BASICS

$\bar{k} = k$, $F = k(X)$

X smooth projective curve geometrically connected (ie, $X \underset{k}{\otimes} \bar{k}$ connected)

G connected semisimple group / k

$\mathfrak{g} = \mathrm{Lie}\, G$

$\chi : \mathfrak{g} \longrightarrow \mathfrak{c} = \mathrm{Spec}\, k[\mathfrak{g}]^G = \mathfrak{g}/\!/G = \mathfrak{t}/\!/W = \mathbb{A}^r$

$f_1, \dots f_r$ = generators of $k[\mathfrak{g}]^G$ of degrees $d_1, \dots d_r$

$d_i - 1$ = exponent

$\mathfrak{c} \simeq \mathbb{A}^r \circlearrowleft \mathbb{G}_m$ weights $d_1, \dots d_r$

$[\mathfrak{g}/G] \longrightarrow \mathfrak{g}/\!/G = \mathfrak{c} \xrightarrow{\varepsilon} \mathfrak{g}$ Kostant section , $\mathfrak{c} \simeq e + \mathfrak{z}_{\mathfrak{g}}(f)$

$\chi : [\mathfrak{g}/G \times \mathbb{G}_m] \longrightarrow [\mathfrak{c}/\mathbb{G}_m]$

LEM: X = projective variety

(a) Y = quasi-projective variety $\Rightarrow$ $\mathrm{Maps}(X,Y)$ = quasi-projective variety

(b) G = algebraic group, $\rho : G \to GL(V)$ representation, $\rho(E) = E \times_G V$

$\mathrm{Maps}(X, [V/G]) \simeq \{(E, \varphi);\ E\ G\text{-torsor over } X,\ \varphi \in H^0(X, \rho(E))\}$

$\square$

$D \in \mathrm{Div}(X)$

$\rho_D \in \mathrm{Pic}(X) = \{\mathbb{G}_m\text{-torsor over } X\}$ $\mathbb{G}_m$-torsor associated with $\mathcal{O}(D)$

$\mathcal{M}$ = stack parametrizing pairs (E, φ) with $\begin{cases} E \in \mathrm{Bun}_G(X) \\ \varphi \in H^0(X, \mathrm{ad}(E) \otimes \rho_D) \end{cases}$

$$\simeq \mathrm{Maps}(X, [\mathfrak{g}/G \times \mathbb{G}_m]) \underset{\mathrm{Maps}(X, B\mathbb{G}_m)}{\times} \rho_D \ni h_{E,\varphi}$$

Compose $(E, \varphi) \in \mathcal{M}$ with $\chi : [\mathfrak{g}/G] \longrightarrow \mathfrak{car}$

$$h_a : X \xrightarrow{h_{E,\varphi}} [\mathfrak{g} / G \times \mathbb{G}_m] \xrightarrow{\chi} [\mathfrak{car} / \mathbb{G}_m]$$

h_a = a pair $(\mathcal{L}_D, a)$ with $a \in \mathcal{A}$

$[\mathfrak{car}/\mathbb{G}_m] = [\mathbb{A}^r / \mathbb{G}_m] \Longrightarrow \mathcal{A} = H^0(X, \mathbb{A}^r \times_{\mathbb{G}_m} \mathcal{O}(D))$

NB: $\mathcal{A} = \bigoplus_{i=1}^{r} H^0(X, \mathcal{O}_X(d_i D))$

$= \bigoplus_{i=1}^{n} H^0(X, \mathcal{O}_X(i D))$ if $G = GL_n$

$f : \mathcal{M} \longrightarrow \mathcal{A}$, $(E, \varphi) \longmapsto a$ = Hitchin map

