Exposé lundi 06 Mars
Théorie de Springer

Références:

1978 • T.A. Springer : A construction of representations of Weyl Groups.
2019 • Pramod. N. Achar : Perverse sheaves & applications to representation theory.
1982 • T.A. Springer : Quelques applications de la cohomologie d'intersection.
1981 • G. Lusztig : Green polynomials and singularities of unipotent classes.
1982 • Beilinson, Bernstein, Deligne, Gabber : Analyse topologique sur les espaces sing.
1981 • Borho - MacPherson, Representations des groupes de Weyl et homologie d'intersection pour les variétés de nilpotents.

Introduction:

• $G = GL_n(\mathbb{C}) \rightsquigarrow \mathfrak{g} = Lie(G) = M_n(\mathbb{C}) \supset \mathcal{N} = \{x \in \mathfrak{g} \mid x^n = 0, n \gg 0\}$

(G acts on $\mathfrak{g}$ via conjugaison $\dashrightarrow$ $\mathcal{N}$ stable under G)

"the nilpotent cone."

$$\left\{\begin{array}{c}G\text{-orbits} \\ \text{in } \mathcal{N}\end{array}\right\} \xleftrightarrow[\text{Jordan blocks}]{1:1} \left\{\begin{array}{c}\text{partitions of} \\ \text{the integer } n\end{array}\right\} \xleftrightarrow[\text{work of Frobenius, Schur...}]{1:1} \left\{\begin{array}{c}\text{finite dimensional irred.} \\ \text{representations of} \\ \mathfrak{S}_n\end{array}\right\}_{/\cong}$$

"nilpotent orbits"

~~• $\mathfrak{S}_n$ the symmetric group. Then finite dimensional representations of $\mathfrak{S}_n$ are in correspondence with partitions...~~

Springer theory explains how to make this combinatorial fact geometric:

- **Step 1:** "Notice that $\mathfrak{S}_n$ = Weyl Group of GL_n."
- **Step 2:** Attach varieties to orbits "Springer fibers"
- **Step 3:** Build an action of $\mathfrak{S}_n$ on the (top) cohomology of those varieties.

We'll follow a modern presentation in the language of Perverse sheaves.

$G \supset (\nabla) \supset (\diagdown)$

Fix $G \supset \underset{\text{Borel}}{B} \supset T \quad \overset{\rightsquigarrow}{\text{Lie}} \quad \mathfrak{g} \supset \mathfrak{b} \supset \mathfrak{h}$

$W = \frac{N_G(T)}{T}$ $\overset{\text{SI}}{\mathfrak{S}_n}$ for GL_n.

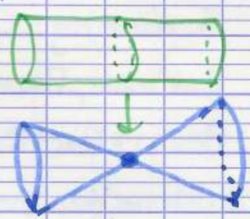
Context : $k = \mathbb{C}$, $G = GL_n$, $\mathfrak{g} = \text{Lie}(G) = M_n$

But we could have other fields, set G to be any connected reductive group.... the setting of GL_n will make things a little bit more concrete.

Variety will mean quasi-projective variety $/\mathbb{C}$.

[redacted]

$\mathcal{N}$ is a closed singular subvariety of $\mathfrak{g}$.



Example : $\cdot\ G = SL_2$

$$\mathcal{N} = \left\{ \begin{pmatrix} a & b \\ c & -a \end{pmatrix} \middle/ \ a^2 + bc = 0 \right\}$$

Consists of two G-orbits $\{0\}$, $\underbrace{\left\{ \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \right\}}_{\text{dense}}$ "regular orbit".

We introduce the Springer resolution of $\mathcal{N}$, but first set

$$\mathcal{F}lag = \left\{ \overset{\mathcal{F}}{\overbrace{\{0\} = F_0 \subset \dots \subset F_{n-1} \subset F_n = \mathbb{C}^n}} \ \middle| \ \begin{cases} \dim(F_i) = i \\ F_i \text{ vector space} \end{cases} \right\}$$

$\mathcal{F}lag$ is a projective variety acted on by G, if $g \in G$:

$$g \cdot \mathcal{F} = gF_0 \subset \dots \subset gF_n$$

It identifies with the quotient G/B:

