École Normale Supérieure de Lyon Unité de Mathématiques Pures et Appliquées

THÈSE

pour obtenir le grade de

DOCTEUR DE L'ÉCOLE NORMALE SUPÉRIEURE DE LYON

Discipline : Mathématiques

Edouard Maurel-Segala

Étude de l'énergie libre de certains modèles matriciels, grandes déviations et relation avec des objets combinatoires.

Directrice de thèse : Alice Guionnet

Jury :

M. Philippe Biane	Rapporteur et Examinateur
M. Damien Gaboriau	Examinateur
Mlle Alice Guionnet	Directrice
M. Michel Ledoux	Examinateur
M. Gilles Schaeffer	Examinateur
M. Ofer Zeitouni	Rapporteur
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Remerciements :

Ma gratitude va tout d'abord à Alice Guionnet qui m'a proposé un sujet d'étude riche et porteur de larges perspectives et qui m'a guidé dans ces travaux. Avoir l'occasion de travailler avec une chercheuse d'une telle renommée, aussi dynamique et enthousiaste fut un honneur. Je ne saurais trop la remercier pour sa disponibilité au long de ces trois années et pour avoir su me motiver dans les moments de doutes et d'incertitudes.

Je remercie Philippe Biane qui a accepté d'être rapporteur et de participer à mon jury. C'est un privilège d'être lu par un chercheur dont les travaux, en particulier ceux de vulgarisation, m'ont initié aux probabilités libres. Les recherches de Ofer Zeitouni en matrices aléatoires et en grandes déviations ont aussi été source d'inspiration. Qu'il soit remercié d'avoir bien voulu examiner ma thèse. C'est un honneur qu'ils m'ont tous deux accordé en acceptant d'être mes rapporteurs.

Les thématiques couvertes par les membres de mon jury sont très diverses. J'apprécie particulièrement que Damien Gaboriau ait fait l'effort de s'intéresser à ces travaux à la limite de son domaine et que Michel Ledoux dont les travaux de grande renommée sur la concentration de la mesure, outil crucial ici, ce soit penché sur ma thèse. Je remercie Gilles Schaeffer d'apporter son expertise en combinatoire, un domaine très présent dans ma recherche. Mes articles trouvent leurs origines dans des travaux de physique théorique et notamment ceux de Jean-Bernard Zuber, ce pourquoi je lui suis reconnaissant d'avoir accepté de faire partie de mon jury de thèse.

Je tiens aussi à exprimer ma gratitude à Amir Dembo pour son accueil chaleureux à Stanford, le temps qu'il m'a consacré, la collaboration initiée. Ce fut une année mathématiquement et personnellement fructueuse. Je remercie Benoît Collins qui a collaboré avec moi à un article commencé à Lyon et poursuivi à distance de l'autre coté de l'Atlantique.

Le cadre de travail qu'offre l'UMPA est exceptionnel. Ce fut un plaisir de faire de la recherche avec tous ses membres dans une atmosphère conviviale, avec des échanges stimulants en salle café, et de bénéficier de la disponibilité extraordinaire des chercheurs de toutes les équipes. Enseigner dans ce laboratoire m'a procuré de belles satisfactions, aux côtés de Julien, Patrick et Tomas qui m'ont aidé à préparer mes TD ; merci aux élèves qui m'ont supporté. Merci aussi à Hélène, Magali et Virginia pour leur aide administrative, Gérard pour son aide informatique. J'ai aussi pu compter sur l'aide infatigable de Pierre-André, Benoît et surtout Aurélien et ma famille lorsque j'étais loin du labo, aux Etats-Unis ou à Paris.

Enfin, je remercie ma famille, entre autre pour la relecture effectuée par mon frère et ma mère, Anah, Antonin, Aurélien, Benoît, Cyril, François, Idriss, Kamil, Manuel, Matthieu, Myriam, Patrick, Pierre, Pierre-André, Sarah, Tomas, Vincent, Vincent, Yann... pour leur amitié et leur soutien.

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Organisation du document :

Ce document reprend les travaux que j'ai accomplis lors de mes 3 années de thèse sous la direction d'Alice Guionnet à l'Ens Lyon. J'ai passé la troisième année de cette thèse en séjour long à l'université de Stanford suite à l'invitation d'Amir Dembo.

Je débuterai par une introduction dont le but est de replacer mes articles dans leur contexte en retraçant brièvement l'histoire des liens entre grandes matrices aléatoires et combinatoire. Je rappellerai aussi les différents résultats obtenus au cours de cette thèse ainsi que des questions soulevées par ces résultats sur lesquelles je travaille actuellement.

Les quatre chapitres suivant cette introduction sont les quatre articles que j'ai écrits pendant ces trois années.

Le premier chapitre est l'article «Combinatorial aspects of matrix models» [GMS06]. C'est un travail réalisé en collaboration avec Alice Guionnet. Le but est de prouver que l'identification au premier ordre entre modèles matriciels à plusieurs matrices et énumérations combinatoires planaires va au-delà de l'identification des séries formelles. Cet article a été publié dans le premier numéro de la revue ALEA (Revue Latino-Américaine de Probabilités et Statistiques). Ce type de liens remonte à l'article fondateur de Brézin, Itzykson, Parisi et Zuber [BIPZ78] et est très utilisé en physique. Le cas des modèles à une matrice avait déjà été traité en détail par Ercolani et McLaughin [EM03] en utilisant des techniques de Riemann-Hilbert. Cet article introduit l'utilisation des équations de Schwinger-Dyson, un outil central développé dans les autres articles.

Le second chapitre est l'article «Second order asymptotics for matrix models» [GMS07] qui a aussi été écrit avec Alice Guionnet, on y étudie le premier ordre de la correction dans la convergence des modèles matriciels puis on le relie aussi à une interprétation combinatoire. Cet article est à paraître dans «Annals of probability». On y prouve aussi un Théorème Central Limite pour la mesure empirique des valeurs propres de modèles matriciels.

Le troisième chapitre est l'article «High order expansion of matrix models and enumeration of maps» [MS06a]. J'y termine cet aspect de l'étude des modèles gaussiens en montrant que, à tous les ordres, l'énergie libre des modèles matriciels s'identifie de manière non-formelle à des séries énumérant des objets combinatoires plongés sur des surfaces de genre d'autant plus grand que l'on va loin dans les corrections à la convergence. Cet article a été soumis.

Le dernier chapitre est l'article «Asymptotics for unitary matrix models» [CGMS06] en collaboration avec Benoît Collins et Alice Guionnet qui est en train d'être terminé. Le but ici est de prouver la convergence d'intégrales sur le groupe unitaire et de mesures qui sont des perturbations de la mesure de Haar. Nous nous inspirons pour cela des articles qui précèdent et de la technologie de l'équation de Schwinger-Dyson. Par ailleurs, après avoir prouvé l'analyticité des intégrales unitaires pour de petits paramètres, nous proposons une interprétation combinatoire nouvelle des coefficients de la série correspondante.

Notations :

1. On notera $\mathcal{M}_N(\mathbb{C})$ l'espace des matrices de taille $N \times N$, $\mathcal{H}_N(\mathbb{C})$ celui des matrices hermitiennes, $\mathcal{U}_N(\mathbb{C})$ celui des matrices unitaires et $\mathcal{A}_N(\mathbb{C})$ celui des matrices antisymétriques. Selon ce qui sera le plus clair, nous noterons $A(ij)$ ou A_{ij} le coefficient de la i-ième ligne, j-ième colonne de la matrice A. On identifiera $\mathcal{H}_N(\mathbb{C})$ à $\mathbb{R}^{N^2}$ via sa base canonique $\{\delta_{ii}, \delta_{ij} + \delta_{ji}, \sqrt{-1}(\delta_{ij} - \delta_{ji})\}_{1 \leqslant i < j \leqslant N}$. Grâce à cette identification nous poserons $d^N A$ la mesure de Lebesgue sur $\mathcal{H}_N(\mathbb{C})$. La mesure de Haar sur $\mathcal{U}_N(\mathbb{C})$ sera notée $\mathbf{m}$. La norme d'une matrice $\|A\|$ est sa norme opérateur.
2. Nous noterons en gras les vecteurs d'éléments. Ainsi $\mathbf{A}$ désignera généralement une vecteur $(A_1, \ldots, A_m)$ de matrices.
3. Pour une mesure μ sur un espace mesuré $(X, \mathcal{A})$ et une fonction mesurable f définie sur X, nous noterons $\mu(f)$ l'espérance de f :

$$\mu(f) = \int_X f(x) d\mu(x)$$

   La topologie la plus utilisée dans ce document est la topologie de la convergence en loi. Une suite de mesures de probabilité $\{\mu_n\}_{n \in \mathbb{N}}$ tend en loi vers μ si pour toute fonction réelle continue bornée f :

$$\hat{\mu}^N(f) \to_{n \to +\infty} \mu(f).$$

   Pour les mesures de probabilités sur $\mathbb{R}$, nous utiliserons aussi la notion de convergence en moments, $\{\mu_n\}_{n \in \mathbb{N}}$ tend en moments vers μ si pour tout entier p,

$$\mu_n(x^p) \to \mu(x^p).$$

4. Pour une matrice A de spectre $\lambda_1, \ldots, \lambda_N$ nous noterons $\hat{\mu}_A^N$ la mesure empirique de ses valeurs propres :

$$\hat{\mu}_A^N = \frac{1}{N} \sum_{i=1}^{N} \delta_{\lambda_i}.$$

5. On notera $C_p, p \in \mathbb{N}$, le p-ième nombre de Catalan défini par récurrence :

$$C_0 = 1,\ C_{p+1} = \sum_{k=0}^{p} C_k C_{p-k}.$$

   Ces nombres sont très utiles dans les problèmes de combinatoire, C_p compte notamment les arbres à p arêtes, les arbres binaires à $2p$ arêtes, les paranthésages équilibrés à p paires de parenthèses, les partitions non-croisées de $\{1, \ldots, 2p\}$ en paires, les marches simples de longueur $2p$ positives et revenant en 0 (appelées chemins de Dick)...
6. On notera $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ l'algèbre complexe des polynômes non-commutatifs en les m variables X_1,...,X_m. Si nous appelons monôme les mots en X_1,...,X_m, $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ est la $\mathbb{C}$ algèbre engendrée par les monômes. Le degré du monôme $X_{i_1} \ldots X_{i_p}$ est p, le degré d'un élément de $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ est le maximum des degrés de ses monômes.

Chapitre 1

Introduction

Le thème de cette thèse est l'étude des liens qui unissent modèles matriciels et fonctions génératrices d'objets combinatoires. Nous verrons en effet qu'une large classe d'intégrales matricielles peuvent être vues comme des fonctions génératrices de nombres de graphes. Réciproquement, il existe une grande variété d'énumérations de graphes qui peuvent s'exprimer de manière très compacte via des intégrales matricielles.

Les matrices aléatoires ont été introduites en physique par Wigner [Wig55] dans les années 50 pour modéliser des opérateurs de mécanique quantique dont la description précise était trop complexe. C'est une vingtaine d'années plus tard que Brézin, Itzykson, Parisi et Zuber dans [BIPZ78] ont mis en lumière leur capacité à capturer des problèmes combinatoires, précisant dans ce cadre les théories de 't Hooft [tH74] sur les développements topologiques. La possibilité d'envisager des énumérations extrêmement variées va rendre les modèles matriciels très populaires dans les années 80 où ils vont se révéler un outil remarquable dans tous les problèmes de surfaces aléatoires qui apparaissent en physique théorique, donnant ainsi un exemple flagrant de ce que Wigner appelait la «déraisonnable efficacité des mathématiques dans les sciences naturelles» [Wig85]. Suite à cette découverte, les modèles matriciels sont devenus un outil privilégié en physique théorique pour tous les problèmes d'énumération de surfaces. Ce n'est que dans les années 90 que les mathématiciens se sont intéressés à cette théorie et aux questions qu'elle soulevait.

Le but de cette introduction est de retracer brièvement l'histoire de ce domaine. N'étant pas physicien, certains travaux ne me sont pas aisément compréhensibles. J'ai cependant essayé de donner un compte-rendu de certains articles venant de la physique car le domaine y trouve son origine. Par ailleurs, il s'agit de travaux très rigoureux mathématiquement et qui ont créé un va-et-vient d'idées particulièrement fertiles entre physiciens théoriciens, probabilistes, spécialistes d'algèbres d'opérateurs... Dans un deuxième temps nous donnerons dans les grandes lignes les résultats obtenus lors de ces trois années de thèse, en particulier sur l'identification non-formelle entre intégrales matricielles et combinatoire. Enfin nous évoquerons quelques questions ouvertes se situant dans le prolongement de cette thèse.

1.1 Contexte

1.1.1 Prémisses

Nous allons tout d'abord évoquer rapidement ce qui a précédé l'identification entre modèles de matrices et combinatoire en essayant d'y détecter les prémisses des liens qui unissent ces deux domaines. Nous verrons ainsi comment la combinatoire est très vite intervenue dans les modèles matriciels et comment l'utilisation d'équations algébriques peut aider à résoudre des problèmes combinatoires.

Premiers résultats en grandes matrices aléatoires

L'observation que des énumérations d'objets combinatoires apparaissent dans l'étude des grandes matrices aléatoires remonte aux origines du domaine et aux modèles les plus simples. Dans les années 50, Wigner eu l'idée de remplacer l'étude de certains opérateurs de mécanique quantique par l'étude d'opérateurs aléatoires. L'intuition derrière cette démarche est que si le système est suffisamment complexe, vu à grande échelle les détails de toutes ses composantes ne sont pas nécessaires pour déduire ses propriétés macroscopiques. Porté par cette philosophie, le modèle exact choisi ne devrait pas avoir d'importance à condition de respecter certaines contraintes du système étudié telles que ses symétries. Nous nous reporterons au livre de Mehta [Meh04] pour une introduction à l'utilisation des matrices aléatoires en physique.

Le modèle le plus simple est celui d'une matrice aléatoire dont les coefficients sont choisis de manière indépendante. Wigner s'intéressa au problème mathématique de comprendre le spectre d'une telle matrice. Dans les années 50 il étudia dans [Wig55] une matrice symétrique dont tous les coefficients ont même valeur absolue mais dont les signes sont choisis de façon indépendante et équiprobable tout en respectant la condition de symétrie. Il prouva que la mesure spectrale de la matrice correctement renormalisée avait une limite déterministe : la loi du semi-cercle. Il constata peu après dans [Wig58] que sa preuve se généralisait à un cadre beaucoup plus général.

[Wigner] Soit X^N une suite de matrices symétriques de taille $N \times N$ telles que

1. $\{X^N(ij) | 1 \leqslant i \leqslant j \leqslant N\}$ est une famille de variables aléatoires indépendantes,
2. pour tout i, j, $E[(X^N(ij))^2] = 1$ et la distribution de $X^N(ij)$ est symétrique,
3. pour tout k, $\sup_{N,i,j} E[(X^N(ij))^k] < +\infty$.

Si nous désignons par $\hat{\mu}^N$ la mesure empirique des valeurs propres de $N^{-\frac{1}{2}} X^N$, alors pour toute fonction continue bornée

$$E[\hat{\mu}^N(f)] \rightarrow_{N\rightarrow+\infty} \sigma(f)$$

où σ est la loi semi-circulaire, i.e. la loi de densité $(2\pi)^{-1}\sqrt{4-x^2}\mathbb{1}_{[-2,2]}$ par rapport à la mesure de Lebesgue. Cette convergence a lieu en loi et en moments.

Notons qu'il existe de nombreuses généralisations de ce théorème. Bai [Bai99] donne une version plus robuste du théorème ainsi qu'une description de méthodes très variées permettant de le prouver. Ainsi, l'hypothèse de symétrie des distributions peut être supprimée et la convergence en espérance de $\hat{\mu}^N$ peut être remplacée par une convergence presque sûre. Il est aussi possible d'affaiblir le contrôle des moments et de conserver la convergence en loi (Remarquons tout de même que l'hypothèse de seconds moment borné reste indispensable pour obtenir la loi du semi-cercle. Ben Arous et Guionnet [BAG07] étudient des matrices à coefficients indépendants n'ayant pas de second moment borné et prouvent la convergence de la mesure empirique sous une autre normalisation ; la loi limite n'est plus la semi-circulaire). Anderson et Zeitouni [AZ06] ont proposé une variante de ce théorème si l'hypothèse de second moments tous égaux est supprimée. Ils prouvent qu'il y a toujours convergence si les $X^N(ij)$ peuvent s'écrire $f(i/N, j/N)Y^N(ij)$ avec f une fonction continue de norme L^2 égale à 1 et la famille des $Y^N(ij)$ est une famille de variables i.i.d.. Cependant la limite n'est pas nécéssairement la loi semi-circulaire.

Ces généralisations illustrent la remarquable robustesse de ce théorème : la loi limite des valeurs propres ne dépend pas du détail de la loi des coefficients.

La méthode utilisée par Wigner est de commencer par prouver la convergence des moments. La convergence en loi vient ensuite par un simple argument de tension. Il montre par de la combinatoire sur les mots que le $2p$-ième moment de la loi empirique $N^{-1}\mathrm{Tr}(X^N)^{2p}$ tend, lorsque N tend vers l'infini, vers le nombre d'arbres à p arêtes. Celui-ci vaut C_p, le p-ième nombre de Catalan. Par symétrie les moments impairs sont nuls. La loi semi-circulaire apparaît alors comme étant l'unique loi de probabilité ayant ces moments. Les nombres de Catalan sont omniprésents en combinatoire où ils énumérent quantité d'objets cruciaux : toutes sortes d'arbres, les paranthésages équilibrés, les chemins de Dick, les partitions non-croisées... En vue de la combinatoire qui va nous intéresser nous allons privilégier une interprétation en terme de cartes.

Combinatoire des cartes simples

La notion essentielle est celle de carte introduite par Tutte [Tut63]. Une carte est un graphe connexe plongé dans une surface compacte orientée de telle façon que les arêtes ne se coupent pas et que les faces (les composantes connexes du complémentaire du graphe sur la surface) soient homéomorphes à des disques. Deux exemples de plongement de graphes sont donnés dans la figure 1.1. Le premier, sur la sphère, et le second, sur une surface de genre 2, sont des cartes. Le troisième en revanche n'est pas une carte car l'une des faces n'est pas homéomorphe à un disque.

Les surfaces compactes orientées sont classées à homéomorphisme près par leur genre, un entier g positif qui compte le nombre de «trous» de la surface. On définit le genre d'une carte comme le genre de la surface dans laquelle le graphe est plongé. On appellera carte planaire une carte sur la sphère (et non sur le plan $\mathbb{R}^2$).

Le sommet d'un graphe plongé dans une surface orientée est caractérisé par sa valence et par un ordre cyclique sur les arêtes. Réciproquement, la donnée d'un ordre cyclique sur les arêtes sortant d'un sommet pour chaque sommet d'un graphe est suffisante pour spécifier le plongement dans la surface (voir Proposition 4.7 dans [Zvo97]). En effet, il est alors facile de définir les faces en suivant le bord des arêtes orientées et de retrouver la surface en recollant des

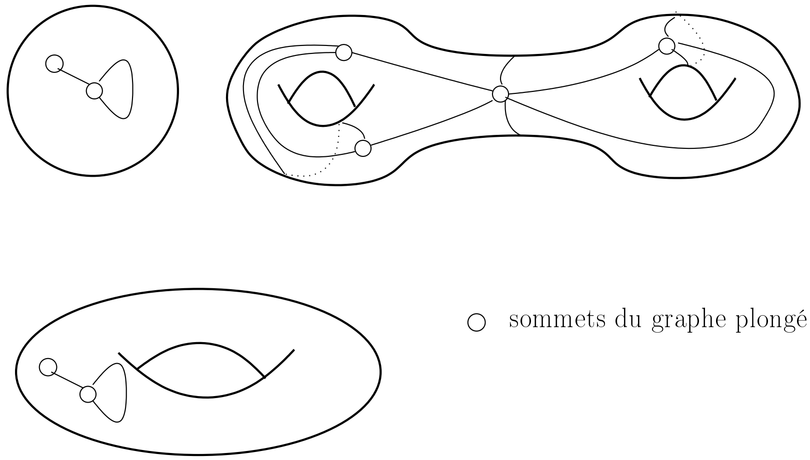

FIG. 1.1 – Deux cartes planaires, une de genre 0, une de genre 2 et un plongement de graphe sur le tore qui ne donne pas de carte.

disques sur ces faces. Conformément aux articles joints, nous appellerons étoile le voisinage d'un sommet de graphe plongé dans une surface munie d'une arête distinguée. Une étoile est déterminée par sa valence qui est le nombre de demi-arêtes qui en sortent. L'une de ces demi-arêtes est distinguée.

Nous nous intéresserons à l'énumération de cartes à homéomorphismes près : deux cartes sont équivalentes s'il existe un homéomorphisme des surfaces qui envoie un graphe plongé sur l'autre en envoyant étoiles sur étoiles. De plus, nous supposerons les étoiles étiquetées, elles ne sont donc pas interchangeables. Par abus de notation, nous désignerons désormais sous le nom de carte une classe d'équivalence de cette relation. L'exemple le plus simple de question à laquelle nous aimerions répondre est : «Quel est le nombre de cartes planaires avec k-sommets de valence 4 ?»

Pour donner un exemple, le nombre de cartes planaires à 2 étoiles de valence 4 est 36. Nous avons représenté dans la figure 1.2 ces cartes. Cependant, chaque dessin représente 4 cartes ce qui est le nombre de possibilités pour la demi-arête distinguée de la seconde étoile.

Une interprétation possible des nombres de Catalan qui s'accorde avec la combinatoire des cartes est la suivante : le p-ième nombre de Catalan est le nombre de cartes planaires à une étoile de valence $2p$ (voir figure 1.3). Il est aisé de vérifier cette égalité : pour construire une telle carte il faut choisir à qui appareiller la demi-arête marquée. Il y a $2p-1$ possibilités et l'arête ainsi formée sépare la carte en deux.

Pour poursuivre la construction, il faut construire une carte à l'intérieur de la boucle et une autre à l'extérieur. Ainsi, nous obtenons la récursion suivante pour le nombre de cartes planaires $\mathcal{M}(X^p)$ que l'on peut construire sur une étoile de valence p :

$$\mathcal{M}(X^p) = \sum_{j=0}^{p-2} \mathcal{M}(X^j)\mathcal{M}(X^{p-2-j}).$$

Cette décomposition inductive est illustrée par la figure 1.4.