$\mathcal{M}_a = f^{-1}(a)$, $a \in \mathcal{A}$

PB: Understand $f_* \overline{\mathbb{Q}_\ell}$

I = universal centralizer

$= \{(x, g) \in \mathfrak{g} \times G ;\ gxg^{-1} = x\}$

= group scheme over $\mathfrak{g}$

LEM: $\exists!$ Abelian group scheme $J \to \mathfrak{car}$ with a map

$\chi^* J \to I$ is invertible over $\mathfrak{g}^{reg}$ $\square$

$h_a^* J$ = group scheme over X, $h_a : X \to [\mathfrak{car}/\mathbb{G}_m]$

$P_a = \{h_a^* J\text{-torsors over } X\}$

LEM : $P_a \circlearrowright \mathcal{M}_a$

PROOF : $(E, \varphi) \in \mathcal{M}_a$

$$h_a : X \xrightarrow{h_{E,\varphi}} [\mathcal{G} / G \times \mathbb{G}_m] \xrightarrow{\chi} [\mathrm{ca} / \mathbb{G}_m]$$

$$h_a^* J = h_{E,\varphi}^* \chi^* J \longrightarrow h_{E,\varphi}^* I = \{\text{autom. of } (E,\varphi) \text{ over } X\}$$

$\Rightarrow$ we can twist (E, φ) by any J-torsor $\square$

EX : $G = GL_n$

$x \in \mathrm{End}\,V$, $a \in k[t]$ char. pol. of x

$Z_{\mathrm{End}\,V}(x) =$ subalg. in $\mathrm{End}\,V$ gen$^{\mathrm{ed}}$ by x

$\Uparrow$

$k[t]/(a) \longrightarrow Z_{\mathrm{End}\,V}(x)$, $t \mapsto x$ isomorphism if x regular matrix

$b \in k[t]$ min. pol, $I_x = k[t]/(b) = k[t]/(a)$ if x regular

$J_a = (k[t]/(a))^\times \longrightarrow Z_{\mathrm{End}\,V}(x)^\times = I_x$ (matrices compagnon)

$\mathrm{Tot}(\mathcal{O}(D)) = \mathrm{Spec}\left(\bigoplus_{i \geq 0} \mathcal{O}(iD)^{-1}\right) \supset Y_a \longrightarrow X$ spectral curve

$\mathcal{M}_a = \{$ torsion free $\mathcal{O}_{Y_a}$-module of generic rank 1 $\}$

$=$ compactified Jacobian of Y_a (if $a \in \mathcal{A}^\heartsuit$)

$\mathcal{P}_a = \{$ invertible $\mathcal{O}_{Y_a}$-module $\} = \mathrm{Pic}(Y_a)$

$=$ Jacobian of Y_a $\circlearrowleft \mathcal{M}_a$ by $\otimes$ $\square$

$\mathfrak{g} \supset \mathfrak{g}^{rs} = \{$ regular semi-simple $\}$

$\chi^{-1}(\mathfrak{car}^{rs}) = \mathfrak{g}^{rs}$

$\mathcal{A}^{\heartsuit} = \{ a \in \mathcal{A} ;$ h_a maps gen^{q} into $\mathfrak{car}^{ssr} \}$

LEM : $\deg(D) \geq 2g$

(a) The Picard stack $P \to \mathcal{A}$ is smooth over $\mathcal{A}^{\heartsuit}$

(b) $\mathcal{M}|_{\mathcal{A}^{\heartsuit}}$ is smooth . It is a $P|_{\mathcal{A}^{\heartsuit}}$ -torsor

$\square$

EX : $G = GL_n$

$\mathcal{A}^{\heartsuit} \subseteq \{ a ; Y_a$ reduced $\}$

$a \in \mathcal{A}^{\heartsuit} \Rightarrow P_a = \mathrm{Pic}(Y_a) \overset{\text{open}}{\subset} \mathcal{M}_a = \overline{\mathrm{Pic}(Y_a)}$

$$\begin{tikzcd} & X \arrow[r, "h_a"] & {[\mathfrak{car}/\mathbb{G}_m]} \\ a \in \mathcal{A}^{\heartsuit} \quad \Rightarrow & U_a \arrow[u, phantom, sloped, "\subset"] \arrow[r] \arrow[ur, phantom, "\square"] & {[\mathfrak{car}^{rs}/\mathbb{G}_m]} \arrow[u, phantom, sloped, "\subset"] \end{tikzcd}$$

$$U_a \neq \emptyset$$

$x \notin U_a \Rightarrow a_x := h_a|_{D_x} \in \mathfrak{car}^{rs}(F_x) \cap \mathfrak{car}(\mathcal{O}_x)$