$\mathcal{F}^{can} = \mathbb{C}e_1 \subset \mathbb{C}e_1 \oplus \mathbb{C}e_2 \subset \dots \subset \mathbb{C}^n = F_n$, with $\mathbb{C}e_1 = F_1$

$$\mathcal{F}lag \xleftrightarrow{1:1} G/B \xleftrightarrow{1:1} \mathcal{B} = \{\text{All Borel subgroups}\}$$

$$\mathcal{F} \longmapsto \text{Stab}_G(\mathcal{F})$$

$$g \cdot \mathcal{F}^{can} \longleftarrow gB \xmapsto{\text{"All borels are conjugated"}} gBg^{-1}$$

$\mathfrak{g}$ also acts on $\mathcal{F}lag$ via $x \in \mathfrak{g}$, $x \cdot \mathcal{F} = xF_0 \subset \dots \subset xF_n$,

and $\mathcal{B}^{Lie} = \{$ All Borel sub-Lie-algebras of $\mathfrak{g} \}$

Then $\mathcal{B} \xrightarrow[1:1]{\text{Lie}} \mathcal{B}^{Lie}$

$$\begin{aligned}
\mathcal{F}lag &\overset{1:1}{\longleftrightarrow} \mathcal{B}^{\text{Lie}} \\
\mathcal{F} &\longmapsto \mathrm{Stab}_{\mathfrak{g}}\mathcal{F} \\
g\cdot\mathcal{F}^{can} &\longleftarrow\!\shortmid\ gbg^{-1}
\end{aligned}$$

Now we can introduce:

$$\begin{aligned}
\widetilde{\mathcal{N}} &= \{(\mathcal{F},x)\in \mathcal{F}lag\times\mathcal{N} \mid x\cdot\mathcal{F}\subset\mathcal{F}\} \\
&\simeq \{(gB,x)\in G/_B\times\mathcal{N} \mid \underset{\substack{\Updownarrow \\ x\,\in\, gbg^{-1}}}{Ad(g)^{-1}(x)\in b}\}
\end{aligned}$$

__Theorem:__ The map $\underset{(\mathcal{F},x)\longmapsto x}{\mu:\widetilde{\mathcal{N}}\to\mathcal{N}}$ is called the springer resolution, if

$x\in\mathcal{N}$ then $\mu^{-1}(x) := \mathcal{B}_x \simeq \{\mathcal{F}\in\mathcal{F}lag \mid x\cdot\mathcal{F}\subset\mathcal{F}\}$

Only depends on the G-orbit of x $\simeq \{gB\in G/_B \mid x\in gbg^{-1}\}$ "Borels containing x"

is called the Springer fiber over x.

- $\widetilde{\mathcal{N}}$ is a smooth variety and $\mu:\widetilde{\mathcal{N}}\to\mathcal{N}$ is a proper map [redacted].

__Example:__ • $G = SL_2$ (cf 1)  ($\underset{x\neq 0.}{\mathcal{B}_x} \simeq \{pt\}$)

In general $\mathcal{B}_0 \simeq G/_B$ "Every Borel contains 0"

But Springer fibers are singular in general.

• $G = SL_3$

$3 = 3 = 2+1 = 1+1+1$ then if $x = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix} = 0 \rightsquigarrow \mathcal{B}_x \simeq G/_B$

0 $\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ $\begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix}$ $x = \begin{pmatrix}0&1&0\\0&0&1\\0&0&0\end{pmatrix} \rightsquigarrow \mathcal{B}_x = \{pt\}$

closed "subregular" orbit, dense "regular" orbit

$x = \begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}$ we can show that it looks like $\mathbb{P}^1 \underset{\{pt\}}{\sqcup} \mathbb{P}^1$

S_1

[redacted]

[redacted]

Goal: Fix $x \in \mathcal{N}$, want to define an action of W on each $H^i(\mathcal{B}_x, \underline{\mathbb{C}})$ ← Cohomology of the constant sheaf $\underline{\mathbb{C}}$ on each fiber

[redacted]

· It is enough to define an action of W on $\tilde{\mathcal{N}}$ stabilising each fiber, not quite obvious to do.

$$\begin{tikzcd} \underline{\mathbb{C}} \quad \tilde{\mathcal{N}} \arrow[d, "\mu"] & \mathcal{B}_x \quad \underline{\mathbb{C}} \arrow[l, hook'] \arrow[d] \\ \mathcal{N} & \{x\} \arrow[l, hook'] \end{tikzcd}$$

What can we say about

"the Springer sheaf" $R\mu_* \underline{\mathbb{C}}$?