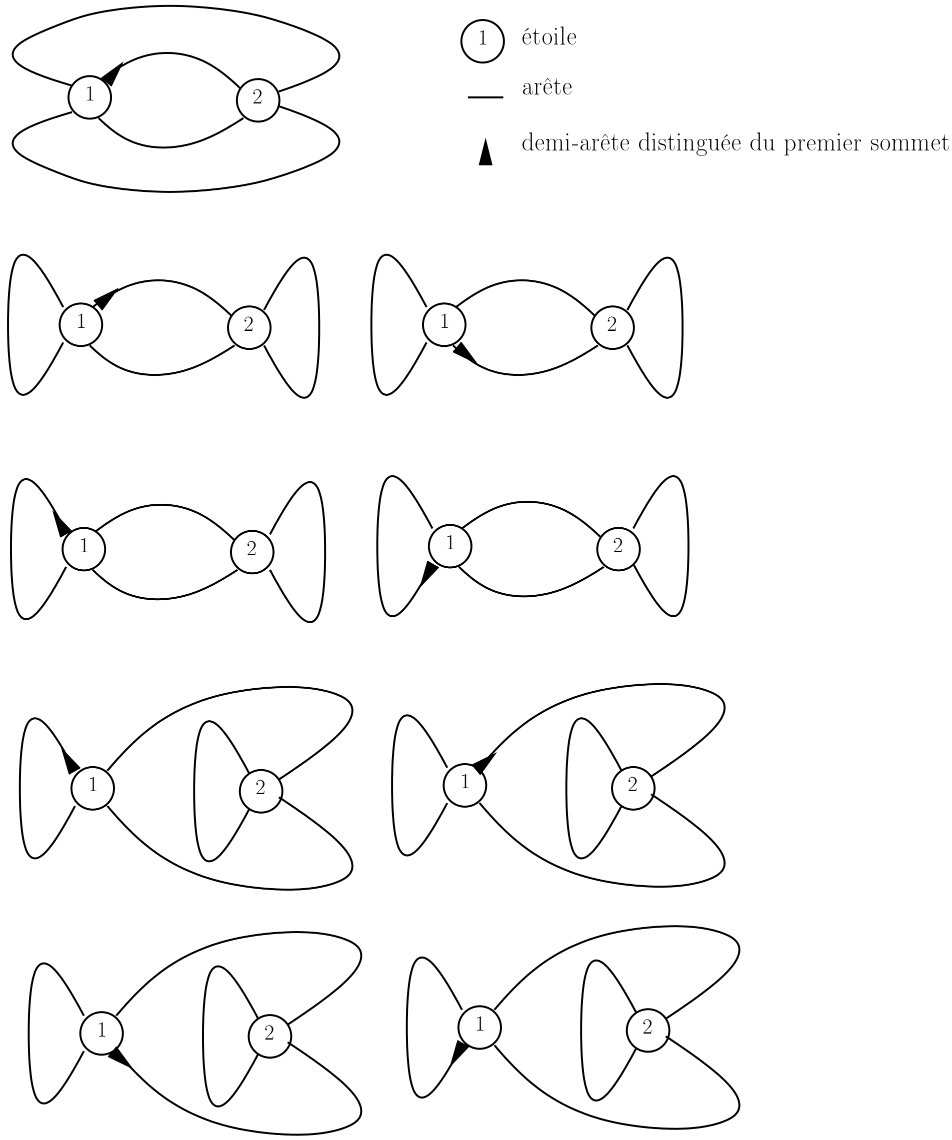

FIG. 1.2 – Cartes planaires à deux étoiles de valence 4 : 36 possibilités

Avec la convention $\mathcal{M}(1) = 0$ et le fait que $\mathcal{M}(X) = 0$, nous nous ramènons aisément à la récursion des nombres de Catalan pour les p pairs. Cette équation qui définit $\mathcal{M}$ par induction est l'exemple le plus simple des équations algébriques utilisées par Tutte pour énumérer les cartes que nous allons rencontrer dans toute la suite.

Plus généralement, pour construire une carte, nous disposerons des étoiles dans une surface puis nous relierons les demi-arêtes deux par deux pour former des arêtes qui ne se croisent pas. On obtient ainsi une carte s'il ne reste plus de demi-arêtes non reliées et que toutes les faces sont homéomorphes à des disques.

Afin de trouver une expression exacte pour un type de carte donné, Tutte découvre des

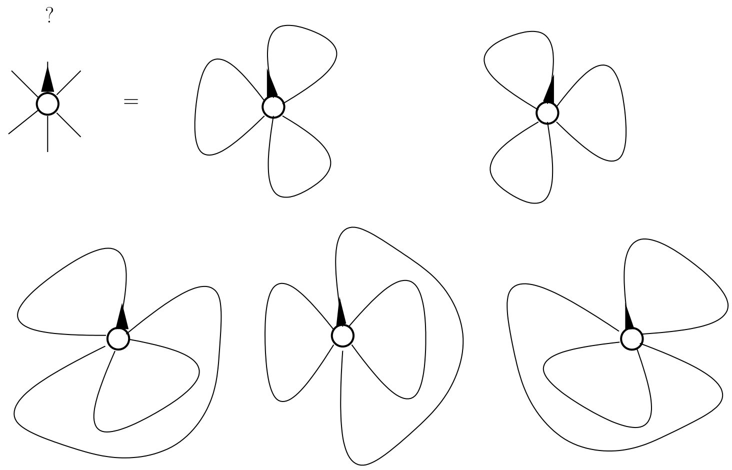

FIG. 1.3 – $C_3 = 5$ vu comme une énumération de cartes.

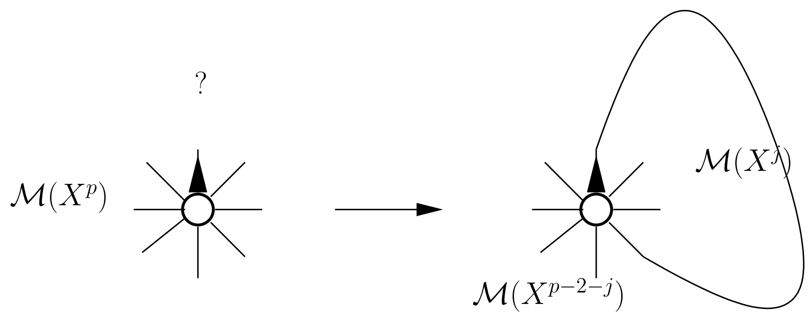

FIG. 1.4 – Relation de récurrence pour les cartes planaires à une étoile.

relations algébriques sur les séries génératrices de ces cartes. L'idée est d'exhiber une décomposition récursive de la carte. Prenons un exemple concret qui se révelera particulièrement pertinent par la suite. Supposons que l'on veuille calculer $\mathcal{M}_k$ le nombre de cartes planaires à k étoiles de valence 4. Comme il n'est pas facile de trouver des relations fermées sur cet ensemble d'objets, introduisons $\mathcal{M}_k(X^p)$ le nombre de cartes planaires à k étoiles de valence 4 et une étoile de valence p. Définissons aussi sa série génératrice :

$$\mathcal{M}(X^p) = \sum_{k\in\mathbb{N}} \frac{(-t)^k}{k!}\mathcal{M}_k(X^p).$$

Afin de trouver une équation algébrique sur cette série, déterminons la manière de construire une telle carte. On dispose sur une sphère une étoile de valence $p+1$ et des étoiles de valence 4. Intéressons-nous maintenant à la demi-arête marquée de l'étoile de valence $p+1$: soit elle est reliée à une étoile de valence 4, soit elle boucle sur l'étoile de valence $p+1$ comme indiquée dans la figure 1.5.

Dans le premier cas nous avons $4k$ choix possibles de la demi-arête avec laquelle la demi-arête distinguée est reliée. Une fois l'arête formée, nous pouvons la contracter, c'est à dire

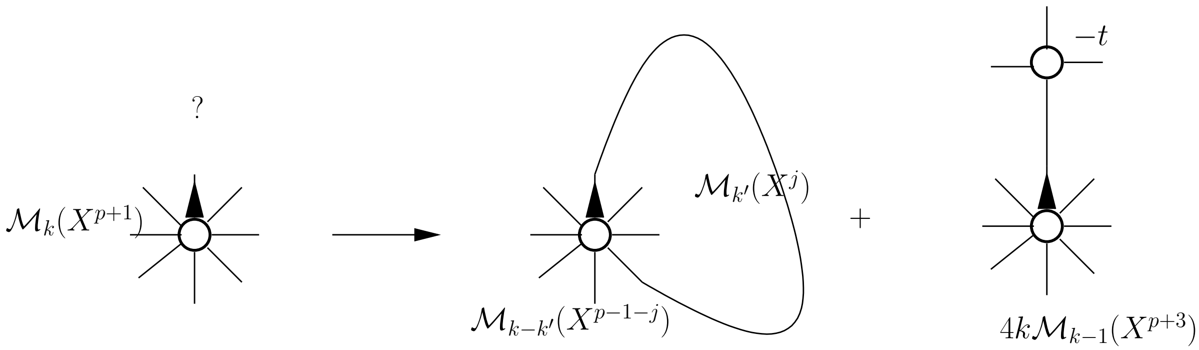

FIG. 1.5 – Relation de récurrence pour les quadrangulations.

diminuer la longueur de l'arête jusqu'à la faire disparaître en recollant les deux sommets qu'elle réunissait, pour obtenir une étoile de valence $p+3$, ce qui conduit à $4k\mathcal{M}_{k-1}(X^{p+3})$ possibilités.

Dans le second cas il faudra finir la construction de la carte à l'intérieur de la boucle et à l'extérieur (Notons que comme nous nous plaçons sur la sphère, le côté sur lequel passe la boucle est sans importance). Il faut alors choisir les étoiles de valence 4 se trouvant à l'intérieur et à l'extérieur de la boucle puis finir la construction, ce qui mène à

$$\sum_{j=0}^{p-1} \sum_{k_1+k_2=k} \binom{k}{k_1} \mathcal{M}_{k_1}(X^j)\mathcal{M}_{k_2}(X^{p-1-j})$$

possibilités. On obtient ainsi :

$$\mathcal{M}(X^{p+1}) = (-t)\mathcal{M}(4X^{p+3}) + \sum_{j=0}^{p-1} \mathcal{M}(X^j)\mathcal{M}(X^{p-1-j}).$$

Il est aisé de voir qu'il existe une unique série formelle en t, solution de cette équation. Ce type d'identité obtenue par décompositions récursives de cartes va jouer un rôle central dans nos travaux. Ce type de relations est le point de départ de Tutte pour trouver dans [Tut62] et [Tut63] des expressions exactes pour l'énumération de triangulations ainsi que pour des classes plus générales de cartes planaires. Ainsi, il obtient une formule explicite pour $\mathcal{M}_k$:

$$\mathcal{M}_k = \frac{2k!4^k3^k}{(k+1)(k+2)}\binom{2k}{k}.$$

1.1.2 Origine de l'analyse de séries combinatoires via les modèles matriciels

L'intérêt pour ce type d'énumérations fut relié en physique théorique par l'article de 't Hooft [tH74] aux théories de jauge de groupe de jauge $\mathcal{U}_N(\mathbb{C})$. Le concept de «développements topologiques» y apparaît naturellement, il s'agit de compter des graphes sur des surfaces énumérées selon leur genre avec un poids N^{-2g} pour les termes de genre g.

Le défi qui se pose alors est celui de trouver une manière de calculer et de comprendre de tels développements.

Perturbation du GUE

C'est en 1978 que Brézin, Itzykson, Parisi et Zuber [BIPZ78] explicitent pour la première fois le lien entre intégrales de matrices et développements topologiques. Le cas étudié dans cet article est celui des quandrangulations : nous cherchons à énumérer les cartes dont tous les sommets sont de valence 4 en pondérant les cartes à k sommets et de genre g par le poids $(-t)^k N^{-2g}/k!$. Le modèle étudié, invariant par l'action de $\mathcal{U}_N(\mathbb{C})$ est celui du **GUE** .

On appelle matrice du **GUE** une matrice

$$X = (X(ij))_{1\leqslant i,j\leqslant N}$$

hermitienne de taille $N \times N$ telle que $\{X(ii)|1 \leqslant i \leqslant N\} \cup \{\sqrt{2}\Re e X(ij), \sqrt{2}\Im m X(ij)|1 \leqslant i < j \leqslant N\}$ est une famille de gaussiennes indépendantes centrées et de variance N^{-1}. De manière équivalente la loi du **GUE** est la mesure de probabilité μ^N sur $\mathcal{H}_N(\mathbb{C})$ ayant pour densité :

$$d\mu^N(A) = \left(\frac{N}{2\pi}\right)^{\frac{N^2}{2}} e^{-\frac{N}{2}\mathrm{Tr}A^2} d^N A$$

où $d^N A$ est la mesure de Lebesgue sur l'espace des matrices hermitiennes.

Comme nous le verrons dans la partie suivante, le calcul gaussien est essentiel dans l'apparition de la combinatoire en grandes matrices aléatoires.

Les matrices du **GUE** vérifient aussi la loi du semi-cercle mais l'intérêt est que certaines perturbations de cette loi sont reliées à des énumérations combinatoires plus riches. Si par exemple nous voulons compter $\mathcal{M}_k^g$ le nombre de cartes de genre g à k étoiles de valence 4 alors l'énergie libre $F_{tX^4}^N$ du modèle matriciel obtenu en perturbant le **GUE** par un potentiel quadratique est relié à cette énumération :

[Brézin-Itzykson-Parisi-Zuber] On a l'identité entre séries formelles en t suivante :

$$F_{tX^4}^N := \frac{1}{N^2} \ln \int_{\mathcal{H}_N(\mathbb{C})} e^{-Nt\mathrm{Tr}A^4} d\mu^N(A) = \sum_{k,g\in\mathbb{N}} \frac{(-t)^k}{N^{2g}k!}\mathcal{M}_k^g.$$

Notons que cette égalité ne peut être qu'au niveau des séries formelles. En effet, la somme sur la droite n'est pas convergente pour $t \neq 0$ et l'intégrale du membre de gauche n'est bien défini que pour les t positifs. Comme observé dans [KNN77], à genre fixé, ce type de série a un rayon de convergence strictement positif et c'est uniquement le fait de resommer sur tous les genres simultanément qui la rend divergente.

Calcul gaussien et combinatoire

Nous allons maintenant essayer de donner une intuition des idées qui conduisent au Théorème 1.1.3. Nous suivrons la démarche de Bessis Itzykson et Zuber [BIZ80]. Zvonkin a aussi écrit une présentation très accessible à destination des mathématiciens [Zvo97].

La base de ce lien est le calcul de Wick : [Calcul de Wick] Si $f_1,\ldots,f_{2p}$ sont des éléments d'un espace gaussien, alors

$$E[f_1 \ldots f_{2p}] = \sum E[f_{i_1} f_{j_1}] \ldots E[f_{i_p} f_{j_p}]$$

où la somme se fait sur toute les partitions de $\{1, \ldots, 2p\}$ en p paires $\{i_1, j_1\},\ldots,\{i_p, j_p\}$ avec $i_1 < i_2 < \cdots < i_p$ et pour tout k, $i_k < j_k$.

Le calcul de Wick permet donc de ramener l'espérance d'un produit de gaussiennes à un produit de covariances.

Revenons maintenant à notre modèle matriciel avec son potentiel $V = tX^4$. Pour se ramener à du calcul gaussien, nous allons développer l'exponentielle. Ainsi la fonction de partition $Z^N_{tX^4}$ se réécrit :

$$Z^N_{tX^4} := \int e^{-Nt\mathrm{Tr}A^4} d\mu^N(A) = \sum_{k\in\mathbb{N}} \frac{(-Nt)^k}{k!} \mu^N[(\mathrm{Tr}A^4)^k].$$

Ici nous avons opéré une interversion somme/intégrale que rien ne permet de justifier, par contre il y a bien égalité des séries formelles en t et c'est ainsi qu'il faut comprendre cette identité.

Un rapide calcul montre que la fonction de covariance dans une matrice du **GUE** est donnée par :

$$\int A^N(ij)A^N(kl)d\mu^N(A) = \frac{\mathbb{1}_{i=l,k=j}}{N}. \tag{1.1}$$

Afin de calculer l'espérance de $(\mathrm{Tr}A^4)^k$ sous cette mesure, nous allons représenter chaque variable $X(ij)$ comme une demi-arête orientée portant une étiquette i sur son bord gauche et une étiquette j sur son bord droit. La variable $A(i_1i_2)A(i_2i_3)A(i_3i_4)A(i_4i_1)$ sera représentée comme une étoile de valence 4 avec les demi-arêtes associées à chacun des coefficients apparaissant dans le produit (voir figure 1.6). Pour calculer l'espérance de $(\mathrm{Tr}A^4)^k$, le calcul de Wick montre qu'il suffit d'envisager toutes les manières d'appareiller deux par deux les variables. En d'autres termes, il faut relier deux par deux les demi-arêtes. Chaque arête représente une paire dans le Théorème de Wick et fait donc apparaître la covariance des variables correspondant aux deux demi-arêtes. D'après (1.1), celle-ci sera non-nulle si et seulement si les indices des bords recollés correspondent (voir figure 1.7). L'appariement de toutes les demi-arêtes forme une carte. Il faut ensuite savoir quel est le poids d'une telle carte. Pour obtenir un terme non nul, il faut que les indices des bords des demi-arêtes correspondent. Comme les bords des arêtes définissent les faces cela revient à demander que tous les indices le long d'une face coïncident, ce qui laisse N choix par faces. Chaque arête représente un couple de Wick et donc une covariance donnant une contribution N^{-1}. Enfin, comme k est le nombre d'étoiles, la contribution d'un appariement est $N^{F-A+S}/S!$ où F est le nombre de faces de la

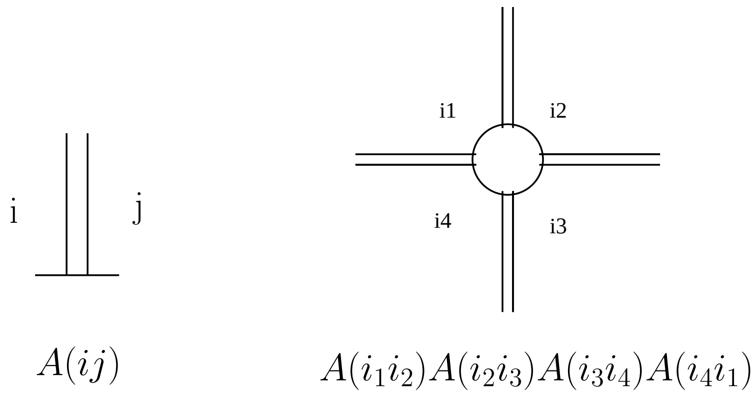

FIG. 1.6 – Représentation des variables en demi-arêtes et des monômes en étoiles

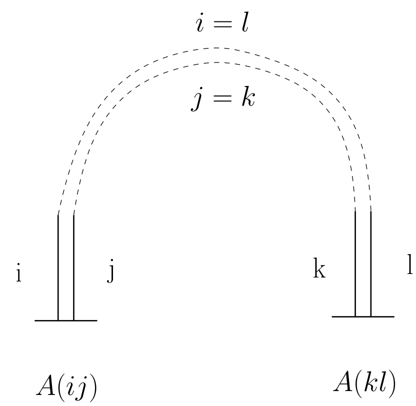

FIG. 1.7 – Règle d'appariement des demi-arêtes : $\mathbb{E}[A(ij)A(kl)] = N^{-1}$ si les indices coïncident

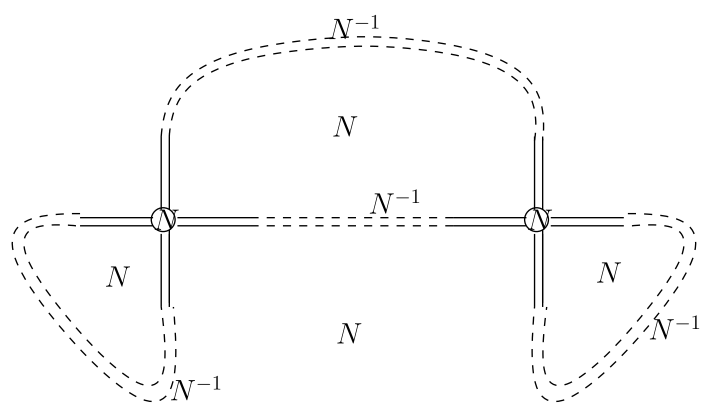

FIG. 1.8 – Appariement de deux étoile de valence 4

multi-carte, A le nombre d'arêtes et S le nombre de sommets (voir figure 1.8). Remarquons que rien ne spécifie ici que l'on connecte tous les sommets, nous obtenons donc une somme sur des unions disjointes de cartes que nous appellerons multi-carte :

$$\mu^N[(N\mathrm{Tr}A^4)^k] = \sum_{\text{multi-cartes à } k \text{ sommets}} \frac{N^{F-A+S}}{N^k k!}.$$

Un fait connu et facile à vérifier (voir [LZ04] par exemple) en combinatoire est, que pour une série génératrice $A(s)$ de graphes étiquetés pondérés telle que le poids d'un graphe est le produit du poid de ses composantes connexes alors la série génératrice des graphes connexes $B(s)$ s'obtient simplement par la relation $B = \ln A$.

Ainsi,

$$\begin{aligned} F^N_{tX^4} &:= \frac{1}{N^2} \ln Z^N_{tX^4} \\ &= \frac{1}{N^2} \ln\{ \sum_{\text{multi-cartes}} N^{F-A+S} \frac{(-t)^S}{S!} \} \\ &= \sum_{\text{cartes}} N^{F-A+S-2} \frac{(-t)^S}{S!} \\ &= \sum_{\text{cartes}} N^{-2g} \frac{(-t)^S}{S!} = \sum_{k\in\mathbb{N}} \frac{(-t)^k}{k! N^{2g}} \mathcal{M}^g_k \end{aligned} \tag{1.2}$$

où nous avons utilisé la formule d'Euler :

$$F - A + S = 2 - 2g.$$

Les mêmes arguments généralisent cette mécanique à quantité d'autres énumérations. On peut ainsi retrouver des cartes à sommets de valence 3 en considérant un potentiel cubique tX^3 ou encore mélanger plusieurs types de sommets. Par exemple, si $V_{t,u} = tX^4 + uX^6$, alors nous avons l'identité entre séries formelles

$$F^N_{V_{t,u}} := \frac{1}{N^2} \ln \int e^{-N\mathrm{Tr}(tA^4+uA^6)} d\mu^N(A) = \sum_{k,l,g\in\mathbb{N}} \frac{(-t)^k(-u)^l}{N^{2g} k! l!} \mathcal{M}^g_{k,l}$$

avec $\mathcal{M}^g_{k,l}$ le nombre de cartes de genre g avec k étoiles de valence 4 et l de valence 6.

Notons que ces techniques permirent à Harer et Zagier [HZ86], en étudiant les moments du **GUE** ($t = u = 0$), de calculer de manière explicite le nombre de cartes à une étoile de valence n sur une surface de genre g.

Analyse du modèle matriciel

L'intérêt de traduire une telle somme combinatoire en intégrale matricielle est que cette dernière est parfois plus facile à analyser. La clé ici est que l'on peut écrire une formule explicite pour les valeurs propres de ce type de modèle. Soit μ^N_V la loi sur les matrices hermitiennes :

$$d\mu^N_V(A) = \frac{1}{Z^N_V} e^{-N\mathrm{Tr}V(A)} d\mu^N(A)$$

où Z_V^N est une constante de normalisation qu'on appelle fonction de partition. L'exemple précédant correspond au cas $V = tX^4$.

Cette loi est invariante par conjugaison, $A \to U^*AU$ par une unitaire ce qui permet de réécrire cette loi sur les valeurs propres, pour toute fonction f invariante par conjugaison :

$$\mu_V^N(f) := \frac{1}{Z_V^N} \int_{\mathcal{H}_N(\mathbb{C})} f(A) e^{-Nt\mathrm{Tr}V(A)} d\mu^N(A) = c_N \int_{\mathbb{R}^N} f(\mathcal{L})\Delta(\mathcal{L})^2 \prod_{i=1}^N e^{-N(V(\lambda_i)+\frac{\lambda_i^2}{2})} d\lambda_i$$

où $\mathcal{L}$ est la matrice diagonale Diag $(\lambda_1, \ldots, \lambda_N)$ et $\Delta(\mathcal{L})$ est le déterminant de Vandermonde des λ_i, $\Delta(\mathcal{L}) = \prod_{i<j}(\lambda_i - \lambda_j)$.

Une analyse de cette expression permet aux auteurs de [BIPZ78] de déduire une formule exacte pour le premier terme du développement topologique des cartes à sommets de valence 4 :

$$F_{tX^4}^0 := \lim_N F_{tX^4}^N = \frac{1}{24}(a^2 - 1)(9 - a^2) - \frac{1}{2} \ln a^2.$$

où a résout

$$12ta^4 + a^2 - 1 = 0. \tag{1.3}$$

De plus, si l'identification n'est pas seulement formelle, alors nous nous attendons à ce que l'énergie libre tende vers la série énumérant les objets planaires :

$$\lim_N F_{tX^4}^N = \sum_k \frac{(-t)^k}{k!} \mathcal{M}_k^0.$$

Les modèles matriciels permettent donc d'obtenir une expression compacte pour l'énumération de certaines cartes.