$K_x = \mathrm{Frac}(\mathcal{O}_x)$, $D_x = \mathrm{Spec}(\mathcal{O}_x)$

$\tilde{a}_x \in (\mathfrak{g} \otimes K_x)^{rs}$ in the Kostant section which lifts a_x

$Gr_{G,x} = G(K_x)/G(\mathcal{O}_x) \supset Sp_{G(\mathcal{O}_x), \tilde{a}_x}$ affine Springer fiber

LEM: $P_{a_x} \overset{(1)}{:=} \{ h_a^* J \text{ torsor on } D_x + \text{triv}^n \text{ on } D_x^\times \}$

$= H^0(D_x^\times, h_a^* J) / H^0(D_x, h_a^* J)$

$\overset{(2)}{\simeq} Z_{G(K_x)}(\tilde{a}_x) / Z_{G(\mathcal{O}_x)}(\tilde{a}_x) \circlearrowright Sp_{G(\mathcal{O}_x), \tilde{a}_x}$

PROOF: Use Beauville-Laszlo's theorem

$\square$

$$\prod_{x \notin U_a} P_{a_x} \xrightarrow[\text{hom}]{gp} P_a = \{ h_a^* J\text{-torsors on } U_a + \text{triv}^n \text{ on } \prod_{x \notin U_a} D_x^\times \} \Big/ \prod_{x \notin U_a} H^0(D_x, h_a^* J)$$

LEM (Ngo's product formula)

$$\mathcal{M}_a \simeq \prod_{x \notin U_a} Sp_{G(\mathcal{O}_x), \tilde{a}_x} \overset{\prod_{x \notin U_a} P_{a_x}}{\times} P_a$$

$\square$

$\mathcal{A}^{ani} \subset \mathcal{A}^{\heartsuit}$ open of a's such that $\pi_0(P_a)$ finite

LEM:

(a) $f : \mathcal{M} \longrightarrow \mathcal{A}$ proper over $\mathcal{A}^{ani}$

(b) $\mathcal{A}^{ani} \neq \emptyset \Rightarrow G$ semi-simple

$\square$

$\widetilde{W} = X_*(T) \times W$

$\mathcal{M}^{par}$ = moduli stack of (E, φ, x, E_x^B)'s with

$(E,\varphi) \in \mathcal{M}$, $x \in X$, E_x^B = B-reduction of E_x

= Parabolic Hitchin map

$f^{par} : \mathcal{M}^{par} \longrightarrow \mathcal{A} \times X$ Hitchin map

Recall the map $\mathcal{M} \times X \longrightarrow [\mathfrak{g}/G] \underset{\mathbb{G}_m}{\times} \rho_D$

$$\Rightarrow \quad \begin{tikzcd} \mathcal{M}^{par} \arrow[r] \arrow[d] \arrow[dr, phantom, "\Box"] & {[\mathfrak{b}/B] \underset{\mathbb{G}_m}{\times} \rho_D} \arrow[d] \\ \mathcal{M} \times X \arrow[r] & {[\mathfrak{g}/G] \underset{\mathbb{G}_m}{\times} \rho_D} \end{tikzcd} \qquad \text{Cartesian square}$$

The Picard stack $\mathcal{P}$ (over $\mathcal{A}$) acts on $\mathcal{M}^{par}$

$\Rightarrow$ Ngo's product formula extends to parabolic setting

LEM: Assume $\deg(D) \geqslant 2g$

(a) $\mathcal{M}^{par}|_{\mathcal{A}^\heartsuit}$ smooth, $\mathcal{M}^{par}|_{\mathcal{A}^\heartsuit} \longrightarrow \mathcal{A}^\heartsuit \times X$ flat

(b) $\mathcal{M}^{par}|_{\mathcal{A}^{ani}}$ smooth, proper over $\mathcal{A}^{ani} \times X$

$\square$

THM [Yun '11]: Assume $\deg(D) \geqslant 2g$

$\exists\ \widetilde{W}$-action on $Rf_!^{par}\, \mathbb{Q}$ over $\mathcal{A}^{ani} \times X$

$\square$

NB: Under Ngo's product formula the $\widetilde{W}$-action is compatible with the affine springer action