If $\mathbb{C}[W] \to \mathrm{End}(R\mu_* \underline{\mathbb{C}})$

then by proper base change we'll have our action of W on the $H^i(\mathcal{B}_x, \underline{\mathbb{C}})$.

Introduce Grothendieck-Springer simultaneous resolution:

$$\begin{tikzcd} \underset{\text{smooth}}{\tilde{\mathfrak{g}}^{r,s}} \arrow[r, hook, "\text{open dense}"] \arrow[d, "\mu|_{\tilde{\mathfrak{g}}^{r,s}}"] & \underset{\text{smooth}}{\tilde{\mathfrak{g}}} \arrow[d, "\mu", "\text{proper}"'] & \underset{\text{smooth}}{\tilde{\mathcal{N}}} \arrow[l, hook', "\text{closed}"'] \arrow[d, "\mu|_{\tilde{\mathcal{N}}}"] & \underset{\text{singular}}{\mathcal{B}_x} \arrow[l, hook'] \arrow[d] \\ \text{connected} \rightarrow \underset{\text{smooth}}{\mathfrak{g}^{r,s}} \arrow[r, hook, "\text{open dense}"] & \underset{\text{smooth}}{\mathfrak{g}} \arrow[d, "\text{car}"] & \mathcal{N} \ (\text{singular}) \arrow[l, hook', "\text{closed}"'] \arrow[d, "\text{car}"] & \{x\} \arrow[l, hook'] \\ \{\text{scindés à racines simples}\} \arrow[r, hook] & \mathbb{C}[X]^{Sn}_{unit} & \{0\} \arrow[l, hook'] \end{tikzcd}$$

less important for us

smooth → $\tilde{\mathfrak{g}} = \{ (\mathcal{F}, x) \in \mathcal{F}lag \times \mathfrak{g} \mid x \cdot \mathcal{F} = \mathcal{F} \} \xrightarrow{\mu} \mathfrak{g}$

$\quad = \{ (\mathfrak{b}', x) \in \mathcal{B}^{lie} \times \mathfrak{g} \mid x \in \mathfrak{b}' \}$

because Affine × Projective $\xrightarrow{\text{proj.}}$ Affine is proper + ...

~~Proposition~~: The map μ is proper and extends the Springer-resolution.
surjective ⇐ $\mathbb{C}$ is algebraically closed

To understand the terminology, we can introduce

car: $\mathfrak{g} \longrightarrow \mathbb{C}[X]^{Sn}_{unit} \simeq \mathbb{A}^n / \mathfrak{S}_n$

$x \longmapsto \chi_x$

$\prod (X - \alpha_i) \longleftarrow (\alpha_1, \dots, \alpha_n)$

Then $\mathcal{N} = \mathrm{car}^{-1}(\{0\})$ and the map μ simultaneously resolves each fiber of the map car

Now if $x \in \mathfrak{g}$, $\mu^{-1}(x) = \{\mathcal{F} \in \mathcal{F}lag \mid x \cdot \mathcal{F} \subset \mathcal{F}\}$ still doesn't have a natural W action:

Example: If $x = \lambda I_n$, $\lambda \in \mathbb{C}$ then $\mu^{-1}(x) \simeq \mathcal{F}lag$ and there is no natural action of $\mathfrak{S}_n$ on the variety of all complete flags.

The idea is to restrict to the regular semi-simple locus, define

$$\mathfrak{g}^{r,s} = \{x \in \mathfrak{g} \mid x \text{ is diagonalizable with } n \text{ distincts eigenvalues}\}$$

Proposition: [redacted] $\mathfrak{g}^{r,s}$ is a dense (Zariski)-open of $\mathfrak{g}$, moreover the map $\mu : \tilde{\mathfrak{g}} \to \mathfrak{g}$ restricts to a Galois covering (in the analytic topology) of $\mathfrak{g}^{r,s}$ of Group W.

Proof: • The density follows from considering the "discriminant" $Res(P,P')$.
• Let $x \in \mathfrak{g}^{r,s}$, then up to conjugating by an element of G, $x \sim \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$ $\lambda_i \neq \lambda_j$, $i \neq j$.

To describe $\mu^{-1}(x)$ we can assume x is of this form.

