) \to P_{i+c} H^{m+\ell}(X, \mathbb{Q})$$

<u>Pf</u> $\gamma \cup -$ maps $\mathrm{Im}\, H^{m-\dim X+r}(Y, {}^p\tau_{\le i}(Rf_*\mathbb{Q}[\dim X - r])$

to $\mathrm{Im}\, H^{m-\dim X+r}(Y, {}^p\tau_{\le i+(c-\ell)}(Rf_*\mathbb{Q}[\dim X - r+\ell]))$

$$\parallel$$

$\mathrm{Im}\, H^{m+\ell-\dim X+r}(Y, ({}^p\tau_{\le i+c} Rf_*\mathbb{Q}[\dim X - r])[\ell])$

which is $P_{i+c} H^{m+\ell}(X, \mathbb{Q})$. ∎

- If $\gamma_i \in H^{\ell_i}$ has ... c_i, then $\gamma_1 \cup \cdots \cup \gamma_s \in H^{\sum \ell_i}$ has ... $\sum c_i$.

<u>Pf</u>. $\overset{\gamma}{\bigcup_{i=1}^{s}} \gamma_i ({}^p\tau_{\le i} Rf_*\mathbb{Q}) \subset {}^p\tau_{\le i+(\sum c_i - \sum \ell_i)}(Rf_*\mathbb{Q}[\sum \ell_i])$. ∎

Now, we use the evaluation map at $x \in X$, get a <u>smooth map</u>. (need to prove) $ev_x : \mathcal{M}^{Dol, \mathcal{L}(p)}_{r,d} \to [\mathfrak{sl}_n / SL_n]$, hence $\mu \circ ev_x$ gives

$$ev_x^* \circ \varphi_\mu = \varphi_{\mu \circ ev_x} \circ ev_x^*$$

which gives (notice $ev_x^{-1}(0) = \widehat{\mathcal{M}}^{Dol, \mathcal{L}}_{r,d}$)

$$\varphi_{\mu \circ ev_x}(\mathbb{C}) = ev_x^* \circ i_* \mathbb{C} = i_* \mathbb{C}$$

$$\Rightarrow \quad \varphi_{\mu \circ ev_x}(IC) = IC_{\widehat{\mathcal{M}}^{Dol, \mathcal{L}}_{d,r}}$$

Hence we need to discuss the relation between vanishing cycles and strong perversity.

<u>Prop</u> Let $g : X \to \mathbb{A}^1$, $X' \subset X_0$ s.t. $\phi_g := \phi_g(IC_X)$ is $IC_{X'}$. Assume further we have a diagram (f proper) (all thing non-singular)

$$\begin{tikzcd} X' \arrow[r, "i"] \arrow[d, "f'"] & X \arrow[d, "f"] \arrow[dr, dashed, "g"] & \\ Y' \arrow[r] & Y \arrow[r, "\nu"'] & \mathbb{A}^1 \end{tikzcd}$$

closed embedding

Then if $\gamma \in H^\ell(X, \mathbb{Q})$ has strong perversity c wrt f, then $i^* \gamma \in H^\ell(X, \mathbb{Q})$ has strong perversity c wrt f'.

[In our case,

$$\begin{tikzcd} \widehat{\mathcal{M}}^{Dol, \mathcal{L}}_{d,r} \arrow[r, "i"] \arrow[d, "f'"] & \widehat{\mathcal{M}}^{Dol, \mathcal{L}(p)}_{d,r} \arrow[d, "f"] \arrow[drr, bend left, "g"] & & \\ \widehat{\mathcal{A}}^{\mathcal{L}} \arrow[r, "i'"] & \widehat{\mathcal{A}}^{\mathcal{L}(p)} \arrow[r, "ev_x"] & \mathfrak{sl}_n /\!/ SL_n \arrow[r, "\mu"] & \mathbb{A}^1 \end{tikzcd}$$

$$\underbrace{ev_x \quad \mu}_{\nu}$$

Then from strong perversity of $c_k(\overset{\mathcal{U}}{\gamma})$ on $\widehat{\mathcal{M}}^{Dol, \mathcal{L}(p)}_{d,r}$ we can deduce ... of $c_k(\overset{\mathcal{U}}{\gamma})$ on $\widehat{\mathcal{M}}^{Dol, \mathcal{L}}_{d,r}$]

Pf We know $\gamma : Rf_* \mathbb{Q} \to Rf_* \mathbb{Q}[\ell]$ satisfies $\gamma({}^p\tau_{\le i} Rf_* \mathbb{Q}) \subset {}^p\tau_{\le i+(c-\ell)}(Rf_* \mathbb{Q}[\ell])$.

