Curious Hard Lefschetz for Char. Varieties.

$g \geq 0 \quad n \geq 1, \quad n' \in (\mathbb{Z}/n\mathbb{Z})^* \qquad \Sigma_g$: genus g surface.

$W_{\leq \bullet}$ weight filtration on $H_c^*(\mathcal{M}_n, \mathbb{C})$.

<u>Thm</u>: (Mellit, conj. by Hausel-R. Villegas)

(i) $H_c^\bullet(\mathcal{M}_n, \mathbb{C})$ is of Tate type ($\Rightarrow W_{\leq \bullet}$ is even)

(ii) $\dim Gr^W_{d-2m} H_c^j(\mathcal{M}_n, \mathbb{C}) = \dim Gr^W_{d+2m} H_c^{j+2m}(\mathcal{M}_n, \mathbb{C})$. $\forall m$

(iii) $\exists$ a symplectic form $\omega \in H^2(M_n, \mathbb{C})$. s.t. $(\wedge \omega)^m$ induces above isomorphism.

(ii) "Curious Poincaré duality"

(iii) "Curious Hard Lefschetz"

<u>Rem</u>: (i) Proof uses a parabolic analog $\mathcal{M}_n^{par}$ of $\mathcal{M}_n$ (add punctures & non-scalar monodromy).

$\Rightarrow$ This holds in generality

(ii) Consequence of $P=W$.

From relative Hard Lefschetz for $\mu : \mathcal{M}_n^{Hitch} \to \mathcal{A}$

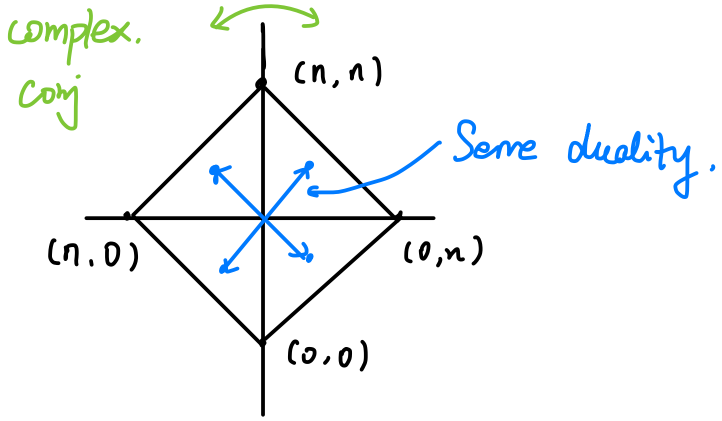
we want $P_{\leq \bullet}$ satisfies CHL & CPD

$\Rightarrow$ to have $P_{\leq k} = W_{\leq 2k}$ enough to show

$$P_{\leq k} \subseteq W_{\leq 2k} \quad (\forall k).$$

§ Recall Hodge Theory notions

X: smooth proj. dim n. $H^j(X;\mathbb{C})$ is a pure Hodge Struc. of weight j. $H^j(X;\mathbb{C}) = \bigoplus_{p+q=j} H^{p,q}(X)$.



(Usual Hard Lefschetz)

$\omega \in H^{1,1}(X.\mathbb{C})$ hyperplane section.

$(\wedge\omega)^k : H^{p,q}(X;\mathbb{C}) \cong H^{n-p,n-q}(X,\mathbb{C})$

(Hodge filtration)

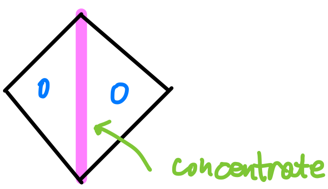
$F_{\geq p} = \bigoplus_{i\geq p} H^{i,j}$ $\quad \bar{F}_{\geq q} = \bigoplus_{j\geq q} H^{i,j}$.

<u>MHS</u>:

$\begin{cases} H \text{ is a } \mathbb{C}\text{-vector space} \\ W_{\leq \bullet} \nearrow \text{ weight filtration} \\ F_{\geq \bullet} \searrow \text{ Hodge filtration.} \end{cases}$

<u>Def</u>: $(H, W_{\leq\bullet}, F_{\geq\bullet})$ is of Tate type if $H^{p,q} \neq \{0\} \Rightarrow p=q$.

i.e.



$\Rightarrow W_{\leq 2k+1} = W_{\leq 2k}$. & $F_{\geq\bullet}$, $\bar{F}_{\geq\bullet}$ splits $W_{\leq\bullet}$.