Le problème vient par contre de la difficulté à analyser le modèle matriciel dans le cas général ou d'analyser les corrections à cette convergence pour connaître les termes suivants du développement topologique. Il n'existe pas de méthode pour traiter le cas général. Toutefois dans le cadre de l'énumération de graphes sans couleurs que l'on a vu jusqu'ici, qui se ramène à des modèles à une matrice, il existe un outil privilégié : l'analyse par polynômes orthogonaux.

La méthode des polynômes orthogonaux

Cette méthode introduite par Bessis [Bes79] lui permet de trouver une expression explicite pour le second terme dans le développement de l'énergie libre pour le potentiel $V = tX^4$,

$$F_{tX^4}^1 := \lim_N N^2(F_{tX^4}^N - F_{tX^4}^0) = \frac{1}{12} \ln(2 - a^2)$$

avec le a défini par (1.3). Si on parvient à identifier le modèle matriciel à la série formelle alors cette formule énumère les cartes à étoiles de valence 4 plongées sur le tore.

La technique utilisée est d'introduire les polynômes orthogonaux dans la densité des valeurs propres. Rappelons ici la formule de la densité des valeurs propres :

$$\Delta(\lambda_1, \ldots, \lambda_N)^2 e^{-N\sum_{i=1}^N V(\lambda_i)+\frac{\lambda_i^2}{2}} \prod_{i=1}^N d\lambda_i.$$

La loi est donc celle de variables indépendantes soumises à une interaction décrite par le carré du déterminant de Vandermonde. C'est celui-ci que nous allons essayer de réécrire en terme de polynômes orthogonaux :

$$\Delta(\lambda_1, \ldots, \lambda_N) = \begin{vmatrix} 1 & \ldots & \lambda_1^{N-1} \\ \vdots & \vdots & \vdots \\ 1 & \ldots & \lambda_N^{N-1} \end{vmatrix} = \begin{vmatrix} 1 & \ldots & h_j(\lambda_1) & \ldots & h_{N-1}(\lambda_1) \\ \vdots & & \vdots & & \vdots \\ 1 & \ldots & h_j(\lambda_i) & \ldots & h_{N-1}(\lambda_i) \\ \vdots & & \vdots & & \vdots \\ 1 & \ldots & h_j(\lambda_N) & \ldots & h_{N-1}(\lambda_N) \end{vmatrix}$$

pour toute famille de polynômes $(h_j)_{j\in\mathbb{N}}$ telle que h_j soit un polynôme de coefficient de plus haut degré x^j. Cette identité s'obtient facilement en redeveloppant le déterminant de droite selon ses colonnes. Nous avons donc une liberté immense dans le choix des h_j. En particulier, il est pertinent de choisir les h_j comme la famille des polynômes orthogonaux vis-à-vis de la mesure de densité $e^{-N(V(x)+\frac{x^2}{2})}dx$. Un calcul élémentaire montre alors que la fonction de partition du système se simplifie :

$$\begin{aligned} Z_V^N &= c_N \int_{\mathbb{R}^N} [\det(h_j^N(\lambda_i))_{1\leqslant i\leqslant N, 0\leqslant j\leqslant N-1}]^2 \prod_{i=1}^{N} e^{-N(V(\lambda_i)+\frac{\lambda_i^2}{2})} d\lambda_i \\ &= c_N N! \prod_{j=0}^{N-1} \int_{\mathbb{R}^N} (h_j^N(x))^2 e^{-N(V(x)+\frac{x^2}{2})} dx \end{aligned}$$

On se ramène ainsi à l'étude de cette famille de polynômes orthogonaux.

Motivations physiques

Les motivations initiales de cette théorie étaient la compréhension des développements topologiques utilisés par 't Hooft [tH74] dans le cadre de la Chromo-Dynamique Quantique. Cependant, la richesse combinatoire des intégrales matricielles les a rendu très populaire dans les années 80 notamment dans l'étude des métriques deux dimensionnelles aléatoires. Le problème est que l'espace des métriques planaires est un espace bien trop grand pour pouvoir chercher directement dessus une mesure raisonnable. Une idée est de discrétiser ces métriques. Les cartes sont des candidats à cette discétisation. On pourra par exemple décider de munir la carte de la distance venant du graphe en mettant une longueur 1 à toutes les arêtes. Cette idée que l'on pouvait regarder les cartes comme des discretisations de métriques continues a en particulier été développée dans [ADF85], [KKM85].

Citons en particulier deux domaines où ces surfaces aléatoires interviennent. En gravité quantique 2D (voir le livre de DiFrancesco, Ginsparg et Zinn-Justin [DFGZJ95]) où on remplace l'espace dans lequel vivent les particules par un espace aléatoire planaire : on quantifie sur l'espace. La théorie des cordes est un autre domaine où l'on souhaiterait sommer sur toutes les métriques possibles. De même que la trajectoire d'une particule ponctuelle est représentée par un objet uni-dimensionnel, la trajectoire d'une corde est de manière naturelle une surface.

1.1.3 Modèles multi-matriciels et cartes colorées

Combinatoire des cartes colorées

Le problème se complique lorsqu'en plus de simplement compter les géométries possibles nous rajoutons de la «matière». Ce cas se pose typiquement en physique statistique sur graphe aléatoire. Le premier exemple est celui du modèle d'Ising. On aimerait répondre à des questions du type «Combien y a t-il de cartes planaires à n sommets de valence 4 munis d'une distribution de spin $\{+,-\}$ sur les sommets et tel qu'il y ait r arêtes reliant un spin $+$ à un spin $-$?»

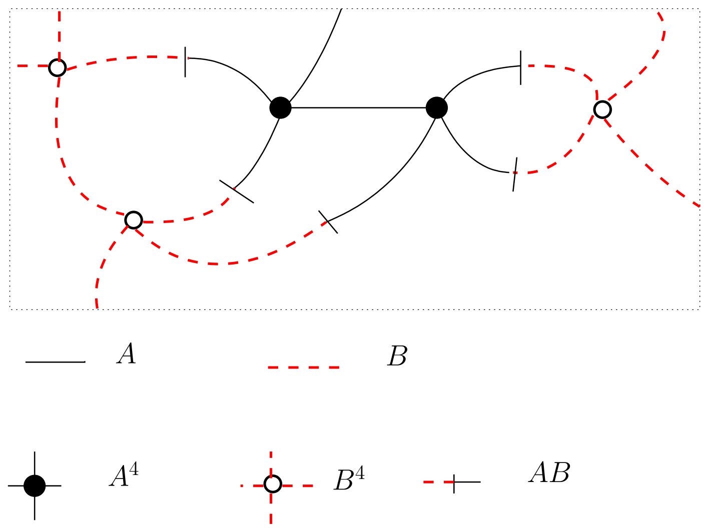

Fig. 1.9 – Configuration d'Ising sur une partie de graphe planaire.

Nous suivrons les idées introduite par Itzykson et Zuber [IZ80] pour répondre à cette question. Nous aurons recours à une astuce consistant à considérer des énumérations de cartes à arêtes colorées. Dans ce cadre plus général, les demi-arêtes des étoiles seront aussi colorées. Du coté du modèle matriciel, nous serons amené à considérer plusieurs matrices. On choisit une matrice A correspondant au spin $+$ et une matrice B correspondant au spin $-$. Nous regarderons des graphes ayant 3 types d'étoiles :

1. Les sites portant un spin $+$ seront représentés par une étoile à 4 demi-arêtes de couleur noire et nous dirons que c'est une étoile de type A^4.
2. De même des étoiles de valence 4 à demi-arêtes blanches (lignes discontinue sur le dessin) appelées étoiles de type B^4 représenteront les sites portant un spin $-$.
3. Enfin pour représenter les interfaces entre les deux spins, nous utiliserons des étoiles de valence 2 à une demi-arête de couleur noire et une de couleur blanche. Nous dirons que ces étoiles sont de type AB.

Nous avons représenté dans la figure 2.5.4 une configuration illustrant ce codage.

Les quantités qui nous intéressent ici sont donc

$$\mathcal{M}^g_{k,l,r} = \text{Card}\left\{ \begin{array}{c} \text{Cartes de genre } g \text{ à } k \text{ étoiles de type } A^4,\ l \text{ étoiles de type } B^4 \\ \text{et } r \text{ étoiles de type } AB \end{array} \right\}$$

$$\mathcal{M}^g(t,u,c) = \sum_{k,l,t\in\mathbb{N}} \frac{(-t)^k}{k!}\frac{(-u)^l}{l!}\frac{c^r}{r!}\mathcal{M}^g_{k,l,r}.$$

Notons que cette série ne compte pas réellement les graphes d'Ising tel qu'on le souhaiterait puisque des étoiles de valence 2 peuvent être liées entre elles mais quitte à resommer sur le nombre de paires de telles liaisons, on peut se ramener à la véritable énumération via un simple changement de variable. Précisément, $\mathcal{M}^g((1-c^2)^2t,(1-c^2)^2u,c)$ énumère les graphes d'Ising comme on le souhaite puisque le coefficient de $(-t)^k(-u)^lc^r$ compte les cartes à $k+l$ étoiles de valence 4 munis d'un distribution de k spins $+$ et l spins $-$ telles que r arêtes relient un spin $+$ à un spin $-$.

Le modèle matriciel qui va être relié à cette énumération s'obtient très facilement à partir du modèle combinatoire. Les étoiles de type A^4 donnent un poids $-t$, celles de type B^4 un poids $-u$ et celles de type AB un poids c ; le potentiel correspondant est donc $V = tA^4+uB^4-cAB$. Au niveau des séries formelles, il est possible de prouver l'identité :

$$F^N_V := \frac{1}{N^2}\ln\int_{\mathcal{H}_N(\mathbb{C})^2} e^{-N\text{Tr}(tA^4+uB^4-cAB)}d\mu^N(A)d\mu^N(B) = \sum_{g\in\mathbb{N}}\frac{1}{N^{-2g}}\mathcal{M}^g(t,u,c).$$

Ici, la mesure sur A et B est donc un produit de loi du **GUE** perturbé par un potentiel correspondant au modèle que l'on désire étudier. À nouveau ici il s'agit d'une identité de séries formelles en les variables t,u,c.

Expliquons les étapes essentielles pour prouver une telle égalité. Comme dans le modèle à une matrice, nous développons l'exponentiel et il faut alors évaluer des espérances de produits de traces :

$$\int_{\mathcal{H}_N(\mathbb{C})^2} t^k\text{Tr}(A^4)^k u^l\text{Tr}(B^4)^k c^r\text{Tr}(AB)^r d\mu^N(A)d\mu^N(B).$$

Ici le calcul de Wick s'applique toujours et pour compter ces termes il suffit de compter à nouveau les appariements. Cependant désormais un appariement aura une contribution uniquement si les variables A et B ne sont jamais reliées. Il suffit donc de demander à ce que les appariements d'étoiles ne relient que les demi-arêtes de la même couleur. Les changements de couleurs auront donc seulement lieu en les étoiles de type AB et nous obtenons bien l'énumération recherchée.

Analyse des modèles multi-matriciels du type Ising

Certains de ces modèles de physique statistique sur graphes aléatoires peuvent donc être représentés par un modèle matriciel. Le problème qui se pose alors est celui de l'analyse du modèle et c'est là que les choses se compliquent. On ne peut plus réduire la loi sur les matrices à une loi sur leurs valeurs propres uniquement ; la matrice de passage entre les bases de diagonalisation des matrices intervient car le modèle n'est pas invariant par conjugaison d'une des deux matrices par une unitaire. Une idée introduite par Itzykson et Zuber dans [IZ80] et devenue très populaire est précisément d'effectuer séparément l'intégration sur la variable angulaire et l'intégration sur les valeurs propres des deux matrices. Si nous regardons un modèle de type Ising, c'est à dire avec un potentiel de la forme $V = V_1(A) + V_2(B) - cAB$, alors,

$$\begin{aligned} Z_V^N &:= \int_{\mathcal{H}_N(\mathbb{C})^2} e^{-N\mathrm{Tr}(V_1(A)+V_2(B)-cAB)} d\mu^N(A) d\mu^N(B) \\ &= \int_{\mathcal{L}\in\mathbb{R}^n, M\in\mathbb{R}^n} \mathrm{IZ}_c(\mathcal{L}, M)\Delta(\mathcal{L})^2\Delta(M)^2 e^{-N\mathrm{Tr}V_1(\mathcal{L})} e^{-N\mathrm{Tr}V_1(M)} d^N\mathcal{L} d^N M \end{aligned}$$

où IZ est l'intégrale d'Itzykson-Zuber qui prend en charge la différence de base de diagonalisation pour les deux matrices :

$$\mathrm{IZ}_c(A, B) = \int_{\mathcal{U}_N(\mathbb{C})} e^{cN\mathrm{Tr}(U^*AUB)} d\mathbf{m}^N(U)$$

où $\mathbf{m}$ est la mesure de Haar sur $\mathcal{U}_N(\mathbb{C})$. Modulo ce terme, l'intégrale se réduit, comme dans le cas à une matrice, à une intégrale sur l'espace des valeurs propres avec l'apparition de carrés de déterminants de Vandermonde.

La difficulté ici est donc l'analyse de IZ . Notons que cette intégrale est invariante par conjugaison de A ou B par une unitaire, nous pouvons donc supposer ces deux matrices diagonales. Ce qui est remarquable, c'est que l'on peut la réexprimer via une expression simple et explicite. Dans [IZ80], utilisant une technique du noyau de la chaleur, les auteurs prouvent que si $A = \mathrm{Diag}\,(a_1, \ldots, a_N)$ et $B = \mathrm{Diag}\,(b_1, \ldots, b_N)$ sont des matrices diagonales sans valeurs propres doubles :

$$\mathrm{IZ}_c(A, B) = \frac{\det e^{cNa_ib_j}}{c^{\frac{N(N-1)}{2}}\Delta(A)\Delta(B)}.$$

L'intégrale peut donc se réécrire de manière exacte comme une somme de ses valeurs en ses points critiques. Cette formule avait été auparavant prouvée de manière indépendante dans un cadre plus abstrait par Harish-Chandra dans [HC57], ce pourquoi elle est aussi appelée intégrale de Harish-Chandra ou intégrale de Harish-Chandra-Itzykson-Zuber. On trouve dans [ZJZ03] une étude des différentes méthodes pour prouver cette identité.

Cependant, afin d'extraire le terme correspondant aux graphes planaires de l'intégrale matricielle il faut pouvoir comprendre la limite en N grand de l'énergie libre et donc de IZ . Ce problème semble beaucoup plus ardu et est loin aujourd'hui encore d'être maîtrisé.

Revenons au premier ordre du modèle d'Ising :

$$F^0_{tA^4+tB^4-cAB} := \lim_N \frac{1}{N^2} \ln \int_{\mathcal{H}_N(\mathbb{C})^2} e^{-N\mathrm{Tr}(tA^4+tB^4-cAB)} d\mu^N(A) d\mu^N(B)$$

En analysant à l'aide du développement en caractère IZ , les auteurs de [IZ80] trouvent les premiers termes du développement de F^0 en c. Ultérieurement, Mehta [Meh81], en utilisant les polynômes orthogonaux, trouve une expression explicite pour F^0 encore simplifiée par Kazakov [Kaz86] :

$$\begin{aligned} F^0_{tA^4+tB^4-cAB} &= \frac{1}{2}\ln\frac{t}{g(t)} + \frac{t^2}{2g^2(t)}\left(\frac{t-1}{2(3t-1)^3} + c^2\frac{t+1}{4(3t-1)} + \frac{c^4}{32}(3t^4-3t^2+1)\right) \\ &\quad -\frac{t}{g(t)}(\frac{1}{3t-1} + \frac{c^2}{4}(1-t^2)) + \frac{1}{2}\ln(1-\frac{c^2}{4}) + \frac{3}{4} \end{aligned}$$

avec

$$g(t) = \frac{t}{(1-3t)^2} - \frac{c^2t+3c^2t^3}{4}. \qquad (1.4)$$

Autres modèles multi-matriciels

L'étude des modèles de physique statistique sur graphes aléatoires ne se limite pas au modèle d'Ising mais se généralise aisément à de nombreux autres modèles. Pour illustrer la richesse des possibilités, citons en particulier le modèle de Potts qui est un modèle où au lieu de pouvoir être dans deux états différents comme dans le modèle d'Ising, les sites peuvent prendre q couleurs différentes. Cette combinatoire est traduite en modèle matriciel en regardant un modèle à $q+1$ matrices. Chaque couleur sera représentée par une matrice $(B_k)_{1\leqslant k\leqslant q}$ et nous considérerons une matrice A supplémentaire qui codera l'interface entre les clusters colorés. Ainsi, nous choisirons un potentiel du type $V = \sum_i P_i(B_i) + P(A) + c\sum_i AB_i$ n'autorisant des changements de couleurs qu'entre l'un des B_i et un site représentant l'interface. Ce modèle introduit par Kazakov [Kaz88] est étudié dans [ZJ00], [EB99].

Un autre modèle populaire est le modèle $O(n)$ qui étudie les énumérations de boucles colorées sur une surface. On se donnera encore une matrice A qui sert à construire la carte et pour chaque couleur une matrice $(B_i)_{1\leqslant i\leqslant n}$. Le potentiel étudié est alors de la forme $V(A) + \sum_i B_i^2 A$ ce qui empêche les clusters de couleur i de brancher et les oblige donc à former des boucles. Ce modèle introduit par Kostov [Kos89] est étudié par exemple dans [EK95].

Des modèles très différents ont été abordés à l'aide de ces techniques. Citons par exemple les travaux de Zinn-Justin et Zuber sur l'énumération des noeuds [ZJZ02] ou l'énumération de cartes selon les degrés des étoiles et des faces simultanément dans l'article de Kazakov, Staudacher, et Wynter [KSW96] et étudié plus récemment par Guionnet et Maïda [GM05b].

Grâce à leur géométrie plus souple et à la réduction sous forme de modèle matriciel, il arrive que ces modèles soient plus faciles à étudier sur graphes aléatoires que dans le réseau $\mathbb{Z}^2$. Un fait remarquable est que si la structure euclidienne et la structure aléatoire ont peu en commun, en 1988 Knizhnik, Polyakov et Zamolodchikov [KPZ88] ont proposé une conjecture qui décrit une formule, appelée depuis KPZ, qui relie des exposants critiques sur graphe aléatoire à leur analogue sur graphe euclidien. C'est une motivation supplémentaire pour l'étude des modèles de matrices. Les travaux de Duplantier [Dup06],[Dup04] donnent des exemples d'utilisation de cette mystérieuse relation. Notamment grâce au travail de Lawler, Schramm et Werner sur le SLE, il est désormais possible de calculer rigoureusement des exposants critiques dans le plan (par exemple [LW99] et la série initiée par [LSW01]). Lorsqu'il est possible de les comparer ces résultats coïncident à ceux trouvés en utilisant les modèles matriciels et KPZ. Cependant il semble qu'il n'existe pour l'heure pas de compréhension mathématique de cette formule.

1.1.4 Approches récentes

Au cours des années 90, le sujet n'a cessé de se populariser notamment avec l'apport de nombreux travaux venant de divers secteurs des mathématiques. Nous allons évoquer quelques-unes des approches qui ont été essayées pour gagner une meilleure compréhension de ces phénomènes.

Convergence des modèles à une matrice

Lorsqu'on cesse d'identifier a priori le modèle matriciel à sa série formelle de nouvelles questions surgissent. La simple question de savoir si le modèle admet une limite est loin d'être triviale.

Les premières avancées ne sont pas liées à l'aspect combinatoire mais s'intéressent plutôt à ces modèles matriciels pour eux-mêmes et prouvent divers résultats de convergences. On remarquera que les deux résultats qui suivent vont non-seulement au delà de l'identification formelle mais aussi au-delà d'un résultat perturbatif qui ne s'appliquerait que pour de petits paramètres. Ici le potentiel V pourra être choisi hors d'un voisinage du potentiel quadratique $x^2/2$.

Dans le cadre des modèles à une matrice, il existe des résultats très précis pour la convergence du premier ordre. Plus particulièrement la quantité à laquelle nous nous intéressons ici est la mesure empirique des valeurs propres. Soit $W : \mathbb{R} \to \mathbb{R}^+$ un potentiel, nous regardons la mesure sur $\mathcal{H}_N(\mathbb{C})$ de densité :

$$d\mu_W^N(A) = \frac{1}{Z_W^N} e^{-N\mathrm{Tr} W(A)} d^N(A)$$

avec

$$Z_W^N = \int_{\mathcal{H}_N(\mathbb{C})} e^{-N\mathrm{Tr} W(A)} d^N(A)$$

la constante de normalisation. Pour faire le lien avec ce qui précède, W joue ici le rôle de $V(x) + x^2/2$

Il faudra bien sûr exiger que W vérifie au moins des conditions de croissance à l'infini pour que Z_W^N soit finie et que la définition de μ_W^N ait un sens. L'enjeu est de comprendre le comportement de $\hat{\mu}^N$, la mesure empirique des valeurs propres sous cette mesure lorsque N tend vers l'infini. En 1997, Ben Arous et Guionnet [BAG97] prouvent un résultat de grandes déviations pour cette convergence. [Ben Arous-Guionnet] Soit $W(x)$ une fonction sur $\mathbb{R}$ différentiable qui tend vers l'infini avec $|x|$ suffisamment (mais pas trop) vite :

$$\limsup_{\delta\to 0} \limsup_{|x|\to+\infty} \sup_{|y|<\delta} \left| \frac{W'(x+y)}{W(x)} \right| < +\infty.$$

Soient Σ et I_W les fonctionelles définies sur l'espace des mesures de probabilités sur $\mathbb{R}$ par :

$$\Sigma(\mu) = \int\int \ln|x-y|d\mu(x)d\mu(y)$$

$$I_W(\mu) = (\mu(W) - \Sigma(\mu)) - \inf_\nu(\nu(W) - \Sigma(\nu)).$$

Alors $\hat{\mu}^N$ satisfait un principe de grandes déviations de bonne fonction de taux I_W. De plus, I_W atteint son minimum en une unique mesure de probabilité σ_W caractérisée par

1. $$W(x) - \int \ln|x-y|d\mu(y) = \inf_\nu(\nu(W) - \Sigma(\nu)) \qquad \sigma_W \text{ p.p.}$$

2. Pour tout x, sauf sur un ensemble de capacité logarithmique nulle (voir [BAG97] pour la définition)
$$W(x) - \int \ln|x-y|d\mu(y) \geqslant \inf_\nu(\nu(W) - \Sigma(\nu))$$

On notera en particulier que ce théorème s'applique dans le cas où W est un polynôme tendant vers $+\infty$ en l'infini, ce qui est important dans le cadre des interprétations combinatoires.

L'année suivante Johansson prouve dans [Joh98] un Théorème Central Limite pour la mesure empirique. [Johansson] Soit W un polynôme tel que le support de σ_W défini dans le théorème précédent soit un intervalle compact $[a, b]$. Pour tout polynôme réel P, sous la mesure μ_W^N, $N(\hat{\mu}^N(P) - \sigma_W(P))$ tend en loi vers une gaussienne $\mathcal{N}(0, \sigma^2(P))$. De plus la fonction de covariance $\sigma^2(.)$ a une description simple. Si les $\tilde{T}_n$ sont les polynômes de Tchebychev modifiés :

$$\tilde{T}_n\left(\frac{(b-a)x}{2} + \frac{a+b}{2}\right) = T_n(x)$$

avec T_n la suite des polynômes de Tchebychev, alors pour tout $m, n \geqslant 1$,

$$\sigma^2(\tilde{T}_n, \tilde{T}_m) = \frac{n\mathbb{1}_{n=m}}{4}.$$

Ce résultat est remarquable par son universalité, la covariance des fluctuations ne dépend du potentiel que via une famille à deux paramètres. Brézin et Zee [BZ93] avaient prédit une telle universalité, nous y reviendrons dans les conclusions de cette introduction.