Ex : $G = GL_n$

$\mathcal{M}^{par} = (x, E, \varphi, E_x^B)$ with

$$\begin{cases} (E,\varphi) = \text{Higgs bundle} \\ E_x^B = \text{flag of rank } n \text{ vector bundles} \end{cases}$$

$$E_n(-x) \subsetneq E_0 \subsetneq E_1 \subsetneq \cdots \subsetneq E_n = E \quad , \quad \mathrm{lgth}(E_{i+1}/E_i) = 1$$

$\mathcal{M}^{par}_{a,x}$ classifies flags $F_n(-x) \subsetneq F_0 \subsetneq F_1 \subsetneq \cdots \subsetneq F_n = F$

with $F_i \in \overline{\mathrm{Pic}(Y_a)}$ and $\mathrm{lgth}(F_{i+1}/F_i) = 1$

$P_a = \mathrm{Pic}(Y_a)$

$\mathcal{L} \cdot (F_0 \subsetneq F_1 \subsetneq \ldots) = (\mathcal{L} \otimes F_0 \subsetneq \mathcal{L} \otimes F_1 \subsetneq \ldots)$

PROOF

$Rf_!^{par}\mathbb{Q}$ not in a single perverse degree

$\Rightarrow$ can't use Lusztig middle extension method

$\Rightarrow$ use Hecke correspondences

$$\begin{tikzcd} & \mathrm{Hecke} \arrow[dl, "a"'] \arrow[dr, "b"] & \\ \mathcal{M}^{par} \arrow[dr, "f^{par}"'] & & \mathcal{M}^{par} \arrow[dl, "f^{par}"] \\ & \mathcal{A} \times X & \end{tikzcd}$$

Hecke parametrizes tuples $(x, E_1, \varphi_1, E_{1,x}^B, E_2, \varphi_2, E_{2,x}^B, \alpha)$

$$\begin{cases} (x, E_i, \varphi_i, E_{i,x}^B) \in \mathcal{M}^{par} \\ \alpha : (E_1, \varphi_1)\big|_{X\setminus\{x\}} \xrightarrow{\sim} (E_2, \varphi_2)\big|_{X\setminus\{x\}} \end{cases}$$

Hecke = ind-algebraic stack with ind-proper maps to $\mathcal{M}^{par}$

LEM:

(a) The fibers of a, b at (x, E, φ, E_x^B) are isomorphic to $Sp_{I_x, \tilde{a}_x}$

with $\tilde{a}_x \in \mathfrak{g} \otimes \mathcal{O}_x$ such that $\chi(\tilde{a}_x) = a_x \in \mathfrak{c}^{\heartsuit}(\mathcal{O}_x)$

(b) Any finite type substack $H \subset \mathrm{Hecke}|_{\mathcal{A}^{\heartsuit} \times X}$ is graph-like$/\mathcal{A}^{\heartsuit} \times X$

PROOF :

$\mathrm{Flag} = G(K)/I$, I = Iwahori

$\gamma \in (\mathfrak{g} \otimes K)^{rs}$

$Sp_{I, \gamma} = \{ gI/I ; g^{-1} \gamma g \in \mathrm{Lie}(I), g \in G(K) \} \subset \mathrm{Flag}$

$K = K_x$, $\mathcal{O} = \mathcal{O}_x$

$I \subset G(\mathcal{O}_x)$ Iwahori associated with Borel $B \subset G$

$Sp_{I,\gamma}$ parametrizes tuples $(E, \varphi, E_x^B, \alpha)$ with

$$\begin{cases} E = G\text{-torsor over } D_x = \mathrm{Spec}(\mathcal{O}_x) \\ \varphi \in H^0(D_x, \mathrm{ad}(E)) \\ E_x^B = B\text{-reduction at } x \\ \alpha : (E, \varphi)|_{D_x^\times} \simeq (D_x^\times \times G, \gamma) \end{cases}$$

$\square$

$\exists$ canonical map $\mathcal{A} \times X \longrightarrow \mathrm{car} \times_{\mathbb{G}_m} \rho_D$