$\mu^{-1}(x) = \{\mathcal{F} \in \mathcal{F}lag \mid x \cdot \mathcal{F} \subset \mathcal{F}\}$

And because $\lambda_i \neq \lambda_j$, $i \neq j$, we have the following:

"Lemme des noyaux"

Lemma: Let V be a subspace of $\mathbb{C}^n$ stable by x, then $V = \bigoplus_{i=1}^{n} V \cap \underbrace{vect(e_i)}_{\dim 1}$ i.e V is a direct sum of ~~eigen~~ some eigenspaces of x.

This implies directly that every $\mathcal{F} \in \mu^{-1}(x)$ is of the form

$$\{0\} \subset \mathbb{C}e_{\sigma(1)} \subset \mathbb{C}e_{\sigma(1)} \oplus \mathbb{C}e_{\sigma(2)} \subset \dots \subset \bigoplus_{i=1}^{n} \mathbb{C}e_{\sigma(i)} = \mathbb{C}^n$$

for some unique $\sigma \in \mathfrak{S}_n$.

This allows us to define an action of $\mathfrak{S}_n$ on $\hat{\mathfrak{g}}^{r,s}$ that is transitive on ~~the~~ each fibers and free.

This implies in return that we get an identification

$$\mathfrak{g}^{1,s} \xrightarrow{\sim} \widetilde{\mathfrak{g}^{1,s}}/W$$

as topological spaces. ■

Now we can make sheaf theory enter the picture.

<u>Recall</u> (from Mattia's talk)

If X is a complex algebraic variety (NOT ASSUMED TO BE SMOOTH!)
(We let X^{an} for the underlying top. space incl the analytic top.)

<u>Local systems</u>: A sheaf $\mathcal{L}$ on X^{an} s.t there exists an open (analytic) cover of $X^{an} = \bigcup U_\alpha$ s.t. $\mathcal{L}|_{U_\alpha} \cong \underline{\mathbb{C}}^d$.

<u>Algebraic Stratifications</u>: $X = \bigsqcup_{s\in\mathcal{S}} X_s$ where

- $\mathcal{S}$ is finite
- For all $s\in\mathcal{S}$, X_s is a smooth locally closed algebraic subv.
- $\forall s\in\mathcal{S}$; $\overline{X}_s = \bigcup_{t\leq s} X_t$

<u>Constructible sheaf wrt $\mathcal{S}$</u>: $\mathcal{F}\in Sh(X^{an})$ s.t
$\forall s\in\mathcal{S}$, $\mathcal{F}|_{X_s}$ is a local system

$\longrightarrow$ $\mathcal{F}\in D^b(X,\mathbb{C})$ is said to be constructible if each $H^k(\mathcal{F})$ is constructible.

(Shift to the left $\left\{ \begin{array}{cccccc} -2 & -1 & 0 & 1 & 2 & 3 \\ & \mathcal{F}_{-1} & \mathcal{F}_0 \to & \mathcal{F}_1 \to & \mathcal{F} & \\ & & [1] & & & \end{array} \right.$ $H^k(\mathcal{F}[1]) = H^{k+1}(\mathcal{F})$)

Introduce $D^b_c(X,\mathbb{C}) \xhookrightarrow[\text{category}]{\text{full sub.}} D^b(X,\mathbb{C})$

[redacted]

$\mathcal{F}\in D^b(X) \rightsquigarrow$ $\mathrm{supp}(\mathcal{F}) = \overline{\bigcup_{i\in\mathbb{Z}} \mathrm{supp}(H^i(\mathcal{F}))}$

Def. of the support

Verdier duality: $\mathbb{D} : D^b_c(X^{an}, \mathbb{C})^{op} \longrightarrow D^b_c(X, \mathbb{C})$ is an involutive functor.