Autour de l'intégrale d'Itzykson-Zuber

Après ces résultats précis sur l'intégrale à une matrice, le défi suivant fut la compréhension de l'intégrale d'Itzykson-Zuber qui permet l'étude des modèles les plus importants en physique statistique sur graphes aléatoires. Pour un aperçu des résultats sur cette intégrale nous recommandons en particulier la lecture de [ZJZ03].

Plus précisément l'enjeu est de comprendre

$$I(\mu_A, \mu_B) := \lim_N \frac{1}{N^2} \ln \int_{\mathcal{U}_N(\mathbb{C})} e^{N\mathrm{Tr}U^*A^NUB^N} d\mathbf{m}(U)$$

où A^N (respectivement B^N) est une suite de matrices diagonales dont la mesure empirique des valeurs propres tend vers μ_A (resp. μ_B). L'existence même de cette limite est un problème très ardu.

En 1994, Matystin dans [Mat94] relie au moyen d'une dérivation formelle cette intégrale aux solutions complexes de l'équation de Hopf :

$$\frac{\partial}{\partial t}f + f\frac{\partial}{\partial x}f = 0$$

pour $f(x,t)$ satisfaisant les conditions au bord

$$\Im m f(x,0) = \frac{d\mu_A(x)}{dx}$$

$$\Im m f(x,1) = \frac{d\mu_B(x)}{dx}.$$

Le premier résultat précis et rigoureux est obtenu en 2002 par Guionnet et Zeitouni [GZ02]. Les auteurs relient la limite de l'intégrale de Itzykson-Zuber à la fonction de taux du mouvement brownien hermitien à condition initiale déterministe. Plus précisément soit $X^N(t)$ la diffusion définie sur $\mathcal{H}_N(\mathbb{C})$ par

$$X^N(0) = A$$
$$dX^N(t) = dH^N(t)$$

où H^N est le mouvement brownien hermitien :

$$\{\sqrt{N}H^N(ii), \sqrt{2N}\Re e H^N(ij), \sqrt{2N}\Im m H^N(ij)\}_{i<j}$$

est une famille de mouvements browniens indépendants. La preuve centrale de l'article [GZ02] montre que si A^N est uniforméméent bornée, $\{t \to \hat{\mu}^N_{X^N(t)} | t \in [0,1]\}$ satisfait un principe de grande déviation de fonction de taux $J(\mu_A, .)$. De là, les auteurs dérivent l'asymptotique de IZ : [Guionnet-Zeitouni] Supposons que la suite de matrice A^N soit uniformément bornée en N et que $\sup_N \hat{\mu}^N_{B^N}(x^2) < +\infty$ alors si A^N (respectivement B^N) est une suite de matrices diagonales dont la mesure empirique des valeurs propres tend vers μ_A (resp. μ_B)

$$I(\mu_A, \mu_B) = -J(\mu_A, \mu_B) + I_{\frac{x^2}{2}} - \frac{1}{2}\mu_A(x^2).$$

où $I_{\frac{x^2}{2}}$ est définie dans l'énoncé du Théorème 1.1.5. Ce résultat est exploité par Guionnet dans [Gui04] pour donner le comportement asymptotique au premier ordre du modèle d'Ising, du modèle de Potts ainsi que d'autres modèles multimatriciels où l'interaction entre les matrices est aussi quadratique.

Une autre piste suivie afin de comprendre l'intégrale d'Itzykson-Zuber est de faire un développement de l'intégrale en c :

$$\int_{\mathcal{U}_N(\mathbb{C})} e^{Nc\mathrm{Tr}U^*A^NUB^N} d\mathbf{m}(U)$$

puis de chercher la limite des coefficients de la série obtenue. Utilisant les travaux de Weingarten [Wei78] sur les limites d'intégrales sur le groupe unitaire, Collins prouva en 2002 dans [Col03] que la limite coefficient par coefficient de la série formelle en c existe et possède une interprétation diagrammatique. Cette interprétation combinatoire que l'on ne détaillera pas ici est retrouvée par Zinn-Justin et Zuber [ZJZ03]. Les auteurs y utilisent une transformation de IZ qui remplace l'intégration le long de la mesure de Haar par une intégration le long de matrices gaussiennes complexes. Le quatrième article sur lequel j'ai travaillé pendant ma thèse en collaboration avec Benoît Collins et Alice Guionnet a pour application l'étude d'intégrales unitaires dont IZ fait partie. En particulier, nous proposons une interprétation combinatoire de ce type d'intégrale. Ces résultats seront exposés plus en détails dans la section 1.2.6.

Probabilités libres

Un autre domaine où l'interprétation combinatoire de mesures de probabilités est important est celui des probabilités libres (voir [VDN92], [Voi00] pour une introduction). En 1991, Voiculescu prouva dans [Voi91] un analogue du théorème de Wigner dans le cadre multi-matriciel. Rappelons que le théorème de Wigner (Théorème 1.1.1) peut s'énoncer comme la convergence du p-ième moment d'une matrice vers $\mathcal{M}(X^p)$ le nombre de cartes planaires sur une étoile à p demi-arêtes. Le théorème de Voiculescu peut-être reformulé de la manière suivante : [Voiculescu] Soient $X_1^N, \ldots, X_m^N$, m suites de matrices aléatoires indépendantes vérifiant chacune les hypothèses du théorème de Wigner. Pour tout entier p, pour tout $i_1, \ldots, i_p$ dans $\{1, , \ldots, m\}$,

$$\frac{1}{N}\mathrm{Tr}(X_{i_1}^N \ldots X_{i_p}^N) \to \mathcal{M}(X_{i_1} \ldots X_{i_p})$$

où $\mathcal{M}(X_{i_1} \ldots X_{i_p})$ est le nombre de cartes colorées planaires que l'on peut construire sur une étoile de valence p dont les demi-arêtes sont respectivement de couleur $i_1, \ldots, i_p$. La figure 1.10 montre un exemple effectif de calcul d'un moment non-commutatif. Ce théorème permet ainsi

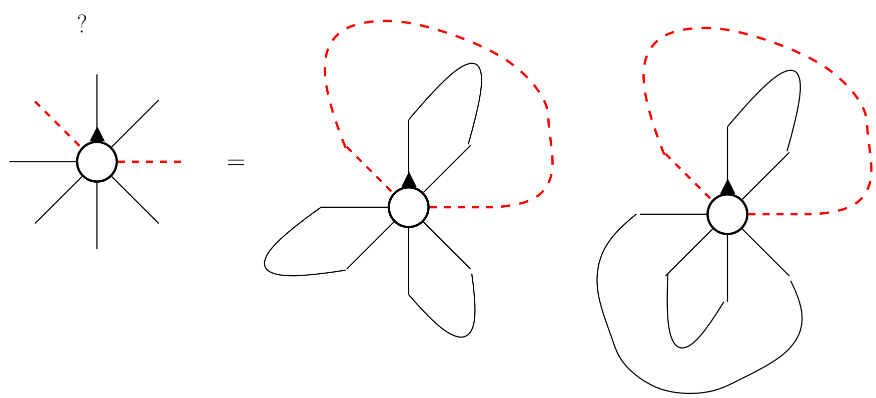

FIG. 1.10 – $\lim_N N^{-1}\mathrm{Tr}(A^2BA^4B) = 2$

de calculer n'importe quel moment non-commutatif de manière aisée. Cette reformulation du théorème en terme de cartes s'inspire des travaux de Nica et Speicher [NS06] qui ont developpé toute une technologie de reformulation des probabilités libres notamment au travers de l'étude des cumulants, en terme de combinatoire de partitions non-croisées. Ces techniques ont notamment été utilisées dans une récente série d'articles par Mingo, Speicher, Sniady et Collins [MS06b], [MŚS07], [CMŚS07] pour comprendre la limite et les fluctuations d'autres modèles de matrices aléatoires de manière combinatoire.

Combinatoire

Si nous nous sommes surtout concentrés jusqu'ici sur l'étude des énumérations combinatoires via les modèles matriciels qui fournissent quantités d'outils d'analyse, ces problèmes peuvent aussi être attaqués de manière purement combinatoire. Dans le domaine des cartes monocolores Bender et Canfield [BC94] ont ainsi obtenu des équations algébriques pour les cartes à étoiles énumérées selon le nombre d'arêtes et ayant des étoiles de type prescrit.

Le cas emblématique du modèle d'Ising sur carte planaire de valence 4 a été étudié par Bousquet-Melou et Schaeffer [BMS02]. Ainsi ils parviennent à retrouver le résultat prouvé par Mehta [Meh81]. La stratégie adoptée par les auteurs est ici de construire une bijection explicite entre les cartes du modèle d'Ising et des familles d'arbres pour lesquelles il est plus facile d'obtenir des équations algébriques sur les séries génératrices.

Équation de boucles

L'idée des équations de boucles aussi appelées équations de Schwinger-Dyson est d'opérer une modification infinitésimale dans le modèle pour faire apparaître des familles d'équations algébriques entre les observables. Comme ces techniques sont une partie importante de cette thèse nous les décrirons plus en détail dans la partie 1.2.2.

Cette technique est utilisée dès l'origine du domaine (même si elle ne porte pas encore ce nom) dans [BIPZ78]. Elle est devenue très populaire en gravité quantique [DFGZJ95]. Ambjørn, Durhuus et Fröhlich [ACKM93], ont montré qu'elles pouvaient être utilisées pour calculer récursivement tous les ordres des observables des modèles matriciels. Des techniques proches ont permis à Albeverio, Pastur et Shcherbina [ASM01] de donner le développement en genre de la transformée de Hilbert de la mesure empirique moyenne.

Les équations de boucles ont été généralisées aux modèles multi-matriciels par Staudacher [Sta93]. Eynard a beaucoup développé cet outil. Il montre notamment comment calculer récursivement les observables des modèles de type Ising [Eyn03a] à partir des équations de boucles. Dans un travail avec Kokotov et Korotkin [EKK05], il calcule de manière explicite la première correction à l'énergie libre. Enfin, dans une récente série d'articles écrits avec Chekhov et Orantin [CEO06, EO05, EO07], les auteurs développent une théorie associant au modèle matriciel une courbe complexe qui contient toute l'information puisque les obervables à tous les ordres peuvent se lire comme des invariants de cette courbe.

Polynômes orthogonaux et technique de Riemann-Hilbert

Rappelons que la fonction de partition du modèle matriciel à une matrice, de potentiel V peut se réécrire explicitement en terme de polynômes orthogonaux :

$$Z_V^N = c_N N! \prod_{j=0}^{N-1} \int_{\mathbb{R}^N} (h_j^N(x))^2 e^{-N(V(x)+\frac{x^2}{2})} dx$$

où h_j^N est le j-ième polynôme orthogonal par rapport à la mesure $\exp(-NV(x))$ et de coefficient directeur x^j. L'analyse des polynômes orthogonaux s'est développée dans les années 90 avec l'utilisation des techniques de Riemann-Hilbert. On pourra consulter [Dei99] pour une introduction et des applications de ces techniques. Un problème de Riemann-Hilbert est la donnée

1. d'un ensemble Σ de courbes lisses orientées de $\mathbb{C}$ s'intersectant de manière transverse et en un nombre fini de points
2. d'une application lisse $v : \Sigma \to \mathcal{GL}_N(\mathbb{C})$ qui si Σ est non borné tend rapidement vers l'identité à l'infini.

Le problème est alors de trouver une application $m : \mathbb{C} \to \mathcal{M}_N(\mathbb{C})$ telle que

1. m est analytique sur $\mathbb{C} - \Sigma$,
2. $m(z) \sim_{z\to+\infty} I$
3. Les discontinuités de m lorsqu'on franchit une composante de Σ sont décrites par v de la manière suivante :
$$m_+(z) = v(z)m_-(z)$$
où m_+ (resp. m_-) est la valeur de m à gauche (resp. à droite) de Σ, ce qui a un sens car les contours sont orientés.

Le lien avec les polynômes orthogonaux se fait en choisissant $\Sigma = \mathbb{R}$, $n = 2$ et

$$v(z) = \begin{pmatrix} 1 & w(z) \\ 0 & 1 \end{pmatrix}.$$

Alors si Y^j est une solution du problème de Riemann-Hilbert correspondant où l'on a modifié la deuxième condition portant sur l'asymptotique en :

$$Y^j(z) =_{z\to+\infty} \left(I + O(\frac{1}{z})\right) \begin{pmatrix} z^j & 0 \\ 0 & z^{-j} \end{pmatrix}.$$

alors $Y_{11}^j(z)$ est le j-ième polynôme orthogonal par rapport à la mesure de densité $w(z)dz$ (ici pour déterminer de manière unique la famille des polynômes orthogonaux nous imposerons au j-ième d'avoir comme terme de plus haut degré z^j).

La stratégie adoptée est, après transformation du problème de matrices aléatoires en un problème de Riemann-Hilbert, de modifier ce dernier en déformant le contour pour le rendre plus simple à résoudre. Ces idées ont été mises en oeuvre avec succès, permettant notamment de prouver des phénomènes d'universalité dans l'espacement des valeurs propres des matrices aléatoires.

Pour les liens avec la combinatoire, le premier résultat liant les modèles de matrices et les énumérations de cartes a été obtenu en utilisant ces techniques par Ercolani et McLaughin [EM03]. [Ercolani-McLaughlin] Soit $V(x) = t_{2p}X^{2p} + \sum_{i=1}^{2p-1} t_i x^i$ un polynôme à coefficient dominant positif et pair. Pour tout g entier, il existe $\varepsilon > 0$ tel que si $\max_{1\leqslant i\leqslant 2p} |t_i| < \varepsilon$ et $\max_{1\leqslant i\leqslant 2p-1} |t_i| < \varepsilon t_{2p}$ alors l'énergie libre du système converge et est égale à la série combinatoire :

$$F_V^N = \frac{1}{N^2} \ln \int_{\mathcal{H}_N(\mathbb{C})} e^{-N\mathrm{Tr} V(A)} d\mu^N(A) = \sum_{\mathbf{k}\in\mathbb{N}^{2p}, h\leqslant g} \frac{1}{N^{2h}} \prod_i (-t_i)^{k_i} k_i! \mathcal{M}_{\mathbf{k}}^h + o(\frac{1}{N^{2g}})$$

où $\mathcal{M}_{\mathbf{k}}^h$ est le nombre de cartes de genre h avec exactement k_i étoiles de valence i. Une large partie de ma thèse a été consacrée à généraliser ce résultat au cadre des modèles à plusieurs matrices. La difficulté est que, mis à part le cas des modèles à une matrice et dans une moindre mesure des modèles du type Ising [BEH02], la méthode des polynômes orthogonaux ne peut fonctionner comme dans cet article. Les méthodes que nous avons utilisées sont donc très différentes.

1.2 Résultats

Le but de cette partie est de décrire les résultats obtenus dans les quatre articles écrits pendant cette thèse. Ces articles étant joints à cette introduction nous n'en donnerons qu'un bref aperçu en soulignant les outils essentiels ainsi que les difficultés particulières rencontrées.

1.2.1 Présentation du modèle et notations

Moments non-commutatifs et états traciaux

Un cadre adéquat pour présenter les relations entre modèles matriciels et combinatoire est celui des probabilités non-commutatives. S'il est en effet naturel de décrire la limite d'un modèle de plusieurs matrices comme un espace de probabilités non-commutatives (voir par exemple [Voi00]). Nous allons essayer de montrer que ces espaces sont aussi remarquablement adaptés pour parler de la combinatoire des cartes colorées. Nous allons essayer de souligner ce dernier point en introduisant simultanément la combinatoire et ces espaces non-commutatifs.

Soit $\mathcal{H}_N(\mathbb{C})$ l'ensemble des matrices hermitiennes de taille $N \times N$. Dans toute la suite, nous nous fixerons un entier m qui est le nombre de matrices de notre modèle. On considérera dans cette partie des m-uplets de matrices aléatoires hermitiennes $\mathbf{A} = (A_1, \cdots, A_m)$. Les modèles que l'on étudiera sont invariants si nous conjuguons toutes les matrices par une matrice unitaire donnée. Dans le cas d'un modèle à une matrice toute la donnée est alors contenue dans la connaissance de la distribution des valeurs propres que l'on peut par exemple étudier via ses moments. Dans le cadre multi-matriciel, l'«angle» (c'est à dire le fait qu'elles ne se diagonalisent pas dans une même base) entre les matrices intervient. C'est pourquoi nous nous intéresserons plutôt à la distribution des moments non-commutatifs de ces matrices. On note $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ l'algèbre des polynômes non-commutatifs à coefficients complexes en les variables $X_1,\ldots,X_m$, c'est à dire les combinaisons linéaires complexes de monômes qui sont des mots en $X_1,\ldots,X_m$. Cette algèbre est munie de l'opérateur de conjugaison défini sur les monômes par :

$$(\lambda X_{i_1} \ldots X_{i_p})^* = \bar{\lambda} X_{i_p} \ldots X_{i_1}$$

pour tout λ complexe et i_j dans $\{1, \ldots, m\}$. Un monôme $X_{i_1} \ldots X_{i_p}$ peut-être à la fois vu comme un moment non-commutatif mais aussi comme le voisinage marqué du sommet d'un graphe à arêtes colorées plongé dans une surface compacte orientée.

Appelons étoile un tel voisinage de sommet. Une étoile est déterminée par la valence du sommet, l'orientation de la surface, la couleur de chacune des arêtes qui en sort et une marque sur l'un de ces germes d'arêtes. Pour construire l'arête associée au monôme $X_{i_1} \ldots X_{i_p}$, traçons p germes d'arêtes ou «demi-arêtes». Nous marquerons la première et nous lui donnerons la couleur i_1 puis en tournant dans le sens horaire autour du sommet nous colorierons la deuxième de couleur i_2 et ainsi de suite jusqu'à la p-ième que nous colorierons couleur i_p. La figure 2.1 dans la partie 2 illustre comment passer des monômes aux étoiles. Il est immédiat qu'on a construit là une bijection entre monômes non-commutatifs et étoiles marquées. La conjugaison agit sur une étoile en prenant son image dans un miroir. Changer la marque de place revient sur un monôme à effectuer une permutation circulaire des variables.

On appelle état tracial une forme linéaire τ sur $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ qui vérifie les propriétés :

$$\tau(1) = 1, \qquad \forall P, Q,\ \tau(PQ) = \tau(QP), \qquad \forall P,\ \tau(PP^*) \geq 0.$$

C'est la généralisation naturelle de la fonction qui donne les moments d'une mesure dans un cadre non-commutatif. Par analogie avec le cas à une variable nous dirons qu'un état tracial τ est à support borné s'il existe $R > 0$ tel que pour tout monôme P,

$$\forall P,\ |\tau(P)| \leqslant R^{\deg P}.$$

On appellera la borne inférieure des R vérifiant cette famille d'inégalités le rayon du support.

L'objet qui nous intéresse est la distribution de notre m-uplet de matrices aléatoires

$$\hat{\mu}^N : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m\rangle & \to & \mathbb{C} \\ P & \to & \frac{1}{N}\mathrm{Tr}P(\mathbf{A}) := \frac{1}{N}\mathrm{Tr}P(A_1, \cdots, A_m). \end{array}$$

C'est une variable aléatoire à valeur dans l'espace des états traciaux. On s'intéressera plus particulièrement à sa limite lorsque N devient grand.

Dérivées non-commutatives

On définit sur notre espace de polynômes deux notions de dérivées. Elles apparaissent de manière naturelle si on dérive un produit de matrices. C'est pour cette raison que la dérivée cyclique fut introduite pour la première fois par Rota, Sagan et Stein [RSS80]. Les dérivées non-commutatives ont été très utilisées par Voiculescu dans sa série de 6 papiers initiés par [Voi93] dans le but de définir un analogue de l'entropie dans un cadre non-commutatif.

Pour $1 \leqslant i \leqslant m$, la dérivée non-commutative ∂_i est un opérateur linéaire de $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ dans $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 2}$ défini par la règle de Leibniz :

$$\partial_i PQ = \partial_i P(1 \otimes Q) + (P \otimes 1)\partial_i Q$$

et pour une variable $\partial_i X_j = \mathbb{1}_{i=j} 1 \otimes 1$. Si P est un monôme, on peut écrire directement l'action de cette dérivée :

$$\partial_i P = \sum_{P = RX_iS} R \otimes S$$

où la somme porte sur toutes les décompositions du monôme P en RX_iS. Cette définition vient naturellement lorsqu'on cherche à dériver un produit de matrices hermitiennes. Si nous identifions de manière canonique $\mathcal{H}_N(\mathbb{C})$ à $\mathbb{R}^{N^2}$ alors pour P un monôme,

$$\left(\frac{\partial}{\partial \Re e A_k(ij)} + \sqrt{-1}\frac{\partial}{\partial \Im m A_k(ij)}\right) P(\mathbf{A})_{ij} = \sum_{P = RX_iS} R(\mathbf{A})(ii) S(\mathbf{A})(jj)$$

On peut voir aussi l'action de cette dérivée graphiquement sur les étoiles. Si nous représentons $R \otimes S$ par deux étoiles : une de type R, une de type S collées en leur sommet et séparées par une boucle sur le sommet qui entoure R (ou S, ce qui est équivalent car topologiquement la

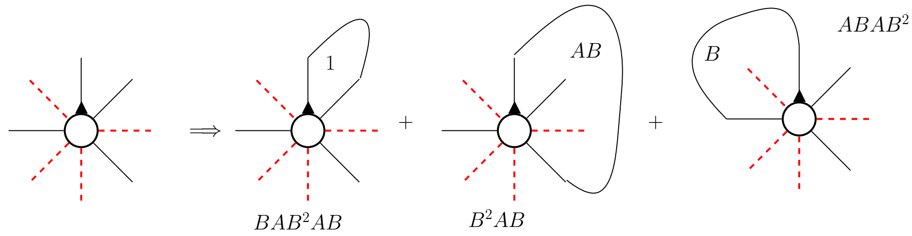

FIG. 1.11 – $\partial_A(ABAB^2AB) = 1 \otimes BAB^2AB + AB \otimes B^2AB + ABAB^2 \otimes B$

seule chose importante est de séparer les deux étoiles) alors cette dérivée consiste à rajouter à une étoile P de toutes les manières possibles une boucle de couleur i partant à gauche de la demi-arête marquée et se recollant sur une autre demi-arête de couleur i. Plus important pour la suite, remplacer X_iP par ∂_iP consiste à relier de toutes les manières possibles la première variable X_i à une autre demi-arête de couleur i (voir la figure 1.11).