$(\mathcal{A}^\heartsuit \times X)^{rs}$ = inverse image of $\mathrm{car}^{rs} \times_{\mathbb{G}_m} \rho_D$

Apply the correspondences construction with

$$X = \mathcal{M}^{par} \quad \text{and} \quad S = \mathcal{A}^\heartsuit \times X \supset U = (\mathcal{A}^\heartsuit \times X)^{rs}$$

LEM:

$\exists$ $\tilde{W}$-action on $\mathcal{M}^{par, rs}$ such Hecke^{rs} = disjoint union of graphs, up to nilpotents

$\square$

PROOF : Assume $G = GL_n$. Recall $p_a : Y_a \longrightarrow X$ Spectral curve

$(a, x) \in (\mathcal{A}^\heartsuit \times X)^{rs} \Rightarrow Y_a$ smooth over x and

$$\mathcal{M}^{par}_{a,x} = \{ \widetilde{\mathcal{F}}_0 \in \overline{\mathrm{Pic}(Y_a)} + \text{ordering } p_a^{-1}(x) = (y_1, y_2, \dots, y_n) \}$$

$w \in W = S_n \Rightarrow w$-action permutes the y_i's

$\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n \Rightarrow \lambda$-action is $\mathcal{F}_0 \mapsto \mathcal{F}_0 \otimes \mathcal{O}_{Y_a}(\sum_{i=1}^n \lambda_i y_i)$

$\square$

CORRESPONDENCES

A correspondence C_{ij} over $X_i \times_S X_j$ is a proper morphism $C_{ij} \xrightarrow{(a_i, a_j)} X_i \times_S X_j$, ie, a diagram

$$\begin{tikzcd} & C_{ij} \arrow[dl, "a_i"'] \arrow[dr, "a_j"] & \\ X_i \arrow[dr, "f_i"'] & & X_j \arrow[dl, "f_j"] \\ & S & \end{tikzcd}$$

EX: $C_{ij} = X_i \times_S X_j$ is a correspondence

We can compose correspondences by setting

$$C_{13} = C_{12} * C_{23} = C_{12} \times_{X_2} C_{23} \longrightarrow X_1 \times_S X_3$$

A cohomological correspondence supported on C_{ij} is an element of

$$\mathrm{Hom}_{C_{ij}}(a_i^* \mathbb{Q}_{X_i}, a_j^! \mathbb{Q}_{X_j})$$

$\exists$ evaluation map

$$\# : \mathrm{Hom}_{C_{ij}}(a_i^* \mathbb{Q}_{X_i}, a_j^! \mathbb{Q}_{X_j}) \longrightarrow \mathrm{Hom}_S(f_{i!} \mathbb{Q}_{X_i}, f_{j*} \mathbb{Q}_{X_j})$$

$$\mathrm{Hom}_{C_{ij}}(a_i^* Q_{X_i}, a_j^! Q_{X_j}) \xrightarrow[\text{adj}]{(a_i,a_j)_* +} \mathrm{Hom}_{X_i \times_S X_j}(p_i^* Q_{X_i}, p_j^! Q_{X_j})$$

$$(a_i,a_j)_* a_i^* = (a_i,a_j)_* (a_i,a_j)^* p_i^* \leftarrow p_i^*$$

$$(a_i,a_j)_* a_j^! \overset{\text{prop.}}{=} (a_i,a_j)_! (a_i,a_j)^! p_i^! \to p_i^!$$

$$\overset{\text{adj}}{=} \mathrm{Hom}_{X_i}(Q_{X_i}, p_{i*} p_j^! Q_{X_j})$$

$$\overset{\text{bc}}{=} \mathrm{Hom}_{X_i}(Q_{X_i}, f_i^! g_{j*} Q_{X_j})$$

$$\overset{\text{adj}}{=} \mathrm{Hom}_S(f_{i!} Q_{X_i}, g_{j*} Q_{X_j})$$

LEM: Assume $X_2 \to S$ proper. $\exists$ composition

$$\begin{array}{c} \mathrm{Hom}_{C_{23}}(a_2^* Q_{X_2}, a_3^! Q_{X_3}) \otimes \mathrm{Hom}_{C_{12}}(a_1^* Q_{X_1}, a_2^! Q_{X_2}) \\ \downarrow \text{comp} \\ \mathrm{Hom}_{C_{13}}(a_1^* Q_{X_1}, a_3^! Q_{X_3}) \end{array}$$