Now if $f : X \to Y$ is an algebraic morphism

$$\begin{tikzcd} D^b_c(X) \arrow[r, shift left, "f_!"] \arrow[d, "\mathbb{D}_X"'] \arrow[dr, phantom, "\circlearrowleft"] & D^b_c(Y) \arrow[l, shift left, "f^!"] \arrow[d, "\mathbb{D}_Y"] \\ D^b_c(X) \arrow[r, shift left, "f_*"] & D^b_c(Y) \arrow[l, shift left, "f^*"] \end{tikzcd}$$

Rmq: If f is proper
$f_! = f_*$ and
" $f_* \circ \mathbb{D} = \mathbb{D} * f_*$ "

Perverse sheaves

Condition de sup. ${}^pD^b_c(X, \mathbb{C})^{\leq 0} = \{\mathcal{F} \in D^b_c(X) \mid \forall i \in \mathbb{Z}, \dim(\mathrm{supp}(H^i(\mathcal{F}))) \leq -i\}$

t-structure $\rightsquigarrow \mathbb{D}$

Condition de cosup. ${}^pD^b_c(X, \mathbb{C})^{\geq 0} = \{ \text{——————} \ \mathbb{D}\mathcal{F} \}$

Heart $\rightsquigarrow$ $\boxed{\mathrm{Perv}(X, \mathbb{C}) \simeq {}^pD^b_c(X)^{\leq 0} \cap {}^pD^b_c(X)^{\geq 0}}$

Example: If X is smooth of dimension d then
$\mathrm{Loc}(X)[d] \underset{\text{sub-cat}}{\hookrightarrow} \mathrm{Perv}(X, \mathbb{C})$
" Shifted local systems "

Explain on the diagram what this yields.

Intermediate extension.

Théorème: $Y \underset{\text{dense}}{\overset{j}{\hookrightarrow}} X \overset{i}{\hookleftarrow} X \setminus Y$

Let $\mathcal{F}^\bullet \in \mathrm{Perv}(Y)$, there exists a perverse sheaf $j_{!*}(\mathcal{F})$ on X uniquely determined up to isomorphism by

Union of the support of its cohomology

- $\mathrm{supp}(j_{!*}(\mathcal{F})) = X$
- $j_{!*}(\mathcal{F})|_Y = \mathcal{F}$ and $\begin{cases} i^!\mathcal{F} \in {}^pD^b_c(Z, \mathbb{C})^{\geq 1} \\ i^*\mathcal{F} \in D^b_c(Z, \mathbb{C})^{\leq -1} \end{cases}$

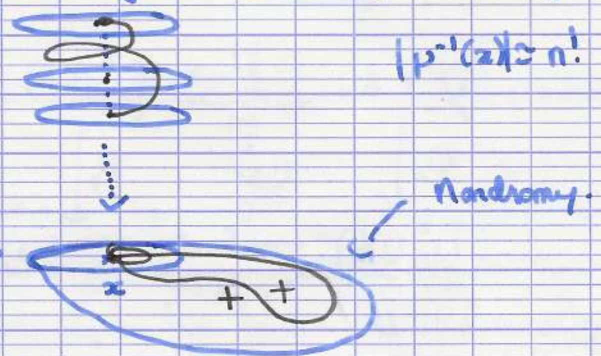
Moreover, $j_{!*} : \mathrm{Perv}(Y) \longrightarrow \mathrm{Perv}(X)$ defines a functor.

This formalism will allow us to "spread the action of $\mathfrak{S}_n$".

Lemma: $\mu_* \underline{\mathbb{C}}_{\tilde{\mathfrak{g}}^{1,s}}[\dim \mathfrak{g}] \simeq \underline{\mathbb{C}[\mathfrak{S}_n]}_{\mathfrak{g}^{1,s}}[\dim \mathfrak{g}]$.

Proof:

$$\begin{array}{c} \tilde{\mathfrak{g}}^{1,s} \longleftrightarrow \\ \downarrow \mu \\ \mathfrak{g}^{1,s} \longleftrightarrow \end{array}$$



It is a local system because ... ∎

Definition:
$$\begin{array}{c} X \\ f \downarrow \\ Y \end{array}$$
surjective morphism is said to be semi-small if Y admits a stratification $Y = \bigsqcup_{s \in \mathcal{S}} Y_s$ s.t.

$$\forall y \in Y_s, \quad \dim(f^{-1}(y)) \leq \frac{1}{2}(\dim(X) - \dim(Y_t))$$

Moreover it is said to be if there exists a dense open s.t. the stratification is adapted and for every $Y_s \subset Y \backslash W$ we have a strict inequality.

$$\begin{array}{c} \quad X \\ \quad \downarrow \\ W \underset{\text{dense}}{\overset{o}{\hookrightarrow}} Y \hookleftarrow Y \backslash W \end{array}$$

" Rmq: Etale morphism are the prototype of semi-small maps "

"Stratification free definition" Rmq: f semi-small $\iff \dim(X \times_Y X) \leq \dim(X)$

f small wrt $W \iff \dim(f^{-1}(Y\backslash W) \times_Y f^{-1}(Y \backslash W)) < \dim(X)$.