On introduit aussi une seconde dérivée, appelée dérivée cyclique qui est un opérateur linéaire sur $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ et défini par $D_i = \text{mult} \circ^t \circ\partial_i$ avec ^{t} l'opérateur de transposition ($(a \otimes b)^t = b \otimes a$) et mult celui de multiplication ($\text{mult}(a \otimes b) = ab$). Comme précédemment nous pouvons écrire l'action de cette dérivée sur les monômes :

$$D_iP = \sum_{P=RX_iS} SR$$

et en donner une interprétation combinatoire : si nous remplaçons l'étoile X_iP par l'étoile D_iPQ cela revient à considérer toutes les manières de relier la demi-arête marquée de X_iP à une demi-arête de Q de la même couleur et de contracter l'arête ainsi formée (voir figure 1.12). Cette dérivée apparaît naturellement lorsqu'on cherche à dériver des traces de produit

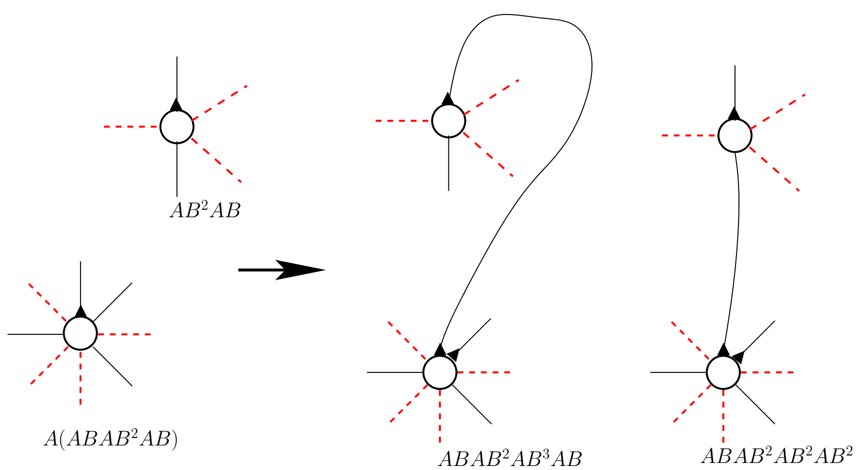

FIG. 1.12 – $(ABAB^2AB)D_A(AB^2AB) = ABAB^2AB^3AB + ABAB^2AB^2AB^2$

de matrices : pour P en monôme en des matrices hermitiennes :

$$\left(\frac{\partial}{\partial \Re e A_k(ij)} + \sqrt{-1}\frac{\partial}{\partial \Im m A_k(ij)}\right) \mathrm{Tr} P(\mathbf{A}) = (D_k P)(\mathbf{A})(ij).$$

Le modèle matriciel

Le modèle qui nous intéresse ici est une petite perturbation du GUE. On choisit les matrices selon la loi :

$$d\mu_V^N(A_1, \ldots, A_m) = \frac{1}{Z_V^N} e^{-N\mathrm{Tr} V(\mathbf{A})} d\mu^N(\mathbf{A})$$

où V est un polynôme non-commutatif et $Z_V^N = \int e^{-N\mathrm{Tr} V(\mathbf{A})} d\mu^N(\mathbf{A})$ est la constante de normalisation. A priori, rien ne dit que ceci définisse une mesure de probabilité car il y a deux obstacles. La premiere est d'assurer que la densité définie ainsi est bien réelle. Si $\mathrm{Tr}V(\mathbf{A})$ est tout le temps réel, nous pouvons supposer, quitte à le remplacer par $1/2(V + V^*)$, sans perte de généralité que V est auto-adjoint ($V = V^*$). Nous imposerons cette condition par la suite, ce qui entraîne immédiatement que pour tout m-uplet de matrices $\mathrm{Tr}V(\mathbf{A})$ est réel. Le second est que, si jamais V décroit trop vite à l'infini, alors Z_V^N est infinie. Par exemple, le potentiel $V = tX^3$ rend l'intégrale définissant Z_V^N divergente. Pour obliger V à bien se comporter à l'infini, nous demanderons une condition de stricte convexité. On peut aussi tirer parti du fait que V vient s'ajouter au potentiel $\frac{X^2}{2}$ qui est déjà convexe. On définit donc la notion de c-convexité pour $c > 0$: on dit que V est c-convexe si V est auto-adjoint et pour tout N la fonction

$$\begin{array}{ccc} \mathcal{H}_N(\mathbb{C})^m & \to & \mathbb{R} \\ \mathbf{A} & \to & \mathrm{Tr}[V(\mathbf{A}) + \frac{1-c}{2}\sum_i A_i^2] \end{array}$$

a une Hessienne positive. Cette condition assure que notre densité est bornée supérieurement par une densité gaussienne, ce qui implique la convergence de l'intégrale définissant Z_V^N. Grâce au lemme de Klein (voir [GZ02]) assurant la convexité de $X \to \mathrm{Tr}f(X))$ pour f fonction réelle convexe, on peut construire des exemples de polynômes vérifiant cette hypothèse :

$$V = \sum_i P_i(\sum_j \alpha_{ij} X_j) + \sum_{k\ell} \beta_{k\ell} X_k X_\ell$$

avec des polynômes réels convexes P_i, des α_{ij}, β_{kl} réels tels que pour tout l, $\sum |\beta_{kl}| < 1 - c$. Notons en particulier que les modèles du type Ising rentrent dans ce cadre.

Finalement, nous demanderons à V d'être «petit», pour cela nous nous fixerons des monômes $q_1, \ldots, q_n$ et nous définirons $V_\mathbf{t} = \sum_{i=1}^n t_i q_i$. Les domaines sur lesquels nous regarderons ces intégrales seront de la forme :

$$B_{\eta,c} = \{\mathbf{t} = (t_1, \ldots, t_n) \in \mathbb{C}^n | V_\mathbf{t} \text{ est } c\text{-convexe et pour tout } i,\ |t_i| < \eta\}.$$

Les quantités qui nous intéressent sont alors la fonction de partition Z_V^N, l'énergie libre F_V^N et le comportement de la distribution empirique $\hat{\mu}^N$ et de sa moyenne $\overline{\mu}^N$:

$$\overline{\mu}^N(P) = \mu_V^N(\hat{\mu}^N(P)) = \int_{\mathcal{H}_N(\mathbb{C})^m} \frac{1}{N} \mathrm{Tr} P(\mathbf{A}) e^{-N\mathrm{Tr} V(\mathbf{A})} d\mu^N(\mathbf{A}).$$

L'énergie libre et la distribution emirique sont liées par la relation :

$$\left.\frac{\partial}{\partial s}\right|_{s=0} F^N_{V+sP} = -\overline{\mu}^N[P].$$

Ainsi, obtenir des résultats sur l'un donne des renseignements sur l'autre. Notre angle d'attaque sera le plus souvent l'étude de la mesure pour déduire des informations sur l'énergie libre. Notons que toutes ces quantités dépendent de la famille de paramètres $\mathbf{t}$ mais, pour alléger les notations, nous ne les ferons pas figurer.

Combinatoire

On s'intéresse à l'énumération de cartes dans une surface donnée avec des étoiles de type donné. Pour y parvenir, si $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ est un polynôme fixé avec q_i des monômes, nous définissons :

$$\mathcal{M}^g(P) = \sum_{\mathbf{k}\in\mathbb{N}} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^g_{\mathbf{k}}(P)$$

où $\mathcal{M}^g_{\mathbf{k}}(P)$ est le nombre de cartes dans une surface de genre g avec une étoile de type P et en plus pour tout i, k_i étoiles de type q_i étiquetées de 1 à k_i. Remarquons qu'on peut aussi reformuler cette série comme une énumération d'objets non-étiquetés :

$$\mathcal{M}^g(P) = \sum_{\mathbf{k}\in\mathbb{N}} \prod_{i=1}^n (-t_i)^{k_i} \mathcal{D}^g_{\mathbf{k}}(P)$$

où $\mathcal{D}^g_{\mathbf{k}}(P)$ est le nombre de cartes dans une surface de genre g avec une étoile de type P qu'on désigne comme étant la racine et en plus, pour tout i, k_i étoiles de type q_i (cette fois indifférentiables les unes des autres).

L'étoile de type P sera appelée racine de la carte.

1.2.2 Premier ordre, équation de Schwinger-Dyson

Le but de cette partie est d'expliquer le résultat principal de [GMS06] : Pour tout $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$, $c > 0$, il existe $\eta > 0$ tel que si $\mathbf{t}$ est dans $B_{\eta,c}$,

$$\lim_N F^N_V = \sum_{\mathbf{k}} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{M}^0_{\mathbf{k}}$$

où $\mathcal{M}^0_{\mathbf{k}}$ est le nombre de cartes planaires avec pour tout i, k_i étoiles de type q_i étiquetées de 1 à k_i.

La première étape de notre stratégie est d'étudier la distribution empirique $\hat{\mu}^N$ plutôt que l'énergie libre. Comme nous l'avons vu les deux objets sont reliés et obtenir des informations sur la mesure permet après une simple intégration d'en obtenir sur l'énergie. Par ailleurs, il s'agit d'un objet a priori plus riche. En réalité, cette démarche est analogue à celle des combinatoriciens qui pour obtenir des résultats dans une série génératrice de cartes à étoiles de valence 4, considèrent la série plus générale où ces cartes sont enracinées en une étoile de valence arbitraire (voir la section 1.1.1).

Le centre de notre étude est l'équation de Schwinger-Dyson qui va créer pour nous le lien entre combinatoire et intégrales matricielles. On dit qu'un état tracial τ satisfait $\mathbf{SD}[V_{\mathbf{t}}]$, l'équation de Shwinger-Dyson de potentiel $V_{\mathbf{t}}$ si pour tout polynôme P et pour tout $1 \leqslant i \leqslant m$,

$$\tau((X_i + D_i V_{\mathbf{t}})P) = \tau \otimes \tau(\partial_i P).$$

Cette équation possède de nombreuses interprétations. Nous verrons son lien avec les modèles de matrices et avec la combinatoire. Son analogue unidimensionnelle est la recherche d'une mesure de probabilité réelle τ ayant une transformée de Hilbert $x + V'(x)$ donnée. Les probabilités libres généralisent cette interprétation en la recherche d'un état tracial tel que la variable conjuguée (notion introduite par Voiculescu dans [Voi98]) de X_i soit $X_i + D_i V$.

Afin de souligner l'importance de cette équation, nous allons développer trois de ces aspects : son lien avec le modèle matriciel, son interprétation combinatoire et enfin ses propriétés en tant qu'équation sur l'espace des états traciaux.

Schwinger-Dyson et modèles matriciels

Comme nous l'avons vu, les opérateurs de différentiation non-commutatifs ∂_i et D_i apparaissent de manière naturelle lorsque l'on se met à differentier des fonctions de produits et traces de matrices. Une simple intégration par parties montre ainsi que pour tout i,P,

$$\mu_V^N[\hat{\mu}^N((A_i + D_i V)P)] = \mu_V^N[\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)].$$

Cette équation est très proche de l'équation de Schwinger-Dyson. Ainsi si nous parvenions à intervertir dans le second terme l'espérance et le produit tensoriel :

$$\mu_V^N[\hat{\mu}^N \otimes \hat{\mu}^N(R \otimes S)] = \mu_V^N[\hat{\mu}^N(R)]\mu_V^N[\hat{\mu}^N(S)]$$

alors $\mu_V^N \circ \hat{\mu}^N = \overline{\mu}^N$ serait solution de l'équation Schwinger-Dyson. Cette interversion n'est pas possible pour N fini. Cependant du fait du choix d'un potentiel convexe, notre mesure vérifie une inégalité de Sobolev logarithmique (voir [ABC+00]) qui garantit des propriétés de concentration de la mesure. Ceci est suffisant pour assurer que l'interversion est vraie asymptotiquement : $\hat{\mu}^N(R)$ et $\hat{\mu}^N(S)$ sont suffisamment proches de leur moyenne.

Il devient ainsi possible de prouver que les points d'accumulation de $\overline{\mu}^N$ satisfont l'équation $\mathbf{SD}[V_{\mathbf{t}}]$ et, par concentration, c'est aussi le cas de ceux de $\hat{\mu}^N$. Enfin, la convexité de V assure aussi que $\overline{\mu}^N$ est tendue et que ses points d'accumulation ont un support strictement borné.

Schwinger-Dyson et combinatoire

Un moyen alternatif au calcul de Wick pour voir émerger la combinatoire des modèles matriciels est de constater que l'équation de Schwinger-Dyson est exactement analogue aux relations de Tutte pour l'énumération des cartes planaires (voir l'exemple que nous avons donné dans la section 1.1.1). Réécrivons cette équation :

$$\tau(X_iP) = \tau \otimes \tau(\partial_i P) - \sum_{j=1}^{n} t_j \tau((D_i q_j)P). \tag{1.5}$$

Nous voulons montrer ici par récurrence que si τ satisfait cette équation, alors pour tout P en tant que série formelle en $\mathbf{t}$,

$$\tau(P) = \mathcal{M}(P).$$

Pour y parvenir, il suffit de voir que ces deux quantités satisfont la même relation d'induction : $\mathbf{SD}[V_{\mathbf{t}}]$. Notons tout d'abord qu'il y a unicité des solutions formelles de cette équation. Cela vient de l'écriture (1.5) où nous calculons $\tau(X_iP)$ en fonction d'un terme où le degré du polynôme a chuté et d'un deuxième où la valuation en les variables $\mathbf{t}$ a augmenté. Nous obtenons donc une récurrence bien fondée sur les coefficients de la famille de séries $\{\tau(P)\}_{P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle}$. Cette récurrence est initialisée par la condition $\tau(1) = 1$. Ainsi vue, l'équation de Schwinger-Dyson (1.5) apparaît comme une un moyen algorithmique de calculer les $\mathcal{M}_{\mathbf{k}}(P)$.

Montrons que $\mathcal{M}$ satisfait la même équation. Soit P un monôme. Afin de calculer $\mathcal{M}(X_iP)$, nous voulons construire les cartes enracinées en une étoile de type X_iP ; pour le montrer, regardons à quoi est reliée la demi-arête issue du X_i : il y a deux possibilités.

1. Soit celle-ci boucle sur P et il faut alors que X_i apparaisse dans $P = RX_iS$. On devra construire une carte dans chaque composante délimitée par cette boucle, donc une carte enracinée en R et une en S :

$$\sum_{P=RX_iS} \mathcal{M}(R)\mathcal{M}(S).$$

   Cette opération est similaire à celle illustrée par la figure 1.11 et correspond au premier terme dans le membre de droite de (1.5).

2. La seconde possibilité est d'être reliée à un sommet de type q_j ce qui rajoute un poids $-t_j$ et n'est possible que si X_i apparaît dans $q_j = RX_iS$. La contraction de l'arête ainsi formée regroupe les demi-arêtes restantes de P et de q pour former une étoile de type PSR :

$$-t_j \sum_{q=RX_iS} \mathcal{M}(PSR).$$

   Cette opération est similaire à celle illustrée par la figure 1.12 et correspond au second terme dans le membre de droite de (1.5).

En mettant bout à bout ces deux possibilités et en utilisant l'écriture en termes de dérivées non-commutatives, nous obtenons que $\mathcal{M}$ est solution de $\mathbf{SD}[V_{\mathbf{t}}]$ et cette équation est simplement la version pour les cartes colorées de la relation de Tutte montrée dans la figure 1.5.

Propriétés de l'équation de Schwinger-Dyson

Arrivé à ce point, si nous faisons le bilan des informations collectées :

1. $\mathcal{M}$ est une série formelle solution de $\mathbf{SD}[V_{\mathbf{t}}]$.
2. Les points d'accumulation de $\hat{\mu}^N$ sont des solutions à support compact de $\mathbf{SD}[V_{\mathbf{t}}]$.

La dernière étape sera donc l'étude de l'unicité des solutions de $\mathbf{SD}[V_{\mathbf{t}}]$.

1. Si τ est une série formelle solution de $\mathbf{SD}[V_{\mathbf{t}}]$ alors pour tout P, la série $\tau(P)$ a un rayon de convergence strictement positif. De plus l'état tracial non formel obtenu est solution de $\mathbf{SD}[V_{\mathbf{t}}]$ et son support est uniformément borné pour $|\mathbf{t}|$ petit.
2. Soit $R > 0$. Il existe $\varepsilon > 0$ tel que pour $|\mathbf{t}| < \varepsilon$, il existe au plus une solution de $\mathbf{SD}[V_{\mathbf{t}}]$ dont le support est borné par R

Ce théorème est suffisant pour conclure que $\hat{\mu}^N$ et $\overline{\mu}^N$ ont un unique point d'accumulation : $\mathcal{M}$. Le résultat sur l'énergie libre est une conséquence simple de ce fait car tous les contrôles sont uniformes.

Notons que la tracialité de $\mathcal{M}$ est triviale puisque la seule différence entre l'étoile X_iP et l'étoile PX_i est la demi-arête en laquelle la racine est marquée. Cependant on peut se passer de cette tracialité pour montrer que $\mathcal{M}$ était solution de $\mathbf{SD}[V_{\mathbf{t}}]$. C'est en fait un corollaire que toute solution est automatiquement traciale. Le résultat analogue pour l'équation qui apparaît dans l'étude des matrices unitaires n'a rien d'évident.

1.2.3 Deuxième ordre

Nous allons maintenant présenter le résultat principal de [GMS07] : Pour tout $V_{\mathbf{t}}$, $c > 0$, il existe $\eta > 0$ tel que si $\mathbf{t}$ est dans $B_{\eta,c}$,

$$F_V^N = F^0(\mathbf{t}) + \frac{1}{N^2}F^1(\mathbf{t}) + o(\frac{1}{N^2})$$

Ce résultat est en fait un corollaire d'un Théorème Central Limite sur la répartition des valeurs propres : Pour tout $V_{\mathbf{t}}$, $c > 0$, il existe $\eta > 0$ tel que si $\mathbf{t}$ est dans $B_{\eta,c}$, pour tout polynôme P,

$$N(\hat{\mu}^N(P) - \mathcal{M}(P)) \to \mathcal{N}(0, \sigma^2(P,P))$$

où σ^2 est la série énumérant les cartes planaires avec deux étoiles de type P :

$$\sigma^2(P,P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_{\mathbf{k}}(P,P)$$

où $\mathcal{M}_{\mathbf{k}}(P,Q)$ est le nombre de cartes dans une surface de genre g avec une étoile de type P, une autre de type Q et en plus, pour tout i, k_i étoiles de type q_i étiquetées de 1 à k_i.

La preuve de ce théorème est détaillée dans [GMS07]. Donnons-en les grandes étapes. Tout d'abord, nous avons besoin de meilleurs contrôles que pour le premier ordre (ceux-ci proviennent essentiellement de la convexité de V et seront détaillés en début de cette partie).

Le point de départ est d'opérer un «changement de variable infinitésimal» outil utilisé en physique notamment pour dériver les équations de boucle. C'est aussi la méthode utilisée par

Johansson pour prouver le Théorème Central Limite dans le cas à une matrice [Joh98]. Cette technique consiste à faire une petite déformation dans la fonction de partition :

$$Z_V^N = \int_{\mathcal{H}_N(\mathbb{C})^m} e^{-N\mathrm{Tr}V(\mathbf{A})} d\mu^N(\mathbf{A}).$$

L'idée est de faire le changement de variable $B_i = A_i + \frac{1}{N}h_i(\mathbf{A})$ pour des fonctions h suffisamment bien choisies (pour des raisons techniques nous ne pouvons pas choisir directement les h_i polynomiaux car ceux-ci ne sont pas bornés).

La jacobienne de ce changement de variable s'écrit :

$$J_h = e^{\frac{1}{N}\sum_i \mathrm{Tr}\otimes\mathrm{Tr}(\partial_i h_i) + \frac{1}{2N^2}\sum_{i,j}\mathrm{Tr}\otimes\mathrm{Tr}((\partial_i h_j).(\partial_j h_i)) + o(1)}$$

et nous avons par ailleurs,

$$N\mathrm{Tr}V(\mathbf{B}) = N\mathrm{Tr}V(\mathbf{A}) + \sum_i \mathrm{Tr}D_i V h_i + \frac{1}{2N}\sum_i \mathrm{Tr}D_{i,j}^2 V.(h_i, h_j) + o(1).$$

Ici D_{ij}^2 est un opérateur différentiel d'ordre ayant une définition analogue à celle de la dérivée non-commutative. Un peu de calcul mène à :

$$\mu_V^N(e^{N(\Xi(h_1,\ldots,h_m))}) = e^{\mathcal{C}(h_1,\ldots,h_m)}(1 + o(1))$$

où $\mathcal{C}$ est quadratique et Ξ est un opérateur linéaire (voir section 3.4.3 de la partie 3 pour une définition précise).

Un peu de technique permet d'obtenir le même résultat pour des h_i polynomiaux. On obtient alors aisément grâce au théorème de Levy un Théorème Central Limite pour les polynômes dans l'image de Ξ. Arrivé à un stade comparable Johansson avait utilisé des techniques analytiques d'inversion de la transformée de Hilbert (en particulier les méthodes développées dans [Tri57]). Ces méthodes ne sont cependant pas accessibles dans le cadre multimatriciel et il faut trouver un moyen d'inverser l'opérateur Ξ pour obtenir le Théorème Central Limite pour des polynômes arbitraires.

C'est ce qui nous a poussé à introduire la famille de normes $\|.\|_A$ pour $A > 2$, sur l'espace des polynômes : si $P = \sum_q \lambda_q q$ est la décompostion de P en monômes :

$$\|P\|_A = \sum_q |\lambda_q| A^{\deg q}.$$

L'intérêt réside ici dans le fait que pour $A > 2$, si $\mathbf{t}$ est petit, nous avons pu prouver que l'opérateur Ξ est continu et inversible sur $\mathbb{C}\langle X_1, \cdots, X_m\rangle_A$, ce qui permet de prolonger le Théorème Central Limite à n'importe quel polynôme.

1.2.4 Ordres suivants et dérivées

Enfin pour achever cet aspect de l'étude des modèles gaussiens, nous allons maintenant décrire les résultats de [MS06a].

Pour tout $V_{\mathbf{t}}$, $c > 0$, $g \in \mathbb{N}$, il existe $\eta > 0$ tel que si $\mathbf{t}$ est dans $B_{\eta,c}$,

$$F_V^N = F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

et $F^g(\mathbf{t})$ énumère les cartes plongées sur des surfaces de genre g. Par ailleurs, ce développement se dérive terme à terme : pour tout $\mathbf{j} = (j_1, \ldots, j_n)$, n uplet d'entiers il existe $\eta > 0$ tel que si $\mathbf{t}$ est dans $B_{\eta,c}$,

$$\frac{\partial^{j_1+\cdots+j_n}}{\partial t_1^{j_1} \ldots \partial t_n^{j_n}} F_V^N = \frac{\partial^{j_1+\cdots+j_n}}{\partial t_1^{j_1} \ldots \partial t_n^{j_n}} F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}} \frac{\partial^{j_1+\cdots+j_n}}{\partial t_1^{j_1} \ldots \partial t_n^{j_n}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

et $\frac{\partial^{j_1+\cdots+j_n}}{\partial t_1^{j_1} \ldots \partial t_n^{j_n}} F^g(\mathbf{t})$ énumère les cartes sur des surfaces de genre g avec au moins j_i étoiles de type q_i.

L'idée directrice est toujours de montrer que certains observables des modèles matriciels satisfont des relations similaires à celles que l'on peut trouver en combinatoire. Cependant il est nécessaire d'augmenter la famille des observables à considérer. D'une part les corrections à la convergence du modèle matriciel font intervenir les moments joints de la mesure empirique, il nous faut alors analyser

$$\nu^N(P_1 \otimes \cdots \otimes P_l) = E_{\mu_V^N}[N^l(\hat{\mu}^N - \mu)^{\otimes l}(P_1 \otimes \cdots \otimes P_l)].$$

D'autre part afin de trouver un ensemble clos de relations pour l'énumération de cartes de genre supérieur, nous sommes là aussi amenés à augmenter les observables. Pour l dans $\mathbb{N}$, une famille de monômes $P_1, \cdots, P_l$ et des une famille de monômes $q_1, \ldots, q_n$, nous définissons les cartes minimales comme étant les multi-cartes formées d'étoiles de type q_i et de l étoiles supplémentaires associées à $P_1, \ldots, P_l$ telles que :

1. chaque composante de la carte contient au moins une des étoiles associé à l'un des P_i,
2. chaque P_i est soit dans une carte non-planaire soit dans la composante d'un autre P_j.

Pour des entiers $\mathbf{k} = (k_1, \cdots, k_n)$, on définira $\mathcal{M}_{\mathbf{k}}^g(P_1 \otimes \cdots \otimes P_l)$ le nombre de telles multi-cartes ayant exactement k_i étoiles de type q_i et dont la somme des genres des composantes est g. C'est en augmentant le nombre d'objets à compter de cette manière que l'on est capable de trouver une famille de relations fermées entre les séries de cartes afin d'appliquer notre technologie.

L'intérêt de cette famille de relations est qu'elle est bien fondée. Elle permet donc encore une fois de calculer récursivement des nombres de cartes. Par ailleurs, en les comparant avec soin à des relations obtenues sur le modèle matriciel, nous pouvons à nouveau rapprocher matrices aléatoires et combinatoire. La seule difficulté est qu'ici comme au premier ordre, la récursion étudiée n'est bien fondée que si l'on parvient à inverser l'opérateur Ξ. En réalité, cet opérateur n'est inversible que sur l'espace limite $\mathbb{C}\langle X_1, \cdots, X_m \rangle_A$ où tous les opérateurs sont à support compact. Or pour obtenir notre résultat, nous avons besoin d'estimés à N fini. C'est ce qui nous conduit à la preuve relativement technique du Lemme 4.6.2 où au lieu d'inverser Ξ, nous utilisons un inverse approximatif où la précision de l'approximation se raffine lorsque N tend vers l'infini, ce qui nous permet d'obtenir le contrôle suffisant pour conclure.