The problem is i^* doesn't preserve perverse truncation, but vanishing cycle functor preserves t-exactness! So we have

$$\phi_g(\gamma) : \left(\phi_\nu(Rf_* \mathbb{Q}_X) \to \phi_\nu(Rf_* \mathbb{Q}_X^{[\ell]}) \right) \overset{\text{base change 1)}}{\downarrow =} \left(Rf_* \phi_g(\mathbb{Q}_X) \to Rf_* \phi_g(\mathbb{Q}_X) \right)$$

$$\overset{\text{for}}{\underset{(i^* f)}{\text{base change}}} = Rf'_* \phi_g(\mathbb{Q}_X) \to Rf'_* \phi_g(\mathbb{Q}_X^{[\ell]}) \quad \text{because } \phi_g(-) \in D^b_c(X').$$

which will satisfies $\phi_g(\gamma)\left({}^p\tau_{\le i} Rf'_* \phi_g(\mathbb{Q}_X)\right) \subset {}^p\tau_{\le i+(c-\ell)} Rf'_* \phi_g(\mathbb{Q}_X[\ell])$

i.e.

$$\phi_g(\gamma)\left({}^p\tau_{\le i} Rf'_* \mathbb{Q}_{X'}\right) \subset {}^p\tau_{\le i+(c-\ell)} Rf'_* \mathbb{Q}_{X'}[\ell]$$

ie, to prove the P=W conj, we need only to show strong perversity k of $c_k(\mathcal{U})$, plutot $c_k(\mathcal{U}) \in H^{2k}(\cancel{X} \times \widehat{M}^{Dol}_{r,d}, \mathbb{Q})$ has strong perversity k wrt $id \times h$.

It will be better if we switch to $\mathcal{L}$-twisted Higgs bundle, denote its moduli space as $\widehat{M}^{Dol,\mathcal{L}}_{r,d}$, because now the Hitchin fibration is $h^{\mathcal{L}} : \widehat{M}^{Dol,\mathcal{L}}_{r,d} \to \widehat{A}^{\mathcal{L}}$ has good support theorem [Chaudouard-Laumon]. As we will see, $\widehat{M}^{Dol,\mathcal{L}}_{r,d}$ is a closed ~~sub~~ immersion in $\widehat{M}^{Dol,\mathcal{L}(p)}_{r,d}$, and its IC sheaf can be identified with a vanishing cycle sheaf wrt. some function $f : \widehat{M}^{Dol,\mathcal{L}(p)} \to \mathbb{A}^1$. We need first the notion of vanishing cycle sheaf.

Def The <u>nearby cycle</u> is defined as a functor (for $f : X \to \mathbb{C}$)

$$\psi_f = i^* R(j \circ \hat{\pi})_* (j \circ \hat{\pi})^{-1}(\mathcal{F}^\bullet) : D_b(X) \to D_b(X_0)$$

where

$$\begin{tikzcd} X_0 \arrow[r, hook, "i"] & X & X_\eta \arrow[l, hook', "j"'] \arrow[d] & \widetilde{X}_\eta \arrow[l, "\hat{\pi}"'] \arrow[d] \\ & & \mathbb{C}^\times & \mathbb{C} \arrow[l, "\exp"'] \end{tikzcd}$$

For all points $x \in X_0$, consider the Milnor ~~fibration~~ fiber $F_x = \mathring{B}_\delta(x) \cap X_t$ whose homotopy type is independent of t for $0 < t \ll \varepsilon$. Then

$$\mathcal{H}^k(\psi_f \mathcal{F})_x \cong \mathbb{H}^k(F_x, \mathcal{F})$$

because $(j \circ \hat{\pi})(\mathring{B}_\delta(x))$ is homeo. to F_x.

$$\varinjlim_{U \ni x} R^k (j \circ \hat{\pi})_* (j \circ \hat{\pi})^{-1}(\mathcal{F})(U) = \varinjlim_{\mathring{B}_\delta(x) \ni x} \cdots = \varinjlim_{\mathring{B}_\delta(x) \ni x} \mathbb{H}^k((j \circ \hat{\pi})^{-1}(\mathring{B}_\delta(x)), \mathcal{F}|\ldots)$$

Def The <u>vanishing cycle</u> is defined as the cone of natural map

$$\phi_f(\mathcal{F}) := \mathrm{cone}(i^* \mathcal{F} \to i^* R(j \circ \hat{\pi})_* (j \circ \hat{\pi})^{-1}(\mathcal{F}))[-1]$$