X a smooth variety,

Théorème: Let $f : X \to Y$ be a proper surjective semi-small [redacted] morphism. Then $Rf_* \mathbb{C}_X[\dim(X)]$ is perverse. Moreover is f is small with respect to ω, then $Rf_* \underline{\mathbb{C}}_X[\dim(X)] \simeq j_{!*}(f_* \underline{\mathbb{C}}_{f^{-1}(\omega)}[\dim(X)])$

$$\begin{tikzcd} f^{-1}(\omega) \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \omega \arrow[r, hook, "j"'] & Y \end{tikzcd}$$

$Y = \bigsqcup Y_s$

Proof: • Notice that $\mathbb{C}_X[\dim(X)]$ is auto-dual wrt to Verdier duality that is $\mathbb{D}_X(\mathbb{C}_X[\dim(X)]) \simeq \mathbb{C}_X[\dim(X)]$, now because f is proper it is enough to check that
$$f_* \mathbb{C}_X[\dim(X)] \in {}^p D^b_c(X)^{\geq 0}.$$

• Let $y \in Y$, then $y \in Y_s$ and for all $i \in \mathbb{Z}$
$$H^i(f_* \mathbb{C}_X[\dim(X)])_y \underset{\text{proper}}{\simeq} H^i(f^{-1}(y), \mathbb{C}_X[\dim(X)]) = H^{\dim(X)+i}(f^{-1}(y), \mathbb{C}_X)$$

Now we have the following cohomological bound:

cf Achar's book (Lemma: If Z is a variety of dim d,
$$H^j(Z, \underline{\mathbb{C}}) = 0 \text{ whenever } j < 0 \text{ or } j > 2d$$)

~~Hence whenever $i \in \mathbb{Z}$ satisfies~~ $\cancel{\begin{cases} i + \dim(X) > 2\dim(f^{-1}(y)) \\ \text{or} \\ i + \dim(X) < 0 \end{cases}}$

Using this bound, $H^i(f_* \mathbb{C}_X[\dim(X)])_y = 0$
unless $0 \leq i + \dim(X) \leq 2\dim(f^{-1}(y)) \leq \dim(X) - \dim(Y_t)$
that is $\dim(Y_t) \leq -i$

Hence $\operatorname{supp}(H^i(f_* \mathbb{C}_X[\dim(X)])) \subset \underbrace{\bigcup \text{ strata de dim } \leq -i}$ •

Now in the small case, we use the caracterisation of intermediate extension.

• $\mathrm{supp}(Rf_*\mathbb{C}_X[\dim(X)]) \supset \mathrm{supp}(H^{-\dim(X)}(Rf_*\mathbb{C}_X[\dim(X)])) = Y$

And if $y \in Y$, $H^{-\dim(X)}(Rf_*\mathbb{C}_X[\dim(X)])_y$

$$\simeq H^0(f^{-1}(y), \mathbb{C}_X) \simeq \mathbb{C}^{\pi_0(f^{-1}(y))} \neq \emptyset$$

•
$$\begin{tikzcd}
f^{-1}(\omega) \arrow[r] \arrow[d] & X \arrow[d, "f"] & f^{-1}(Y\setminus\omega) \arrow[l] \arrow[d] & f^{-1}(y) \arrow[l, hook'] \arrow[d] \\
\omega \arrow[r, hook, "j"'] & Y & Y\setminus\omega \arrow[l, hook', "i"] & \{y\} \arrow[l, hook']
\end{tikzcd}$$

We need to check $\begin{cases} i^* f_* \mathbb{C}_X[\dim(X)] \in {}^pD^b_c(Y\setminus\omega)^{\leq -1} \\ i^! f_* \mathbb{C}_X[\dim(X)] \in {}^pD^b_c(Y\setminus\omega)^{\geq 1} \end{cases}$

Notice that

$$i^*(\underbrace{\mathbb{D}_Y f_* \mathbb{C}_X[\dim(X)]}_{\simeq\, f_*\mathbb{C}_X[\dim(X)]})$$
$$\|$$
$$\mathbb{D}_Y(i^! f_* \mathbb{C}_X[\dim(X)])$$