1.2.5 Intégrales divergentes

L'identification de l'intégrale matricielle et de la série combinatoire peut donc se faire de façon non-formelle sous deux conditions : la perturbation doit être petite et convexe.

La première restriction peut être partiellement levée. Dans un article plus récent Guionnet et Shlyakhtenko [SG07] prouvent qu'il n'y a pas de rupture d'analyticité de la distribution empirique au premier ordre tant que l'on reste dans le domaine de convexité de V. Ainsi, cela prouve que dans le domaine de convergence de la série génératrice des cartes, si les paramètres rendent V convexe, alors l'intégrale matricielle coïncide avec cette série. L'analyse se fonde sur l'écriture de la loi du modèle comme une mesure invariante d'une diffusion matricielle.

L'hypothèse de convexité du potentiel peut être relativement restrictive. Ainsi pour énumérer des triangulations, nous aimerions considérer un potentiel $V = tX^3$, mais celui-ci non seulement ne vérifie aucune convexité, mais transforme la fonction de partition en intégrale divergente. Il existe cependant un moyen de passer outre cette difficulté. Il est toujours possible de rendre le modèle matriciel bien défini en décidant, par exemple, de restreindre l'espace aux matrices de norme plus petite qu'une borne donnée. Ainsi pour $L > 0$ on peut définir :

$$Z_{L,V}^N = \int_{\|A_i\|<L} e^{-N\mathrm{Tr}V(\mathbf{A})} d\mu^N(\mathbf{A}).$$

Le fait remarquable est que pour un choix raisonnable de L (ni trop petit ni trop grand) la limite de l'énergie libre $F_{V,L}^N := N^{-2} \ln Z_{L,V}^N$ est indépendante du choix précis de L et correspond à l'interprétation combinatoire attendue. Pour tout $V_{\mathbf{t}} = \sum_i t_i q_i$, si on se donne $L > 2$ alors il existe $\eta > 0$ tel que si pour tout i, $|t_i| < \eta$ alors,

$$\lim_N F_{V,L}^N = \sum_{\mathbf{k}} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{M}_{\mathbf{k}}^0$$

où $\mathcal{M}_{\mathbf{k}}^0$ est le nombre de cartes planaires avec pour tout i, k_i étoiles de type q_i étiquetées de 1 à k_i. Ce résultat se généralise aussi à l'ordre suivant, voir Théorème 3.1.4 dans la partie 3.

Donnons maintenant l'intuition dérrière ce théorème. On doit fixer L plus grand que 2 le rayon du support de la semi-circulaire pour laisser intact le domaine où vivent asymptotiquement les valeurs propres. La constante L étant fixée, pour t suffisamment petit, la densité

$$\mathbb{1}_{||A||<L} e^{-N\mathrm{Tr}(tA^3+A^2/2)}$$

devient log-concave ce qui permet d'appliquer dans les grandes lignes les techniques précédentes. La seule difficulté est alors de contrôler le terme de bord.

1.2.6 Modèles unitaires

Le but ici est d'étudier à l'aide de la technologie des équations de Schwinger-Dyson les perturbations de la mesure de Haar sur le groupe unitaire.

Appelons $\mathbf{m}$ la mesure de Haar sur $\mathcal{U}_N(\mathbb{C})$ et donnons nous p suites de matrices déterministes $A_1^N, \ldots, A_p^N$. Nous supposerons

1. La norme opérateur des A_i^N est uniformément bornée par une constante C indépendante de N. On pourra supposer sans perte de généralité que $C = 1$.
2. Les A_i^N admettent une distribution limite i.e. il existe sur $\mathbb{C}\langle A_1, \cdots, A_m\rangle$ un état tracial τ, tel que pour tout P dans $\mathbb{C}\langle A_1, \cdots, A_m\rangle$,
$$\frac{1}{N}\mathrm{Tr}P(A_i^N) \to_{N\to+\infty} \tau(P).$$

Par ailleurs nous supposerons sans perte de généralité que les A_i^N sont hermitiens.

Appelons $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$ l'algèbre des polynômes complexes en les variables non-commutatives $A_1,\ldots,A_p,U_1,U_1^{-1},\ldots,U_m,U_m^{-1}$ muni de l'adjonction définie par $A_i^* = A_i$, $U_i^* = U_i^{-1}$. L'objet de notre étude est la mesure sur $\mathcal{U}_N(\mathbb{C})^m$ obtenue en perturbant la mesure de Haar par un potentiel V dans $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$:
$$d\mu_V^N(\mathbf{U}) = e^{N\mathrm{Tr}V(\mathbf{U},\mathbf{A})}d\mathbf{m}^{\otimes m}(U_1, \ldots, U_m).$$
En particulier nous aimerions comprendre la limite de l'état tracial $\hat{\mu}^N : P \to N^{-1}\mathrm{Tr}P(\mathbf{U}, \mathbf{A})$ sous cette loi.

Notons que l'intégrale de IZ (voir section 1.1.3) entre dans ce cadre puisqu'elle est l'énergie libre du modèle obtenue en choisissant $V = cU^*A_1UA_2$. Nous avons vu l'importance de cette intégrale en physique statistique sur graphes aléatoires. Sans chercher à obtenir de résultats non perturbatifs comme dans [GZ02], l'enjeu est de voir si l'on peut obtenir des résultats dans un voisinage de zéro en c et si les conclusions de [Col03] dépassent le cadre formel pour former une série convergente.

Soulignons deux différences essentielles entre ce cadre et le cadre gaussien.

1. La première différence est d'ordre technique : les matrices unitaires, à la différence des gaussiennes, sont bornées ce qui conduit à de nombreuses simplifications techniques. Nous aurons par exemple beaucoup moins de problèmes pour prouver des résultats de tension.
2. Nous sommes désormais privés du calcul de Wick pour nous donner une intuition d'une interprétation combinatoire. À la différence de l'article [ZJZ03] qui ramenait le calcul de IZ à une intégration gaussienne, nous allons utiliser l'équation de Schwinger-Dyson pour trouver une interprétation combinatoire. Cette interprétation est nouvelle et n'a été donnée à notre connaissance nulle part ailleurs.

Suivant les idées de [GMS06, GMS07, MS06a] nous introduisons des opérateurs de dérivation non-commutatives. L'opérateur ∂_i est défini par la règle de Leibniz et par
$$\partial_i U_j = \mathbb{1}_{i=j} 1 \otimes U_i,$$
$$\partial_i U_j^* = -\mathbb{1}_{i=j} U_i^* \otimes 1.$$
La dérivée cyclique D_i vaut encore mult $\circ^t \circ \partial_i$.

Ainsi pour un monôme P dans $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$,
$$\partial_i P = \sum_{P=RU_iS} RU_i \otimes S - \sum_{P=RU_i^*S} R \otimes U_i^* S,$$

$$D_iP = \sum_{P=RU_iS} SRU_i - \sum_{P=RU_i^*S} U_i^*SR.$$

Ces dérivées proviennent de la dérivation sur le groupe unitaire. Ainsi pour toute matrice antisymétrique,

$$\frac{\partial}{\partial s}\mathrm{Tr}P(U_ie^{sA})|_{s=0} = \mathrm{Tr}D_iPA.$$

Le lien entre modèle matriciel et solutions de l'équation de Schwinger-Dyson avait été prouvé en 2003 par Biane [Bia03]. Une démarche similaire à celle du cas gaussien nous permet de préciser ce résultat dans notre cadre de petites perturbations. Soit $V = \sum_{i=1}^n t_iq_i$ un potentiel dans $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$. Il existe $\varepsilon > 0$ tel que si pour tout i, $|t_i| < \varepsilon$ et $V^* = V$ alors pour tout polynôme P en les variables non-commutatives A_i,U_i,U_i^*, $\hat{\mu}^N(P)$ converge presque sûrement et en espérance vers $\mu(P)$, μ étant l'unique solution de l'équation de Schwinger-Dyson :

$$\forall i, \forall P, \quad \mu\otimes\mu(\partial_iP) + \mu(D_iVP) = 0.$$

De plus, μ est analytique en les t_i et est aussi l'unique solution de l'équation de Schwinger-Dyson vue comme une équation sur les séries formelles. En particulier, ce résultat identifie, dans un régime haute température, la limite «réelle» de IZ trouvée dans [GZ02] et la solution formelle de [Col03].

Le défi est ensuite de parvenir à trouver une interprétation combinatoire de $\mu_{\mathbf{t}}(P)$ à partir de l'équation de Schwinger-Dyson. C'est le travail que nous avons réalisé dans la partie 5.5. L'interprétation n'est pas aussi simple que dans les modèles gaussiens puisqu'on doit envisager des cartes avec de multiples manières de relier les objets entre eux : ainsi deux structures vivent sur la carte, des arêtes orientées similaires aux arêtes du cas gaussien et de nouvelles arêtes que nous avons appelées «dotted edges». Par ailleurs la manière de calculer le poids d'une carte est moins explicite en particulier la contribution donnée par les ensembles de «dotted edges» n'a pu être donnée que via le nombre de manières de construire ces ensembles.

1.3 Conclusion et perspectives

Les résultats de cette thèse permettent donc dans un régime haute température de prouver la convergence de modèles matriciels proche de la mesure du **GUE** . La première application est l'extrême régularité de la limite de ces modèles puisque ceux-ci dépendent analytiquement de leurs paramètres. De plus, puisque la série associée énumère des cartes, nous avons prouvé un lien allant au-delà des séries formelles entre intégrales matricielles et énumération combinatoire.

Le fait que ces énumérations soient liées à des modèles de matrices montre qu'elles ont une structure supplémentaire. Dans le cas à une matrice, l'existence d'une mesure de probabilité derrière ces énumérations combinatoires est essentielle dans de nombreux papiers (à commencer par [BIPZ78]) pour trouver des expressions exactes.

Dans le cadre que nous avons examiné, celui des modèles à plusieurs matrices, la probabilité est remplacée par un état tracial. Ainsi nous obtenons la propriété de positivité de ces séries combinatoires $\mathcal{M}(PP^*) \geqslant 0$ dont il n'existe pas de preuves directes a priori. Nous pouvons espérer que de nombreux progrès sont encore à faire dans l'utilisation de ce résultat.

Pour conclure je proposerai quelques questions ouvertes qui s'inscrivent dans le prolongement de cette thèse.

1.3.1 Combinatoire des cartes non-orientables

Dans la droite ligne de mes travaux de thèse, j'étudie en ce moment le même type de résultats peut être prouvé pour des modèles de matrices symétriques ou symplectiques.

Ces modèles ne bénéficient pas des simplifications miraculeuses du cas hermitien (l'article de Johansson [Joh98] illustre pour le cas à une matrice le fait qu'il existe un nombre important de raccourcis pour le cas complexe). Il s'agit désormais d'étudier des déformations du GOE (Ensemble Gaussien Orthogonal) et du GSE (Ensemble Gaussien Symplectique), notamment du point de vue combinatoire. Le GOE a été étudié par Goulden et Jackson dans [GJ97]. Les auteurs y étudient les cartes à une étoile. Mulase et Waldron ont écrit un article sur la dualité en GOE et GSE [MW03], un phénomène dont il serait intéressant de voir les implications dans notre cadre.

Nous nous attendons toujours à trouver des développements topologiques :

$$\sum_{\chi\in\mathbb{Z},\chi\leqslant 2} N^{\chi-2}\mathcal{M}^{\chi}$$

avec $\mathcal{M}^{\chi}$ qui compte des cartes de caractéristique d'Euler χ. Mais désormais la somme ne portera plus nécessairement sur des cartes dessinées sur des surfaces orientables. Ainsi χ pourra prendre des valeurs impaires et nous aurons à considérer des cartes sur le plan projectif ou la bouteille de Klein.

1.3.2 Modèles matriciels complexes

Une autre question que j'ai un peu examinée avec ma directrice de thèse est celle de l'étude de modèles matriciels à potentiel complexe. C'est un cas extrêmement important puisqu'on en trouve en particulier un exemple au coeur de la preuve d'une conjecture de Witten par Kontsevich [Kon92]. Cependant, les intégrales oscillantes impliquées sont extrêmement difficiles à contrôler. Il semble que les outils utilisés pour les potentiels auto-adjoints ne puissent s'appliquer ici. Une stratégie serait d'utiliser la récente technologie développée par Guionnet et Shlyakhtenko dans [SG07] en regardant la mesure matricielle comme la mesure invariante d'une diffusion.

1.3.3 Transport infinitésimal de semi-circulaires

Les limites des modèles matriciels fournissent des exemples non triviaux d'états traciaux dans le voisinage du produit libre de lois semi-circulaires. Une autre famille d'états traciaux naturels dans ce voisinage est l'ensemble des états que l'on peut obtenir par un petit transport de la loi semi-circulaire. Plus précisément si S_1,...,S_m sont m semi-criculaires libres et $(F_i)_{1\leqslant i\leqslant m}$ est une famille d'éléments de $\mathbb{C}\langle S_1, \cdots, S_m\rangle$, nous pouvons considérer l'état tracial σ_F qui est la loi de

$$S_1 + F_1, \ldots, S_m + F_m.$$

La question est alors de savoir si lorsque les F_i sont petits pour une norme raisonnable, nous allons pouvoir trouver un potentiel V tel que σ_F s'écrive comme limite du modèle matriciel correspondant. Réciproquement, nous pouvons nous demander quelles sont les limites τ_V de modèles matriciels qui peuvent s'écrire comme de petites modifications de la semi-circulaire.

1.3.4 Basse temperature

Dans le cadre d'un groupe de travail à Stanford en compagnie de géomètres, Eliashberg et Ionel, nous avons commencé avec Guionnet et Dembo à étudier de nouveaux liens entre modèles matriciels et géométrie qui proviennent de théorie des cordes. Schématiquement, des conjectures de Witten indiquent que toutes les théories devraient coïncider ce qui fournit des égalités non triviales entre des invariants d'objets géométriques. Les preuves rigoureuses qui existent de ces identifications utilisent les modèles matriciels. Nous avons suivi les notes de Marino [Mar04] qui présente en particulier la théorie de Chern-Simons qui peut se décrire via un modèle matriciel dans sa limite à «basse température». Le modèle que l'on cherche à comprendre est le suivant :

$$Z_V^N(\beta) = \int_{\mathcal{H}_N(\mathbb{C})} e^{-\frac{N}{\beta}TrV(A)} d^N(A).$$

L'objectif est de comprendre ce qui se passe pour de petites valeurs de β. Les valeurs propres devraient se concentrer en les minima de V. Si V n'a qu'un unique minimum, nous pouvons nous ramener à une étude proche de celle des déformations du **GUE** . Le cas à plusieurs minima semble beaucoup plus ardu à traiter, à l'exception des modèles possédant des symétries remarquables permettant de déterminer quelle masse de l'ensemble des valeurs propres se concentre autour de chaque minimum. Par exemple, le cas d'un potentiel pair à double puit comme $V(x) = (x^2 - 1)^2$ peut être analysé. Nous pouvons ainsi montrer la convergence de la mesure empirique des valeurs propres et de l'énergie libre à tous les ordres. La simplification donnée par ce modèle est que la répartition des valeurs propres tombant dans chaque puit est donnée par la symétrie. Le problème se complique lorsque cette répartition n'est pas donnée a priori.

1.3.5 Modèles matriciels et partition de type B

Il est bien connu suite aux travaux de Voiculescu que la loi limite de matrices indépendantes est décrite par la notion de liberté. Un article de Biane, Goodman et Nica [BGN03], motivée par des considérations algébriques, introduit et étudie la notion de cumulants et partition de type B. Cette approche permet de définir une notion abstraite de liberté de type B pour des variables non-commutatives. Cette notion est équivalente à une forme de liberté classique avec amalgamation. Sur une suggestion de Shlyakhtenko, j'aimerais regarder si cette liberté peut aussi apparaître naturellement comme la limite en grande dimension d'observables de matrices indépendantes. La limite étant décrite par la notion classique de liberté, il est possible que cette B-liberté intervienne pour les corrections à cette convergence.

1.3.6 Autour de l'universalité

Un phénomène surprenant apparaît lorsque l'on regarde les fluctuations au premier ordre de modèles matriciels. Si nous regardons le théorème prouvé par Johansson (Théorème 1.1.6), les fluctuations des modèles matriciels au premier ordre ont une certaine universalité. D'après le papier de Johansson celle-ci n'apparaît qu'au premier ordre et uniquement pour le modèle complexe, et non pas pour l'orthogonal et le symplectique. Ce résultat est analogue à l'universalité de la fonction de corrélation à deux points connexe qui avait été prédite par Brézin et Zee dans [BZ93]. Les auteurs y justifiaient cette universalité en invoquant le fait que la renormalisation consistant à supprimer une ligne et une colonne de la matrice avait pour unique point fixe le **GUE** .

Ce résultat semble avoir d'intéressantes conséquences sur la combinatoire des cartes planaires car nous pouvons par ailleurs identifier la covariance du Théorème Central Limite à des énumérations combinatoires. Ainsi si $V = \sum_i t_i x^i$, posons

$$C_V(x^p, x^q) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} (-t_i)^{k_i} \mathcal{M}_{\mathbf{k}}(x^p, x^q)$$

avec $\mathcal{M}_{\mathbf{k}}(x^p, x^q)$ le nombre de cartes planaires à k_i étoiles de valences i et deux autres étoiles : une de valence p, une de valence q. On énumère donc les cartes planaires à deux racines de type fixé.

L'universalité va nous fournir des identités sur les C_V. Pour simplifier, supposons que V est un polynôme pair (mais un enchaînement analogue pourrait être réalisé dans le cas général). La mesure limite des valeurs propres $\sigma_{x^2/2+V}$ est à support compact $[-\alpha; \alpha]$. D'après les résultats de Johansson, C_V vérifie l'identité suivante :

$$C_V(\tilde{T}_1, \tilde{T}_1) = C_V(\frac{x}{\alpha}, \frac{x}{\alpha}) = \frac{1}{4}$$

d'où

$$\alpha^2 = 4C_V(x,x).$$

Ce résultat est prouvé par des méthodes combinatoires dans [DF06]. En substituant cette expression de α dans n'importe quelle autre relation d'orthogonalité des polynômes de Tchebychev, nous pouvons déduire des identités entre les séries, par exemple :

$$C_V(\tilde{T}_2, \tilde{T}_2) = C_V(2(\frac{x}{\alpha})^2 - 1, 2(\frac{x}{\alpha})^2 - 1) = \frac{1}{2}$$

permet d'obtenir

$$C_V(x^2, x^2) = 2C_V(x,x)^2.$$

Ainsi la série des cartes planaires avec deux sommets de valence 2 marqués s'exprime en fonction de celle avec deux sommets de valence 1 et ce via une relation indépendante du potentiel V (pourvu qu'il soit pair), c'est à dire indépendante des poids et des types de sommets autorisés. Cette relation est la plus simple que l'on puisse obtenir mais ce rapprochement en crée beaucoup d'autres. Ce qui est remarquable est le fait qu'elles sont indépendantes des poids accordés aux différentes valences possibles. La question se pose de savoir s'il existe derrière des bijections naturelles entre les graphes correspondants. On peut aussi se demander si la compréhension de ce phénomène nous donnerait des informations sur le cas coloré.

Chapitre 2

Combinatorial Aspects of matrix models

Ce chapitre est l'article [GMS06] écrit en collaboration avec Alice Guionnet et publié dans la revue ALEA (Revue Latino-Américaine de Probabilités et Statistiques).

Abstract

We show that under reasonably general assumptions, the first order asymptotics of the free energy of matrix models are generating functions for colored planar maps. This is based on the fact that solutions of the Schwinger-Dyson equations are, by nature, generating functions for enumerating planar maps, a remark which bypasses the use of Gaussian calculus.

2.1 Introduction

It has long been used in combinatorics and physics that moments of Gaussian matrices have a valuable combinatorial interpretation. The first result in this direction is due to [Wig58] who proved that the trace of even moments of a $N \times N$ Hermitian matrix A with i.i.d centered entries with covariance N^{-1} converge as N goes to infinity towards the Catalan numbers which enumerate a large variety of combinatorial objects such as non crossing pair-partitions, rooted trees or Dick paths. If one restricts to Gaussian entries, that is matrices following the law μ_N of the **GUE** which is the probability measure on the set $\mathcal{H}_N(\mathbb{C})$ of $N \times N$ Hermitian matrices with density

$$\mu_N(dA) = \frac{1}{Z_N} e^{-\frac{N}{2}\mathrm{Tr}(A^2)} \prod_{1 \leqslant i \leqslant j \leqslant N} d\Re e(A_{ij}) \prod_{1 \leqslant i < j \leqslant N} d\Im m(A_{ij}),$$

it occurs that the corrections to this convergence count graphs which can be embedded on surface of higher genus, a fact which was used by [HZ86]. This enumerative property was fully developed after 't Hooft, who considered generating functions of such moments. For instance, c.f. [Zvo97], we have the formal expansion

$$F^N_{tx^4} = \frac{1}{N^2}\log\int e^{-Nt\mathrm{Tr}(A^4)}d\mu_N(A) = \sum_{g\geqslant 0} N^{-2g}\sum_{k\geqslant 1}\frac{(-t)^k}{k!}C(k,g)$$

with

$$C(k,g) = \mathrm{Card}\{\text{ maps with genus } g \text{ with } k \text{ vertices of valence } 4\}$$

Here, maps are connected oriented diagrams which can be embedded into a surface of genus g in such a way that edges do not cross and the faces of the graph (which are defined by following the boundary of the graph) are homeomorphic to a disc. The valence of the vertices comes from the quartic potential. The counting is done up to equivalent classes, i.e. up to homeomorphism. Let us stress that the above equality is only formal and should be understood in the sense that all the derivatives at the origin on both sides of the equality match, i.e. for all $k\in\mathbb{N}$,

$$(-1)^k\partial_t^k F^N_{tx^4}|_{t=0} = \sum_{g\geqslant 0}\frac{1}{N^{2g}}C(k,g).$$

This equality can be proved thanks to Wick's formula since $\partial_t^k F^N_{tx^4}|_{t=0}$ depends only on moments of gaussian variables (note above that the sum is in fact finite).

Such expansions can be generalized to arbitrary polynomial functions (to enumerate maps with vertices of different degrees) and to several-matrices integrals to enumerate colored maps. More precisely, let V be a polynomial of m non-commutative variables, $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with some monomials q_i and complex parameters $\mathbf{t} = (t_i)_{1\leqslant i\leqslant n}$ such that $\mathrm{Tr}V_{\mathbf{t}}(A_1,\cdots,A_m)$ is real for any self adjoint matrices $A_1,\cdots,A_m$. Then, the free energy expands formally into

$$F^N_{V_{\mathbf{t}}} = \frac{1}{N^2}\log\int e^{-N\mathrm{Tr}(V_{\mathbf{t}}(A_1,\cdots,A_m))}\prod_{i=1}^n d\mu_N(A_i) = \sum_{g\geqslant 0}\frac{1}{N^{2g}}F_g(t_1,\cdots,t_n)$$

where for $g\in\mathbb{N}$, F_g is a generating function for the enumeration of colored maps of genus g related to the monomials $(q_i)_{1\leqslant i\leqslant n}$ (see section 2.2.5).