compatible with evaluation maps, ie, $\mathrm{comp}(x,y)_\# = x_\# \circ y_\#$

$\square$

Now let X be smooth equidimentional of dimention d

with $f : X \to S$ proper

$C \xrightarrow{(a,b)} X \times_S X$ correspondence

Fix $\mu : C \times C \to C$ proper map which is associative

$\mu_* \circ \text{comp}$ = an associative multiplication on $\mathrm{Hom}_C(a^* \mathbb{Q}_X, b^! \mathbb{Q}_X)$

C is graph-like $/S \overset{\text{def}}{\iff}$

$$\begin{cases} * \ \exists\, U \subset S \text{ open } / \ a|_U,\ b|_U : C_U \to X_U \text{ are étale} \\ * \ \dim C_U \leq d,\ \dim \mathrm{Im}(C \setminus C_U \to X \times_S X) < d \end{cases}$$

$$\Rightarrow \mathrm{Hom}_C(a^* \mathbb{Q}_X, b^! \mathbb{Q}_X) = H^0(C, \mathbb{D}_C[-2d]) = \underset{\substack{\rotatebox{90}{$\in$} \\ [C]}}{H^{BM}_{2d}(C, \mathbb{Q})}$$

LEM: Assume C is graph-like

Then the evaluation map $H^{BM}_{2d}(C, \mathbb{Q}) \xrightarrow{\#} \mathrm{End}_S(f_* \mathbb{Q}_X)$

is an algebra homomorphism which factorizes as

$$\begin{tikzcd} H^{BM}_{2d}(C, \mathbb{Q}) \arrow[rr, "\#"] \arrow[dr, "res"'] & & \mathrm{End}_S(f_* \mathbb{Q}_X) \\ & H^{BM}_{2d}(C_U, \mathbb{Q}) \arrow[ur, dashed, "alg.hom."'] & \end{tikzcd}$$

$\square$

EX: $X = \tilde{\mathfrak{g}}$, $S = \mathfrak{g} \supset U = \mathfrak{g}^{rs}$

$f : \tilde{\mathfrak{g}} \to \mathfrak{g}$ Grothendieck Springer

$\mathbb{Q} W \longrightarrow H^{BM}_{top}(St)$, $w \longmapsto [St_w]_\#$ alg. hom.

$\Rightarrow W$-action on $f_* \mathbb{Q}$

AFFINE SPRINGER REPRESENTATION

$I \subset P \subset G(k)$ standard parahoric

$L_P =$ Lévi factor

$Sp_{P,\gamma} \subset Flag_P = G(k)/P$

$$ev_P : Sp_{P,\gamma} \longrightarrow [Lie(L_P)/L_P]$$
$$gP/P \longmapsto \pi_P(g^{-1}\gamma g) \bmod L_P$$

$H_*(Sp_\gamma, \mathbb{Q}) =$ colimit because Sp_γ of ∞ type

THM [Lusztig]: $\exists\ \tilde{W}$-action on $H_*(Sp_\gamma)$

PROOF: Assume G s.c., hence $\widetilde{W} = \langle s_i \rangle$

We have a Cartesian diagram

$$\begin{tikzcd} Sp_{I,\gamma} \arrow[r, "ev_I"] \arrow[d, "p"'] & {[\widetilde{Lie(L_P)}/L_P]} \arrow[d, "q"] \\ Sp_{P,\gamma} \arrow[r, "ev_P"] & {[Lie(L_P)/L_P]} \end{tikzcd}$$

$$W_P \circlearrowright Rq_* \mathbb{D} \overset{bc}{\Longrightarrow} W_P \circlearrowright Rp_* \mathbb{D}$$

$$\Longrightarrow W_P \circlearrowright H_*^{BM}(Sp_{I,\gamma})$$

$W_P = \{1, s_i\} \Longrightarrow s_i$-action

$W_P = \langle s_i, s_j \rangle \Longrightarrow$ relation s_i / s_j

$\square$