Hence, we have an exact sequence

$$\cdots \to \mathbb{H}^k(\mathring{B}_\delta(x) \cap X_0, \mathcal{F}) \to \mathbb{H}^k(\mathring{B}_\delta(x) \cap X_t, \mathcal{F}) \to \mathcal{H}^{k+1}(\phi_f(\mathcal{F}))_x \to \cdots$$

From this, we know that at least for $\mathcal{F} = \mathbb{C}$, $\mathrm{supp}(\mathcal{H}^k(\phi_f(\mathbb{C}))_x) \subset X_0 \cap \{x \mid df = 0\}$ ~~sing~~

<u>Ex</u> $\mu = B(\cdot,\cdot) : \mathfrak{sl}_n /\!/ SL_n \to \mathbb{A}^1$ satisfies

$$\phi_\mu(\mathbb{C}[\dim \mathfrak{sl}_n]) = i_* \mathbb{C}, \quad i : \{0\} \to \mu^{-1}(0),$$

because the only critical point is 0, and locally μ is given by $c(\sum z_i^2)$. The Milnor fiber is given by $\sum |z_i|^2 \leq \delta$, $\sum z_i^2 = t$. Hence $\dim \mathcal{H}^0(\phi_\mu(\mathbb{C}[\dim \mathfrak{sl}_n])) = 1$.

this $\phi_g(\gamma)$ will be $i^*\gamma$: $Rf^!_* \mathbb{Q}_X \to Rf^!_* \mathbb{Q}_X[l]$, because

$$\phi_g(\gamma) = \phi_g\left(\mathbb{Q}_X \xrightarrow{\gamma} \mathbb{Q}_X[l]\right) = \phi_g(\mathbb{Q}_X) \xrightarrow{\gamma \otimes \phi_g(\mathbb{Q}_X)} \phi_g(\mathbb{Q}_X[l]) = i^*\phi_g(\mathbb{Q}_X) \xrightarrow{i^*\gamma \otimes \phi_g(\mathbb{Q}_X)} i^*\phi_g(\mathbb{Q}_X[l])$$

$$(\text{under some shift}) = i^* \mathbb{Q}_{X'} \xrightarrow{i^*\gamma} i^* \mathbb{Q}_{X'}[l] .$$

Appendix:

Properties of ψ_g & ϕ_g :

chasing diagram

- Base change : $X \xrightarrow{g} Y \xrightarrow{f} \mathbb{A}^1 \Rightarrow \phi_f(Rg_*\mathcal{F}) \cong Rg_*(\phi_{f\circ g}\mathcal{F})$. for g proper)
- functoriality : $X \xrightarrow{g} Y \xrightarrow{f} \mathbb{A}^1 \Rightarrow g^*\phi_f(\mathcal{F}) \cong \phi_{f\circ g}(g^*\mathcal{F})$ for g smooth
- They are t-exact (hard!)

Pf. First we can show ϕ_f, $\psi_f[-1]$ commutes with Verdier dual functor, i.e

$$\phi_f(\mathbb{D}\mathcal{F}) = \mathbb{D}\phi_f(\mathcal{F}) \Leftarrow$$

$$\text{cone}\left(i^*\mathbb{D}\mathcal{F} \to i^* R(j\circ\hat{\pi})_*(j\circ\hat{\pi})^*\mathbb{D}\mathcal{F}\right)[-1] = \text{cone}\left(\mathbb{D}(i^!\mathcal{F}) \to \mathbb{D}(i^! R(j\circ\hat{\pi})_!(j\circ\hat{\pi})^!\mathcal{F})\right)[-1]$$

And then we prove the right t-exactness of $\psi_f[-1]$ (then right t-exactness of ϕ_f follow automatically). Assume f is ~~given by an alg. family~~ regular outside 0. Then local monodromy theorem says monodromy operator satisfies $(T^k-1)^n = 0$. By changing to a k-covering of $D-\{0\}$, we can assume $(T-1)^n = 0$. We have

$$i^*Rj_*j^* \to \psi_f \xrightarrow{T-1} \psi_f \to .$$

$i \geq 0$

$$\text{If } \mathcal{F} \in \text{Perv}(X) \Rightarrow {}^pH^i(\psi_f\mathcal{F}) \xrightarrow{T-1} {}^pH^i(\psi_f\mathcal{F}) \to {}^pH^{i+1}(i^*Rj_*j^*\mathcal{F}) = 0$$

surj & nilpotent

$$\Rightarrow {}^pH^i(\psi_f\mathcal{F}) = 0 \quad \Rightarrow \psi_f\mathcal{F} \in {}^pD^{\leq -1}(X) .$$