Eg of varieties of Tate type.

- $X = \mathbb{C}$ $\quad H_c^{\bullet}(X;\mathbb{C}) = H_c^2(X;\mathbb{C}) = H^{1,1}(X,\mathbb{C})$ of wt 2.
- $X = \mathbb{C}^\times$ $\quad H_c^0(X;\mathbb{C}) = \{0\}$
  $H_c^1(X;\mathbb{C}) = \mathbb{C}$ all of weight 0.
  $H_c^2(X;\mathbb{C}) = \mathbb{C}$ all of weight 2.

ws all of $(\mathbb{C}^*)^i \times (\mathbb{C})^j$ are of Tate type.

X smooth proj. $Y \subseteq X$ sub-var. $[Y] \in H^{p,p}(X)$ if $\dim Y = p$.

<u>CHL vs. HL</u>

usual HL: ω is of $\begin{cases} \text{coh deg} = 2. \quad F, \bar{F}\text{-deg} = (1,1) \\ \text{weight} = 2 \end{cases}$

curious HL: ω is of $\begin{cases} \text{coh deg} = 2. \quad F. \bar{F} \text{ deg} = (2,2) \\ \text{weight} = 4. \end{cases}$

<u>Def</u>: let V be a f. dim vector space. a pair $(G_{\leq \bullet}, \omega)$ w/.
$\begin{cases} \omega \in \mathrm{End}(V) \\ G_{\leq \bullet} \nearrow \text{ filtration on } V; \quad \omega \cdot G_{\leq k} \subseteq G_{\leq k+2}. \end{cases}$

satisfies CHL w/. middle deg $2d$. if $\forall i$

$$\omega^i \cdot G_{\leq 2d-2i} / G_{\leq 2d-2i-1} \xrightarrow{\simeq} G_{\leq 2d+2i} / G_{\leq 2d+2i-1}.$$

[Same def. for $G_{\geq \bullet} \searrow$ filtration]

<u>Rem</u>: If ω is nilpotent. $\forall 2d$, $\exists$ a unique decreasing filtration. s.t. $(G_{\geq \bullet}, \omega)$ satisfies CHL of mid. deg $2d$.

<u>Def</u>: If X is a complex quasi-proj. variety. $\omega \in H^2(X; \mathbb{C})$. then we say $(X; \omega)$ satisfies CHL of mid deg $2d$ if. $(H_c^*(X; \mathbb{C}), W_{\leq}, \wedge \omega)$ does

<u>Rmk</u>: Assume that (X, ω) satisfies CHL of mid degree $2d$. & X is of Tate type. Then since $F_{\geq \bullet}$ splits $W_{\leq \bullet}$,

one can show $(H_c^*(X;\mathbb{C}), F_{\geq \cdot}, \omega)$ satisfies

CHL $\Rightarrow$ $F_{\geq \cdot}$ is completely determined by $\wedge\omega$.

<u>Basic example</u>:

$$((\mathbb{C}^\times)^{2n}, \omega) \qquad \omega = \sum_{\substack{i<j \\ i,j=1,\cdots,2n}} \omega_{i,j} \frac{dx_i}{x_i} \wedge \frac{dx_j}{x_j}.$$

$(\omega_{i,j})$ non-deg.

middle deg $= 2n$.

Weak stratification:

X: topological space. $\mathcal{P}$: poset. (finite).

$X = \coprod_{\sigma \in \mathcal{P}} X_\sigma$ is a weak stratification. if. $\forall \sigma \in \mathcal{P}$.

$\overline{X_\sigma} \subseteq X_{\leq \sigma}$.

ws $\exists$ a refinement of $\mathcal{P}$ into totally-ordered poset.

$\Rightarrow$ get a (usual) weak stratification $X = \coprod_i X_i$

$\overline{X_i} = \coprod X_{\leq i}$

Prop. let X be a quasi-proj. alg. variety. w/. a weak stratification.

- X is of Tate type if all X_i are.
- let $\omega \in H^2(X, \mathbb{C})$. If $\exists d$. st. $\forall i$ $(X, \omega|_{X_i})$ satisfies CHL w/. mid dim d. then so does (X, ω)

Idea of Proof of Thm:

(a) Introduce $\mathcal{M}_n^{par}$ (Σ w/. punctures)

(b) When $\exists$ a puncture w/. regular semi-simple monodromy, construct $\mathcal{M}_n^{par}$ into cells ("braid varieties") whose strata are $(\mathbb{C}^*)^{2d-2i} \times \mathbb{C}^i$ for some i.