Therefore it suffices to check one or another because Verdier duality exchanges

$${}^pD^b_c(Y\setminus\omega)^{\leq -1} = {}^pD^b_c(Y\setminus\omega)^{\leq 0}[1]$$
$${}^pD^b_c(Y\setminus\omega)^{\geq 1} = {}^pD^b_c(Y\setminus\omega)^{\geq 0}[-1]$$

[redacted]

~~In fact RHom(F[1], $\mathbb{C}_X$[1]) = RHom(F, $\mathbb{C}_X$~~

We can check that

$$i^* f_* \mathbb{C}_X[\dim(X)] \in {}^pD^b_c(Y\setminus\omega)^{\leq -1} \simeq \{$$

i.e. we must check

$$\forall y \in Y\setminus\omega,\ \forall i \in \mathbb{Z},\ \dim(\mathrm{supp}(H^i(F))) < i$$

and it follows from the same computation with a strict inequality ~~because~~ inherited of smallness ■

Back to Springer

$$\begin{tikzcd}
\underline{\mathbb{C}} \quad \tilde{g}^{1,s} \arrow[r, hook, "\circ"] \arrow[d, "\mu"] & \tilde{g} \ \underline{\mathbb{C}} \arrow[d, "\mu"] & \tilde{\mathcal{N}} \ \underline{\mathbb{C}} \arrow[l, hook', "/"'] \arrow[d, "\mu|_{\tilde{\mathcal{N}}}"] & \mathcal{B}_x \ \underline{\mathbb{C}} \arrow[l, hook'] \arrow[d] \\
g^{1,s} \arrow[r, hook, "j \atop \circ", "\text{dense}"'] & g & \mathcal{N} \arrow[l, hook', "i"'] & \{x\} \arrow[l, hook']
\end{tikzcd}$$

$\mu_* \mathbb{C}_{g^{1,s}} \simeq \underline{\mathbb{C}[W]}$ $\qquad j_{*!}(\underline{\mathbb{C}[W]}) \simeq R\mu_* \underline{\mathbb{C}}$

Theorem: • μ is proper surjective small wrt $g^{1,s}$.
(easier) → • $\mu|_{\tilde{\mathcal{N}}}$ is semi-small for the stratification of $\mathcal{N}$ w.r.t nilpotent orbits.

cf Achar's Book

Proof: Omitted, involves the Steinberg variety (not that hard though)

Consequence:
Let $\sigma \in \mathfrak{S}_n$, it determines $\sigma : \underline{\mathbb{C}[\mathfrak{S}_n]} \to \underline{\mathbb{C}[\mathfrak{S}_n]}$
By functoriality of $j_{*!}$ it determines

$$\begin{array}{ccc} j_{*!}(\sigma) : \ j_{*!}\underline{\mathbb{C}[\mathfrak{S}_n]} & \to & j_{*!}\underline{\mathbb{C}[\mathfrak{S}_n]} \\ \wr\| & & \wr\| \\ R\mu_* \underline{\mathbb{C}}_{\tilde{g}} & \longrightarrow & R\mu_* \underline{\mathbb{C}}_{\tilde{g}} \end{array}$$

And hence for all $x \in \mathcal{N}$ and all $i \in \mathbb{Z}$,

$$\begin{array}{ccc} \sigma : \ H^i(R\mu_* \underline{\mathbb{C}}_{\tilde{g}})_x & \to & H^i(R\mu_* \underline{\mathbb{C}}_{\tilde{g}})_x \\ \wr\| \ \text{PBC} & & \wr\| \ \text{PBC} \\ H^i(\mathcal{B}_x, \underline{\mathbb{C}}) & \longrightarrow & H^i(\mathcal{B}_x, \underline{\mathbb{C}}). \end{array}$$

"original theorem" (Theorem (Springer correspondance) ← coeff in $\mathbb{Q}$ rather than $\mathbb{C}$

8.3.8 ~~For each $x \in \mathfrak{S}$~~ Every irreducible $\mathbb{Q}[W]$-module occurs inside the $\mathbb{Q}[W]$-module $H^{2\dim(\mathcal{B}_x)}(\mathcal{B}_x, \mathbb{Q})$ for some $x \in \mathcal{N}$ (unique up to conjugaison...)