The interest in such formal expansions lies in the hope to be able to estimate the free energy F^N_V when N goes to infinity by probability techniques, henceforth finding formulae for the generating functions $(F_g)_{g\geqslant 0}$. Such a strategy can only be validated if the expansion is not only formal, but for reasonable (small but non zero) parameters $\mathbf{t} = (t_1,\cdots,t_n)$, for all $k\in\mathbb{N}$ and for N large enough,

$$F^N_{V_{\mathbf{t}}} = \sum_{g=0}^k\frac{1}{N^{2g}}F_g(t_1,\cdots,t_n) + o(\frac{1}{N^{2k}}).$$

This means that one can invert the limits of t small and N large in the expansion.

Our aim is to look beyond this formal approach and try to justify this inversion of limits.

In the case of one matrix integrals, this problem is quite well understood at any level of the expansion and for any reasonable potentials V (see [ASM01] and [EM03] for instance).

Several matrix models are much harder. In the physics literature, the focus is mostly on a few specific integrals ; we refer the interested reader to the reviews [DFGZJ95, GDS91]. In the mathematical literature, fewer matrix integrals could be analyzed and only their first order asymptotics could be derived (see [MM91, Meh81] and [Gui04, GM95]). Even for these last integrals, the relation of their first order asymptotics with the related enumeration problem was not yet established rigorously. In combinatorics, another road was opened by [BMS02], following the ideas of [Tut63], to enumerate colored planar maps ; instead of studying matrix models, they used directly bijection between maps and well labeled trees.

To establish such a relation, we shall study an even more interesting quantity than the free energy, namely, the limiting empirical distribution of matrices ; for $A_1, \cdots, A_m \in \mathcal{H}_N(\mathbb{C})^m$, it is defined as the linear form on the set $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ of polynomials of m non-commutative variables so that

$$\hat{\mu}^N_{A_1,\cdots,A_m}(P) = \frac{1}{N}\mathrm{Tr}(P(A_1, \cdots, A_m)) \quad \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle.$$

Let μ^N_V be the Gibbs measure on $\mathcal{H}_N(\mathbb{C})^m$ given by

$$\mu^N_V(dA_1, \cdots, dA_m) = e^{-N^2 F^N_V} e^{-N\mathrm{Tr}(V(A_1,\cdots,A_m))} \prod_{i=1}^m d\mu_N(A_i)$$

with F^N_V as above. We shall prove that for reasonnable polynomials V, for all polynomials P, $\mu^N_V(\hat{\mu}^N_{A_1,\cdots,A_m}(P))$ converges and its limit is a generating function for maps. We prove this in two steps

• First, we study the solution τ in the algebraic dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ of the so-called Schwinger-Dyson equations **SD[V]** :

$$\tau_V(1) = 1, \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \forall i \in \{1, \cdots, m\}$$

$$\tau_V((X_i + D_i V)P) = \tau_V \otimes \tau_V(\partial_i P).$$

Here, ∂_i and D_i are respectively the non-commutative derivative and the cyclic derivative with respect to the i^{th} variable (see paragraph 2.2.2). We give sufficient conditions on V so that solutions to this equation exist and are unique.

Moreover, we relate solutions to **SD[V]** with generating functions of planar maps. Let us describe these planar maps. We associate to $(X_i)_{1\leqslant i\leqslant m}$ m half-edges of different colors, and to a monomial $q(\mathbf{X}) = X_{i_1}\cdots X_{i_p}$ a star with p colored half-edges by ordering clockwise the half-edges corresponding to $X_{i_1}, \cdots, X_{i_p}$. Such a star is said to be of type q. Note that it has a distinguished half-edge, the first one, X_{i_1}, and its half-edges are oriented by the above clockwise order (one should imagine the star to be fat, each half-edge made of two parallel segments which have opposite orientation, the whole orientation being given by the clockwise order). This defines a bijection between non-commutative monomials and stars. Alternatively, a star can be seen as an oriented circle with colored dots and one marked dot. A map is a connected graph with colored stars, each half-edge of each star being glued with exactly one half-edge of the same color and orientation and the edges obtained in this way do not cross each other (see a more precise description of the planar maps we enumerate in subsection 2.2.5). As edges around a vertex are cyclically ordered, one can find a canonical embedding of this graph on a surface. The map is said to be planar if we obtain a sphere by this construction.

We can now relate Schwinger-Dyson's equation and maps enumeration :

Let $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with t_i in $\mathbb{C}$ and q_i monomials. For all $R > 2$, there exists an open neighborhood $U \subset \mathbb{C}^n$ of the origin (a ball of positive radius) such that :

- For $\mathbf{t} \in U$, there exists a unique $\tau_{\mathbf{t}} \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ which is a solution to $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ and such that for all p, for all $i_1, \cdots, i_p$ in $\{1, \cdots, m\}$, $|\tau_{\mathbf{t}}(X_{i_1} \cdots X_{i_1})| \leqslant R^p$.
- For all P monomial in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\mathbf{t} \to \tau_{\mathbf{t}}(P)$ is analytic on U and for all $k_1, \cdots, k_n$ integers, $(-1)^{\Sigma k_i} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_{\mathbf{t}}(P)|_{\mathbf{t}=0}$ is the number of maps with k_i stars of type q_i and one of type P.

Hence, $(\tau_{V_{\mathbf{t}}})_{|t| \leqslant \varepsilon}$ are generating function for the enumeration of colored planar maps and Schwinger-Dyson's equations can be viewed as the generating differential equations to enumerate colored planar maps. This is due to the fact that the action of the derivatives ∂_i and D_i on monomials, under the above bijection between stars and monomials, produces natural operations on planar maps.

• Then, we shall see (see section 2.3) that, under some appropriate assumptions on V, $\hat{\mu}^N_{A_1, \cdots, A_m}$ converges almost surely under $\mu^N_V(dA_1, \cdots, dA_m)$ towards a solution τ_V to the Schwinger-Dyson equations $\mathbf{SD[}V\mathbf{]}$.

We show under rather general assumptions that the limit points of $\hat{\mu}^N_{A_1, \cdots, A_m}$ will solve a weak form of the Schwinger-Dyson equation (see section 2.3.1) which turns into its strong form if the limit points are compactly supported, i.e have all the moments of monomial functions of degree d bounded by R^d for some finite constant R. For small t_i's, this proves that $\hat{\mu}^N_{A_1, \cdots, A_m}$ converges almost surely towards the solution of **SD[V]** if we know that the limit points of $\hat{\mu}^N_{A_1, \cdots, A_m}$ satisfy such a bound. We then give sufficient conditions to obtain such an a priori estimate.

For instance, if we consider a convex potential V (see section 2.3.2) we have :

Let $a \in [0, 1]$, let U_a be the set of t_i's for which $V + \frac{a}{2} \sum_i X_i^2 = V_{\mathbf{t}} + \frac{a}{2} \sum_i X_i^2$ is convex, then there exists $\varepsilon > 0$ such that for $(t_i)_{1 \leqslant i \leqslant n} \in U_a \cap B(0, \varepsilon)$, $\hat{\mu}^N_{A_1, \cdots, A_m}$ converges in $L^1(\mu^N_{V_{\mathbf{t}}})$ and almost surely to the unique solution to $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ as described in theorem 2.1.1

Remark : For the one matrix model, [EM03] assumed that $V_{\mathbf{t}} = \sum_{i=1}^{2D} t_i X_i$ with $t_{2D} > \gamma \sum_{i=1}^{2D-1} |t_i|$ and $t_{2D} < T$ for some $\gamma, T > 0$. Note that if T, γ are large enough the hypothesises of Theorem 2.1.2 are satisfied.

For general potential V, we obtain a similar result provided we add a cut-off (see section 2.3.3).

Coming back to the free energy of matrix models, we conclude (see Theorem 2.3.3) that when the empirical distribution of matrices converges towards the solution to Schwinger-Dyson's equations, the free energy is also a generating function of the associated planar maps.

As a consequence, we can apply these results to the study of Voiculescu's microstates entropy (see section 2.4) and show that the microstates entropy can be estimated at the solutions to **SD[V]** when the t_i's are small enough.

Finally, we compare diverse approaches to the enumeration of planar maps by either using matrix models or combinatorics techniques.

The results of this paper are clearly known, at least at a subconscious level, by physicists, but we could not find any proper reference on the subject. However, we want to emphasize that the use of Schwinger-Dyson's equations is well spread in physics. This paper is rather elementary but provides a mathematical framework to the study of matrix models and related map enumeration. We hope it will demystify this interesting field of physics to mathematicians, or at least to probabilists. The generalization of the techniques developed in this paper to higher order expansions is the subject of a forthcoming article.

2.2 Schwinger-Dyson's equations and combinatorics

2.2.1 Tracial states

Let $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ be the set of polynomial functions in m self-adjoint non-commutative variables. We endow $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ with the involution given for all $z \in \mathbb{C}$, all $i_1, \cdots, i_p \in \{1, \cdots, m\}$ and all $p \in \mathbb{N}$, by

$$(zX_{i_1} \cdots X_{i_p})^* = \bar{z}X_{i_p} \cdots X_{i_1}.$$

We will say that P in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ is self-adjoint if $P^* = P$.

For any $R > 0$, completing $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ for the norm

$$||P||_R = \sup_{\mathcal{A}\ C^*-\text{algebra}} \sup_{\substack{a_1,..,a_m \in \mathcal{A},\\ a_i = a_i^*, \|a_i\|_{\mathcal{A}} \leqslant R}} \|P(a_1, \cdots, a_m)\|_{\mathcal{A}}$$

produces a C^*-algebra $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R = (\mathbb{C}\langle X_1, \cdots, X_m\rangle, ||.||_R, *)$.

We let $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ be the set of self-adjoint linear forms on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ (i.e linear forms such that $\tau(a^*) = \overline{\tau(a)}$), and denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*_R$ the subset of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ of continuous forms with respect to the norm $||.||_R$, i.e the topological dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle_R$.

We let $\mathcal{M}^m$ be the set of laws of m bounded self-adjoint non-commutative variables, that is the subset of elements τ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ such that

$$\tau(PP^*) \geqslant 0, \quad \tau(PQ) = \tau(QP) \quad \forall P, Q \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \quad \tau(1) = 1. \tag{2.1}$$

For any $R < \infty$, $\mathcal{M}^m_R = \mathbb{C}\langle X_1, \cdots, X_m\rangle^*_R \cap \mathcal{M}^m$ is a compact metric space for the weak*-topology by Banach-Alaoglu theorem. Elements of $\mathcal{M}^m = \cup_{R \geqslant 0}\mathcal{M}^m_R$ are said to be compactly supported, by analogy with the case $m = 1$ where they are indeed compactly supported probability measures. A family $(\tau_t)_{t \in I}$ of elements of $\mathcal{M}^m_R$ for some $R < \infty$ is said to be uniformly compactly supported.

To deal with variables which do not have all their moments, we will sometimes change the set of test functions and, following [CDG01], consider instead of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ the complex vector space $\mathcal{C}_{st}^m(\mathbb{C})$ generated by the Stieljes functionals

$$ST^m(\mathbb{C}) = \{ \prod_{1\leqslant i\leqslant p}^{\rightarrow} (z_i - \sum_{k=1}^m \alpha_i^k \mathbf{X}_k)^{-1} | \quad z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{R}, p \in \mathbb{N}\} \tag{2.2}$$

where $\prod^{\rightarrow}$ is the non-commutative product. We can give to $ST^m(\mathbb{C})$ an involution and a norm

$$||F||_\infty = \sup_{\mathcal{A}C^*-\text{algebra}} \sup_{a_i=a_i^*\in\mathcal{A}} ||F(a_1,\cdots,a_m)||_\infty$$

where the supremum is taken on unbounded operators affiliated with $\mathcal{A}$, which turns it into a C^*-algebra. We denote $\mathcal{C}_{st}^m(\mathbb{R}) = \{G = F + F^*, F \in \mathcal{C}_{st}^m(\mathbb{C})\}$. We let $\mathcal{M}_{ST}^m$ be the set of linear forms on $\mathcal{C}_{st}^m(\mathbb{C})$ which satisfy (2.1) (but with functions of $\mathcal{C}_{st}^m(\mathbb{C})$ instead of $\mathbb{C}\langle X_1,\cdots,X_m\rangle$). If one equips $\mathcal{M}_{ST}^m$ with its weak topology, then $\mathcal{M}_{ST}^m$ is a compact metric space (see [CDG01]).

2.2.2 Non-commutative derivatives

We define the non-commutative derivative with respect to the variable X_i,

$$\partial_i : \mathbb{C}\langle X_1,\cdots,X_m\rangle \to \mathbb{C}\langle X_1,\cdots,X_m\rangle \otimes \mathbb{C}\langle X_1,\cdots,X_m\rangle$$

given by the Leibnitz rule

$$\partial_i(PQ) = \partial_i P \times (1\otimes Q) + (P\otimes 1)\times \partial_i Q$$

for any $P, Q \in \mathbb{C}\langle X_1,\cdots,X_m\rangle$ and the condition

$$\partial_i X_j = \mathbb{1}_{i=j} 1\otimes 1.$$

In other words, if P is a non-commutative monomial

$$\partial_i P = \sum_{P=P_1X_iP_2} P_1\otimes P_2$$

where the sum runs over over all possible decomposition of P as $P_1X_iP_2$. This definition can be extended to $\mathcal{C}_{st}^m(\mathbb{C})$ by keeping the above Leibnitz rule (but with P, Q in $\mathcal{C}_{st}^m(\mathbb{C})$) and

$$\partial_i(z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1} = \alpha_i (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1} \otimes (z_i - \sum_{k=1}^m \alpha_k \mathbf{X}_k)^{-1}.$$

We also define the cyclic derivative D_i as follows. Let $m : \mathbb{C}\langle X_1,\cdots,X_m\rangle^{\otimes 2} \to \mathbb{C}\langle X_1,\cdots,X_m\rangle$ (resp. $\mathcal{C}_{st}^m(\mathbb{C})\otimes\mathcal{C}_{st}^m(\mathbb{C}) \to \mathcal{C}_{st}^m(\mathbb{C})$) be defined by $m(P\otimes Q) = QP$. Then, we set

$$D_i = m\circ\partial_i.$$

Note that, if P is a non-commutative monomial,

$$D_i P = \sum_{P=P_1X_iP_2} P_2 P_1.$$

2.2.3 Schwinger-Dyson's equation

Let V be in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ and consider the following equation on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$; we will say that $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies the Schwinger-Dyson equation with potential V, denoted in short **SD[V]**, if and only if for all $i \in \{1, \cdots, m\}$ and $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$,

$$\tau(1) = 1, \quad \tau \otimes \tau(\partial_i P) = \tau((D_i V + X_i)P) \qquad \mathbf{SD}[\mathbf{V}]$$

These equations are called Schwinger-Dyson's equations in physics, but in free probability, one would rather say that the conjugate variable (or alternatively the non-commutative Hilbert transform) $\partial_i^* 1$ under τ is equal to $X_i + D_i V$ for all $i \in \{1, \cdots, m\}$.

2.2.4 Uniqueness of the solutions to SD[V] for small parameters

Let

$$V(X_1, \cdots, X_m) = V_{\mathbf{t}}(X_1, \cdots, X_m) = \sum_{i=1}^{n} t_i q_i(X_1, \cdots, X_m)$$

where the q_i's are fixed monomials of m non-commutative indeterminates and $\mathbf{t} = (t_1, \cdots, t_n)$ are complex parameters.

In this paragraph, we shall consider solutions to **SD[$V_{\mathbf{t}}$]** which satisfy a compactness condition that we shall discuss in the following subsections. Let $R \in \mathbb{R}^+$ (We will always assume $R \geqslant 1$ without loss of generality).

(H(R))*An element $\tau \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ satisfies* **(H(R))** *if and only if for all $k \in \mathbb{N}$,*

$$\max_{1 \leqslant i_1, \cdots, i_k \leqslant m} |\tau(X_{i_1} \cdots X_{i_k})| \leqslant R^k.$$

In the sequel, we denote D the degree of V, that is the maximal degree of the $q_i's$; $q_i(X) = X_{j_1^i} \cdots X_{j_{d_i}^i}$ with, for $1 \leqslant i \leqslant n$, $\deg(q_i) =: d_i \leqslant D$ and equality holds for some i.

The main result of this paragraph is

If we fix $R > 0$ then, there exists $\eta > 0$ such that for all $\mathbf{t} \in \mathbb{C}^n$ such that $|\mathbf{t}| := max_i |t_i| < \eta$, **SD[$V_{\mathbf{t}}$]** has at most one solution which satsifies **(H(R))**.

Remark : Note here that it could be believed at first sight that the solutions to **SD[V]** are not unique since they depend on the trace of high moments $\tau(q_j P)$. However, our compactness assumption **(H(R))** gives uniqueness because it forces the solution to be in a small neighborhood of the law $\tau_0 = \sigma^m$ of m free semi-circular variables, so that perturbation analysis applies. We shall see in Theorem 2.7 that this solution is actually the one which is related with the enumeration of maps.

Proof.

Let us assume we have two solutions τ and τ'. Then, by the equation **SD[V]**, for any monomial function P of degree $l-1$, for $i \in \{1, \cdots, m\}$,

$$(\tau - \tau')(X_i P) = ((\tau - \tau') \otimes \tau)(\partial_i P) + (\tau' \otimes (\tau - \tau'))(\partial_i P) - (\tau - \tau')(D_i V P)$$

Hence, if we let for $l \in \mathbb{N}$

$$\Delta_l(\tau,\tau') = \sup_{\text{monomial } Q \text{ of degree l}} |\tau(Q) - \tau'(Q)|$$

we get, since if P is of degree $l-1$,

$$\partial_i P = \sum_{k=0}^{l-2} p_k^1 \otimes p_{l-2-k}^2$$

where p_k^i, $i = 1,2$ are monomial of degree k or the null monomial, and $D_i V$ is a finite sum of monomials of degree smaller than $D-1$,

$$\begin{aligned}\Delta_l(\tau,\tau') &= \max_{P \text{ of degree } l-1} \max_{1\leqslant i\leqslant m} \{|\tau(X_iP) - \tau'(X_iP)|\} \\ &\leqslant 2\sum_{k=0}^{l-2} \Delta_k(\tau,\tau')R^{l-2-k} + C|\mathbf{t}| \sum_{p=0}^{D-1} \Delta_{l+p-1}(\tau,\tau')\end{aligned}$$

with a finite constant C (which depends on n only). For $\gamma > 0$, we set

$$d_\gamma(\tau,\tau') = \sum_{l\geqslant 0} \gamma^l \Delta_l(\tau,\tau').$$

Note that under **(H(R))**, this sum is finite for $\gamma < (R)^{-1}$. Summing the two sides of the above inequality times γ^l we arrive at

$$d_\gamma(\tau,\tau') \leqslant 2\gamma^2(1-\gamma R)^{-1} d_\gamma(\tau,\tau') + C|\mathbf{t}| \sum_{p=0}^{D-1} \gamma^{-p+1} d_\gamma(\tau,\tau').$$

We finally conclude that if $|\mathbf{t}|$ is small enough so that we can choose $\gamma \in (0, R^{-1})$ so that

$$2\gamma^2(1-\gamma R)^{-1} + C|\mathbf{t}| \sum_{p=0}^{D-1} \gamma^{-p+1} < 1$$

then $d_\gamma(\tau,\tau') = 0$ and so $\tau = \tau'$ and we have at most one solution. Taking $\gamma = (2R)^{-1}$ shows that this is possible provided

$$\frac{1}{R^2} + C|\mathbf{t}| \sum_{p=0}^{D-1} (2R)^{p-1} < 1$$

so that when R goes to $+\infty$ we need to take $|\mathbf{t}|$ of order at most R^{2-D}.

□

2.2.5 Combinatorics

In this paragraph we describe the combinatorial objects we are considering. A star is the neightbourhood of a vertex in a planar graphs i.e. it is a vertex with some half-edges coming out of it. Theses half-edges are ordered in the clockwise order starting from a distinguished one. We associate to each $i \in \{1, \cdots, m\}$ a different color. Then, we define a bijection between oriented edges-colored stars with a distinguished half-edge and non-commutative monomials as follows. For any $i \in \{1, \cdots, m\}$, we associate to X_i a half-edge of color i. We shall say that a star is of type $q(X_1, \cdots, X_m) = X_{i_1} \cdots X_{i_l}$ if it is a star with l half-edges which we color clockwise; the first half-edge will be of color i_1, the second of color i_2 ... etc ... until the l^{th} half-edge is colored with color i_l. Note that this star possesses a distinguished half-edge, the one corresponding to X_{i_1}, and an orientation, corresponding to the clockwise order (see figure 2.1 for an example). By convention, the star of type $q = 1$ is simply a point.

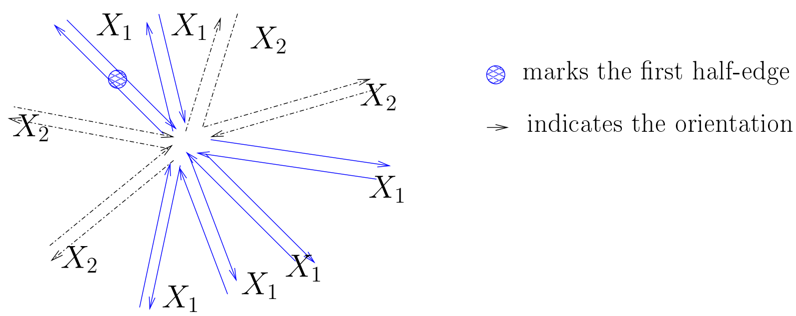

FIG. 2.1 – The star of type $q(X) = X_1^2 X_2^2 X_1^4 X_2^2$

A map is a connected graph whose vertices are colored stars, each half-edge is glued with exactly one half-edge of the same color and orientation. Because the edges coming out of a star are cyclically ordered, we can define the faces of this graph and thus find an embedding of this graph on an orientated surface in such a way that edges do not cross each other (see [Zvo97]). A map is planar if this surface has genus zero, i.e. is the sphere. Planar maps can be thought as graphs embedded on the sphere up to homeomorphism. Maps are only considered up to an homeomorphism of the sphere. Now we will be interested in enumerating maps with a fixed set of stars, we define for q_i the family of monomials which appear in V and $\mathbf{k} = (k_1, \cdots, k_n)$ a family of integers :

$$\mathcal{M}_{\mathbf{k}} = \sharp\{\text{planar maps build with } k_i \text{ stars of type } q_i\}.$$

and

$$\mathcal{M}_{\mathbf{k}}(P) = \sharp\{\text{planar maps build with } k_i \text{ stars of type } q_i \text{ and one of type } P\}.$$

In that set, each star is labeled and has a marked half-edge (which corresponds to its first variable) so that for example if $V = X^4$, $\mathcal{M}_2 = 36$.

Due to the fact that everything is labeled, we enumerate lots of very similar objects. A way to avoid this problem is to look at the maps as they are enumerated by combinatorialists (see [BMS02]). The idea is to forget every label and to add a root : a distinguished star with a marked half-edge. We will say that a map is rooted at a vertex of type P if its root is of type P with the marked half-edge the first one in the above construction of a star from a monomial. We can define for P a monomial, $\mathbf{k} = (k_1, \cdots, k_n)$,

$$\mathcal{D}_{\mathbf{k}}(P) = \sharp\{\text{maps with } k_i \text{ stars of type } q_i \text{ rooted at a vertex of type } P\}$$

To go from these rooted maps to the previous one we only have to label each star and be careful about the symmetry of the stars in order to specify a half-edge by star. More precisely, let us define the degree of symmetry $s(q)$ of a monomial q as follows. Let $\omega : \mathbb{C}\langle X_1, \cdots, X_m\rangle \rightarrow \mathbb{C}\langle X_1, \cdots, X_m\rangle$ be the linear function so that for all $i_k \in \{1, \cdots, m\}$, $1 \leqslant k \leqslant p$

$$\omega(X_{i_1} X_{i_2} \cdots X_{i_p}) = X_{i_2} \cdots X_{i_p} X_{i_1}$$

and, with $\omega^p = \omega \circ \omega^{p-1}$, define

$$s(q) = \sharp\{0 \leqslant p \leqslant \deg\,(q) - 1 | \omega^p(q) = q\}.$$

We easily see that for all monomial P, distinct monomials q_i (but one of them may be equal to P), and integers k_i :

$$\mathcal{D}_{k_1,\dots,k_n}(P) = \frac{\mathcal{M}_{k_1,\dots,k_n}(P)}{\Pi_{i=1}^n k_i! s(q_i)^{k_i}} \tag{2.3}$$

2.2.6 Graphical interpretation of Schwinger-Dyson's equations

We shall now make an assumption on the solutions of Schwinger-Dyson's equation $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ when the parameters belong to an open convex neighborhood of the origin, namely

(H) *There exists a convex neighborhood $U \in \mathbb{C}^n$, a finite real number R and a family $\{\tau_{\mathbf{t}}, t \in U\}$ of linear forms on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ so that for all $\mathbf{t}$ in U, $\tau_{\mathbf{t}}$ is a solution of* $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ *which satisfies* **H(R)**.