(c) Construct a (tautological) class $\omega \in H^2(\mathcal{M}_n^{par})$ whose restriction to each cell satisfies CHL.

$\Rightarrow$ CHL + Tate for $\mathcal{M}_n^{par}$

(d) Use a Springer argument to deduce the case of $\mathcal{M}_n$.

Braid varieties:

$$\begin{array}{ccc} Br_n & & \sigma_i \\ \big\downarrow & & \big\downarrow \\ W & & s_i \end{array} \qquad \pi_1\left(\mathbb{C}^n/\Delta_{big}/W=\mathfrak{S}_n\right) = \left\langle \sigma_i^{\pm 1} \;\middle|\; \begin{array}{l} (\sigma_i\sigma_{i+1})^3 = Id \\ (\sigma_i\sigma_j)^2 = Id \quad \text{if } |i-j|>1 \end{array} \right\rangle$$

Br_n^+ monoid generated by $\{\sigma_i\}$.

$\exists$ <u>Canonical</u> lifting $W \longrightarrow Br_n^+$

$$\underbrace{\pi = s_{i_\ell} \cdots s_{i_1}}_{\text{reduced expression.}} \rightsquigarrow \tilde{\pi} = \sigma_{i_\ell} \cdots \sigma_{i_1}$$

$\forall\ \beta \in Br_n^+ \rightsquigarrow \tilde{S}_\beta$ braid variety.

$\beta \in Br_n^+ \quad \beta = \sigma_{i_\ell} \cdots \sigma_{i_1}$ reduced. exp.

$$f_\beta : \mathbb{C}^\ell \longrightarrow GL_n$$
$$(z_1, \cdots, z_\ell) \longmapsto M_{i_\ell}(z_\ell) \cdots M_{i_1}(z_1). \qquad M_j(z_j) = \begin{pmatrix} 1 & & & & \\ & \ddots & & & \\ & & 1 & & \\ & & & \boxed{\begin{matrix} 0 & 1 \\ 1 & z_j \end{matrix}} & \\ & & & & \ddots \\ & & & & & 1 \end{pmatrix}$$

Eg. If $\beta = \overset{\frown}{\pi}$ for $\pi \in W$. then:

$$f_\beta : \mathbb{C}^\ell \xrightarrow{\sim} \pi \cdot U_\pi^- \qquad U_\pi^- = U^- \cap \pi^{-1} U^-$$

(Bruhat cell)

$$\overset{\frown}{S_\beta} = f_\beta^{-1}(B) = \{ (z_1, \cdots, z_\ell) \mid M_{i_\ell}(z_\ell) \cdots M_{i_1}(z_1) \subseteq B \}.$$

Prop. $\overset{\frown}{S_\beta}$ is independent of reduced expression of β.

Cell decomposition of $\overset{\frown}{S_\beta}$.

$$T : \overset{\frown}{S_\beta} \longrightarrow W^{\ell(\beta)+1}.$$
$$(\underline{z}) \longmapsto (p_0(\underline{z}), p_1(\underline{z}), \cdots, p_\ell(\underline{z}))$$

$$M_{i_k}(z_k) \cdots M_{i_1}(z_1) \in B\, p_k(\underline{z})\, B.$$

FACT. $p(\underline{z})$ is a walk in W. $\begin{cases} p_0(\underline{z}) = 1 \\ p_\ell(\underline{z}) = 1. \end{cases}$

$$p_{k+1}(\underline{z}) = \begin{cases} s_{i_{k+1}}\, p_k(\underline{z}) & \text{if } \ell(s_{i_{k+1}} p_k(\underline{z})) > \ell(p_k(\underline{z})) \\ s_{i_{k+1}}\, p_k(\underline{z}) & \text{else.} \\ \text{or } p_k(\underline{z}) & \end{cases}$$

Analogy of Hecke alg.

Thm: $\forall \beta$, $\mathbb{C}_f = \tau^{-1}(f) \subseteq \overset{\frown}{S_\beta}$ is isomorphic to

$$(\mathbb{C}^*)^{\#\text{stays}} \times \mathbb{C}^{\#\text{up}}.$$