Note that up to take a smaller set U, we can assume that the conclusions of Theorem 2.2.1 are valid, i.e for all $\mathbf{t} \in U$ there exists an unique solution to $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ which satisfies **H(R)**.

The central result of this article is then

Assume that **(H)** is satisfied. Then

1. For any $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\mathbf{t} \in U \rightarrow \tau_{\mathbf{t}}(P)$ is $\mathcal{C}^\infty$ at the origin in the sense that for all $\overline{k} = (k_1, k_2, \cdots, k_n) \in \mathbb{N}^n$ there exists $\varepsilon_{\overline{k}} > 0$ so that $\partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_{\mathbf{t}}(P)$ exists on $U_\varepsilon = U \cap B(0, \varepsilon)$ with $B(0, \varepsilon) = \{\mathbf{t} \in \mathbb{C}^n : |\mathbf{t}| \leqslant \varepsilon\}$.
2. We let $\tau^{\overline{k}}(P) = (-1)^{k_1+\dots+k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_n}^{k_n} \tau_{\mathbf{t}}(P)|_{\mathbf{t}=0}$. Then, we have for all polynomial P and all $i \in \{1, \cdots, m\}$,

$$\tau^{\overline{k}}(X_i P) = \sum_{\substack{0 \leqslant p_j \leqslant k_j \\ 1 \leqslant j \leqslant n}} \prod_{j=1}^n \binom{k_j}{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k}-\overline{p}}(\partial_i P) + \sum_{1 \leqslant j \leqslant n} k_j \tau^{\overline{k}-1_j}((D_i q_j)P) \tag{2.4}$$

where $1_j(i) = \mathbb{1}_{i=j}$ and $\tau^{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$.

3. Moreover the $\mathcal{M}_{\mathbf{k}}(P)$'s satisfy the same family of equations (2.4) than the $\tau_{\mathbf{k}}(P)$'s. Hence, for any monomial $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, any $k_1, \cdots, k_n \in \mathbb{N}$,

$$\tau^{\overline{k}}(P) = \mathcal{M}_{\overline{k}}(P).$$

Proof.

• The smoothness of $\mathbf{t} \to \tau_{\mathbf{t}}$ comes as in the proof of Theorem 2.2.1 from Schwinger-Dyson's equations and induction on the degree of the test polynomial function. Denote $V = V_{\mathbf{t}}$, $\tau = \tau_{\mathbf{t}}$ and take $\mathbf{t} = (t_1, \cdots, t_n), \mathbf{t}' = (t'_1, t'_2, .., t'_n) \in U$. By **SD[V]**,

$$\begin{aligned}(\tau_{\mathbf{t}} - \tau_{\mathbf{t}'})[(X_i + D_i V_{\mathbf{t}})P] = (\tau_{\mathbf{t}} - \tau_{\mathbf{t}'}) \otimes \tau_{\mathbf{t}}(\partial_i P) + \tau_{\mathbf{t}'} \otimes (\tau_{\mathbf{t}} - \tau_{\mathbf{t}'})(\partial_i P) \\ + \tau_{\mathbf{t}'}[(D_i V_{\mathbf{t}'} - D_i V_{\mathbf{t}})P]\end{aligned}$$

By our finite moment assumption, we deduce that if P is a monomial function of degree $l-1$, for any $i \in \{1, \cdots, m\}$,

$$|\tau_{\mathbf{t}}[(X_i + D_i V_{\mathbf{t}})P] - \tau_{\mathbf{t}'}[(X_i + D_i V_{\mathbf{t}})P]|$$

$$\leqslant 2\sum_{k=0}^{l-2} \max_{Q \text{ monomial of degree } \leqslant k} |\tau_{\mathbf{t}}[Q] - \tau_{\mathbf{t}'}[Q]| R^{l-2-k} + \sum_{1\leqslant i\leqslant n} |t_i - t'_i| R^{l+D-1}.$$

Thus we deduce that for any $p \in \mathbb{N}$,

$$\begin{aligned}\Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) &= \max_i \max_{P \text{ monomial of degree } p-1} |\tau_{\mathbf{t}}(X_i P) - \tau_{\mathbf{t}'}(X_i P)| \\ &\leqslant 2\sum_{k=0}^{l-2} \Delta_k(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) R^{l-2-k} + \sum_{i=1}^{n} |t_i| \Delta_{l+d_i-1}(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) + \sum_{1\leqslant i\leqslant n} |t_i - t'_i| R^{l+D-1}.\end{aligned}$$

Now, let $\gamma \in (0, R^{-1})$ and let's sum both sides of this inequality multiplied by γ^l to obtain, with $d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) = \sum_{l\geqslant 0} \gamma^l \Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'})$,

$$\begin{aligned}d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) &\leqslant 2(1-\gamma R)^{-1}\gamma^2 d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \\ &+ \sum_{i=1}^{n} |t_i| \gamma^{-d_i+1} d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) + (1-\gamma R)^{-1} \sum_{1\leqslant i\leqslant n} |t_i - t'_i| R^{D-1}.\end{aligned}$$

Since by definition $\Delta_l(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant 2R^l$, $d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'})$ is finite for $\gamma R < 1$ we arrive at

$$(1 - 2\gamma^2(1-R\gamma)^{-1} - \sum_{1\leqslant i\leqslant n} |t_i| \gamma^{-D+2}) d_\gamma(\tau_{\mathbf{t}}, \tau_{\mathbf{t}'}) \leqslant (1-R\gamma)^{-1} \sum_{1\leqslant i\leqslant n} |t_i - t'_i| R^{D-1}.$$

Now, for ε small enough, we can find $\gamma = \gamma(|t|) > 0$ so that

$$1 - 2\gamma^2(1-R\gamma)^{-1} - \sum_{1\leqslant i\leqslant n} |t_i| \gamma^{-D+2} > 0$$

and so

$$\sum_{l\geqslant 0}\gamma^l\Delta_l(\tau_{\mathbf{t}},\tau_{\mathbf{t}'})\leqslant C(\mathbf{t})\sum_{1\leqslant i\leqslant n}|t_i-t'_i|$$

which implies that for all $l\in\mathbb{N}$

$$\Delta_l(\tau_{\mathbf{t}},\tau_{\mathbf{t}'})\leqslant C(\mathbf{t})\gamma^{-l}\sum_{1\leqslant i\leqslant n}|t_i-t'_i|$$

so that for any monomial function P, $\mathbf{t}\to\tau_{\mathbf{t}}(P)$ is Lipschitz in $U_\varepsilon:=U\cap B(0,\varepsilon)$ for ε small enough. Moreover, we have proved that there exists $\eta_0(\varepsilon)=\gamma^{-1}<\infty$, so that

$$\Delta_l(\tau_{\mathbf{t}},\tau_{\mathbf{t}'})\leqslant C_0(\varepsilon)\eta_0(\varepsilon)^l|\mathbf{t}-\mathbf{t}'| \text{ with } |\mathbf{t}-\mathbf{t}'|=\max_{1\leqslant i\leqslant n}|t_i-t'_i|. \tag{2.5}$$

Consequently, $\tau_{\mathbf{t}}$ is almost surely differentiable in U_ε and the derivative satisfies

$$\partial_{t_k}\tau_{\mathbf{t}}[(X_i+D_iV_{\mathbf{t}})P]+\tau_{\mathbf{t}}[D_iq_kP]=\partial_{t_k}\tau_{\mathbf{t}}\otimes\tau_{\mathbf{t}}(\partial_iP)+\tau_{\mathbf{t}}\otimes\partial_{t_k}\tau_{\mathbf{t}}(\partial_iP) \tag{2.6}$$

for almost all $\mathbf{t}\in U_\varepsilon$. Since $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ is countable, these equalities hold simultaneously for all $P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle$ almost surely, let U'_ε be this subset of U_ε of full probability.

Inequality (2.5) implies that

$$\max_{1\leqslant k\leqslant m}\max_{P \text{ monomial of degree } l}|\partial_{t_k}\tau_{\mathbf{t}}(P)|\leqslant C_0(\varepsilon)\eta_0(\varepsilon)^l$$

for all $\mathbf{t}\in U'_\varepsilon$. This bound in turn shows that we can redo the argument as above to see that for $|\mathbf{t}|$ small enough, $\mathbf{t}\to\partial_{t_k}\tau_{\mathbf{t}}(P)$ is Lipschitz. Indeed, if we set

$$\Delta_1(l)=\Delta_l^1(\partial\tau_{\mathbf{t}},\partial\tau_{\mathbf{t}})=\max_{1\leqslant k\leqslant m}\max_{P \text{ monomial of degree } l}|\partial_{t_k}\tau_{\mathbf{t}}(P)-\partial_{t_k}\tau_{\mathbf{t}'}(P)|$$

we get, for $\mathbf{t}',\mathbf{t}\in U'_\varepsilon$,

$$\Delta_1(l)\leqslant 2\sum_{k=0}^{l-2}\Delta_1(k)R^{l-2-k}+C_0(\varepsilon)|\mathbf{t}-\mathbf{t}'|l\eta_0(\varepsilon)^l+\sum_{i=1}^n|t_i|\Delta_1(l+d_i-1)$$

so that we get that by summation, for $\gamma<\min(R^{-1},\eta_0(\varepsilon)^{-1})$,

$$(1-2(1-R\gamma)^{-1}\gamma^2-\sum_{i=1}^n|t_i|\gamma^{-d_i+1})\sum_{l\geqslant 0}\Delta_1(l)\gamma^l\leqslant\gamma^2C_0(\varepsilon)(1-\gamma\eta_0(\varepsilon))^{-2}|\mathbf{t}-\mathbf{t}'|.$$

Hence, again, we can choose $\eta_1(\varepsilon)<\infty$ big enough so that there exists $C_1(\varepsilon)<\infty$ so that if ε is small enough

$$\Delta_1(l)\leqslant C_1(\varepsilon)\eta_1(\varepsilon)^l|\mathbf{t}-\mathbf{t}'|.$$

In particular, this shows that we can extend $\mathbf{t}\in U'_\varepsilon\to\partial_{t_k}\tau_{\mathbf{t}}(P)$ for all monomial functions P continuously in U_ε and so the equality (2.6) holds everywhere. Now, we can proceed by induction to see that $\mathbf{t} \to \tau_{\mathbf{t}}(P)$ is $\mathcal{C}^\infty$ differentiable in a neighborhood of the origin. More precisely, for any $\overline{k} = (k_1, \cdots, k_n)$ there exists $\varepsilon = \varepsilon_{\overline{k}} > 0$ so that on U_ε,

$$\tau_{\mathbf{t}}^{\overline{k}}(P) = (-1)^{k_1 + \cdots + k_n} \partial_{t_1}^{k_1} \cdots \partial_{t_m}^{k_n} \tau_{\mathbf{t}}(P)$$

exists and furthermore satisfies the equation

$$\tau_{\mathbf{t}}^{\overline{k}}((X_i + D_i V_{\mathbf{t}})P) = \sum_{\substack{0 \leqslant p_i \leqslant k_i \\ 1 \leqslant i \leqslant n}} \prod_{i=1}^n \binom{k_i}{p_i} \tau_{\mathbf{t}}^{\overline{p}} \otimes \tau_{\mathbf{t}}^{\overline{k} - \overline{p}}(\partial_i P) + \sum_{1 \leqslant j \leqslant m} k_j \tau_{\mathbf{t}}^{\overline{k} - \mathbb{1}_j}((D_i q_j)P)$$

Applying this result at the origin, we obtain the second point.

• We now show the combinatorial interpretation of (2.4). It is based on the observation that the $\{\tau^{\overline{k}}(P), \overline{k} \in \mathbb{N}^n\}$ and the $\{\mathcal{M}_{\overline{k}}(P), \overline{k} \in \mathbb{N}^n\}$ satisfy the same inductive relations (2.4).

Let us first interpret graphically $\tau^0 = \tau_0$. τ_0 satisfies by definition **SD[0]** which is well known to have a unique solution given by the law of m free semi-circular variables (see [Voi91]). Then, $\tau_0(X_{i_1} \cdots X_{i_k})$ can be computed for instance using cumulants techniques as developed by [Spe97] ; it counts the number of planar maps which can be constructed from the star associated to $X_{i_1} \cdots X_{i_k}$ by gluing together the half-edges of the star colorwise. We prove again this result by induction over the degree of the monomial function. We put $\mathcal{M}_{\mathbf{k}}(1) = \mathbb{1}_{\mathbf{k}=0}$ by convention and then we start the induction. Let $i \in \{1, \cdots, m\}$ and $P = X_i Q$. To compute $\mathcal{M}(X_i Q)$, we break the edge between the distinguished half-edge X_i and the other half-edge of Q with which it was glued, then erasing these two half-edges. Since the maps are planar, this decomposes the planar map into two planar maps (see figure 2.2) corresponding respectively to the stars Q_1, Q_2 for any possible choices of Q_1, Q_2 so that $Q = Q_1 X_i Q_2$. Hence

$$\mathcal{M}(X_i Q) = \sum_{Q = Q_1 X_i Q_2} \mathcal{M}(Q_1)\mathcal{M}(Q_2).$$

Thus, if $\mathcal{M}(R) = \tau_0(R)$ for all monomial of degree strictly smaller than P,

$$\mathcal{M}(X_i Q) = \sum_{Q = Q_1 X_i Q_2} \tau_0(Q_1)\tau_0(Q_2) = \tau_0 \otimes \tau_0(\partial_i Q)$$

which completes the argument since the right hand side is exactly $\tau_0(X_i Q)$.

We now consider the general case ; let us assume that for $|\overline{k}| \leqslant M$, the graphical interpretation has been obtained for all monomial and that for $|\overline{k}| = M + 1$, it has been proved for monomial of degree smaller or equal to L. By the preceding, we can take $M \geqslant 1$ and $L \geqslant 1$ since for all $\overline{k} \neq 0$, $\tau^{\overline{k}}(1) = 0$. Again, we shall show that $\mathcal{M}(P, (q_1, k_1), \cdots, (q_n, k_n))$ satisfies the same induction relation than $\tau^{\overline{k}}(P)$.

Let us consider a star of type $X_i P$ (rooted at the half-edge X_i, with its inner orientation) with P a monomial of degree less than L and $|\overline{k}| = \sum k_i = M + 1$. Now, in order to compute $\mathcal{M}(X_i P, (q_1, k_1), \cdots, (q_n, k_n))$, we break the edge between the distinguished half-edge X_i (which has color i) and the other half-edge with which it was glued.

The first possibility is that it was glued with an edge of the star P. Then, since the maps are planar, this decomposes the map in two planar maps. If this half-edge was given by the

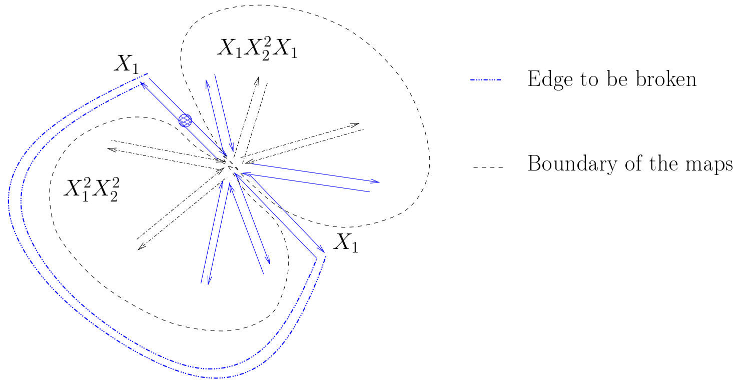

FIG. 2.2 – Decomposition $P(X) = X_1X_2^2X_1^4X_2^2$ into $X_1X_2^2X_1 \otimes X_1^2X_2^2$

X_i so that $P = P_1X_iP_2$, one of this planar map contain the star of type P_1 and the other the star of type P_2, which have also a distinguished half-edge and are oriented. If one of this planar map is glued with k_j stars of type q_j , $0 \leqslant k_j \leqslant n$, the other map is glued with the remaining stars, that is $k_j - p_j$ stars of type q_i. There are $\prod_{j=1}^n \binom{k_j}{p_j}$ ways to choose p_j among k_j stars of type q_j for $1 \leqslant j \leqslant n$ (recall here that stars are labeled). Since we do that for all (P_1, P_2) so that P have the above decomposition, we obtain the planar maps corresponding actually to the stars associated with the monomials of $\partial_i P$. Note that the case where one of the monomial in $\partial_i P$ is the monomial 1 shows up when $P = X_iQ$ or QX_i for some monomial Q and the weight corresponds then to the case where we glue the first half-edge X_i in X_iP with its left or right neighbor. In this case, none of these two half-edges can be glued with another star, and there is only one possibility to glue these two half-edges otherwise, which corresponds to the weight $\tau^{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$.

Hence, the number of planar maps corresponding to this configuration is given by

$$\sum_{\substack{0\leqslant p_j \leqslant k_j \\ 1\leqslant j\leqslant n}} \sum_{P=P_1X_iP_2} \prod_{1\leqslant j\leqslant n} \binom{k_j}{p_j} \mathcal{M}_{p_1,\cdots,p_n}(P_1)\mathcal{M}_{k_1-p_1,\cdots,k_n-p_n}(P_2)$$

$$= \sum_{\substack{0\leqslant p_j \leqslant k_j \\ 1\leqslant j\leqslant n}} \prod_{1\leqslant j\leqslant n} \binom{k_j}{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k}-\overline{p}}(\partial_i P)$$

where we finally used our induction hypothesis.

The other possibility is that this edge is glued with a star of type q_j for some $j \in \{1, \cdots, n\}$. In this case, erasing the edge means that we destroy a star of type q_j and replace the star of type X_iP and the star of type q_j by a single bigger star. If $q_j = Q_1X_iQ_2$, we replace the two stars of type X_iP and q_j by a single one of type Q_2Q_1P (see figure 2.3). Since we do that with all the possible edges of color i in q_j, we find that we can glue all monomials appearing in D_iq_j, and so the corresponding weight is given by $\tau^{\overline{k}-\mathbb{1}_j}(D_iq_jP)$ times k_j, the number of ways to choose one star among k_j of type q_j.

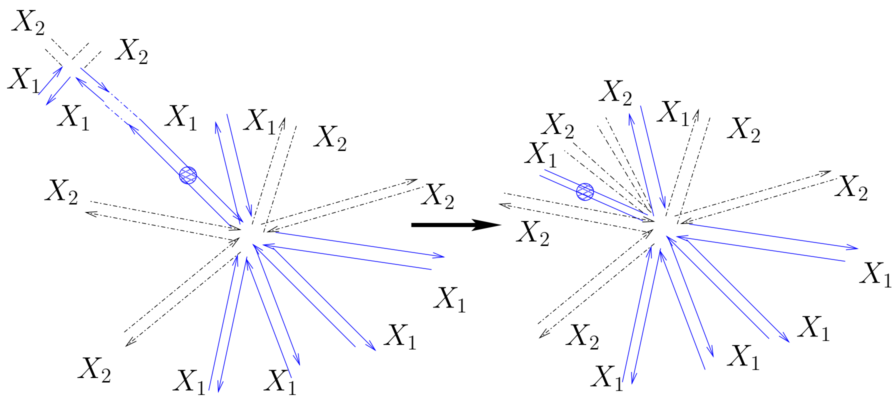

FIG. 2.3 – The merging of $q(X) = X_1^2 X_2^2 = X_1 X_1 X_2^2$ and $X_1 P$ into $X_1 X_2^2 P$

Hence, by induction, we proved that the number of planar maps with k_j stars of type q_j and one of type $X_i P$ is given by

$$\begin{aligned}\mathcal{M}_{\overline{k}}(X_iP) &= \sum_{\substack{0\leqslant p_j \leqslant k_j\\ 1\leqslant j\leqslant m}} \prod_{j=1}^{n} \binom{k_j}{p_j} \tau^{\overline{p}} \otimes \tau^{\overline{k}-\overline{p}}(\partial_i P) + \sum_{1\leqslant j \leqslant m} k_j \tau^{\overline{k}-\mathbb{1}_j}((D_i q_j)P) \\ &= \tau^{\overline{k}}(X_iP)\end{aligned}$$

for all $i \in \{1, \cdots, m\}$. This shows that the graphical interpretation holds for all L and $|\overline{k}| \leqslant M+1$. We can start the induction since we know that $\tau^{\overline{k}}(1) = \mathbb{1}_{\overline{k}=0}$. This completes the proof.

□

Remark : Note that this graphical approach can be generalized to matrix models with more complex potentials involving tensor products. For example, one can consider a potential V which is a sum of monomials and of tensor products of monomials :

$$V_{\mathbf{t}} = \sum_i t_i q_i^1 \otimes \cdots \otimes q_i^d$$

and the associated measure with density $Z_N^{-1} e^{-N^{2-d}(\mathrm{Tr})^{\otimes d} V_{\mathbf{t}}}$ with respect to $\mu_N^{\otimes m}$. Then one can write the generalized Schwinger Dyson's equation :

$$\tau \otimes \tau(\partial_i P) = \tau(X_i P) + \sum_{k,j} t_k \tau^{\otimes d_k}(q_k^1 \otimes \cdots \otimes D_i q_k^j P \otimes \cdots \otimes q_k^d)$$

The previous results remain valid up to a graphical interpretation of the new term. For example $q^1 \otimes \cdots \otimes q^k$ will be a bunch of k loops, the first one containing the half-edges of the star of q^1, in the clockwise order, the first of which is the marked one, the second one the half-edges of q^2 ... The additional constraint being that vertices which will be placed in a loop can not be linked to any vertices in an other loop.

2.2.7 Existence of an analytic solution to Schwinger-Dyson's equation

The aim of this section is to prove that for all monomials $(q_j)_{1\leqslant j\leqslant n}$, there exists a convex neighborhood of the origin (actually an open ball) and a finite constant R so that hypothesis **(H)** of section 2.2.6 is satisfied. Moreover, we show that it depends analytically on $\mathbf{t}$ in a neighborhood of the origin. Let $V_{\mathbf{t}}$ be as before. There exists an open neighborhood $U\subset\mathbb{C}^n$ of the origin (a ball of positive radius) such that for $\mathbf{t}\in U$, there exists $\tau_{\mathbf{t}}\in\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ satisfying $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ such that :

- $\mathbf{t}\to\tau_{\mathbf{t}}$ is analytic on U, i.e. there exists $(\tau^{\overline{k}},\overline{k}\in\mathbb{N}^n)$ in $\mathbb{C}\langle X_1,\cdots,X_m\rangle^*$ such that for all P in $\mathbb{C}\langle X_1,\cdots,X_m\rangle$, t in U,

$$\tau_{\mathbf{t}}(P)=\sum_{\overline{k}\in\mathbb{N}^n}\prod_{1\leqslant i\leqslant n}\frac{(-t_i)^{k_i}}{k_i!}\tau^{\overline{k}}(P) \tag{2.7}$$

  and the series converges absolutely on U.
- $\tau^{\overline{k}}(P)=(-1)^{\Sigma k_i}\partial_{t_1}^{k_1}\cdots\partial_{t_n}^{k_n}\tau_{\mathbf{t}}(P)|_{\mathbf{t}=0}=\mathcal{M}_{\overline{k}}(P)$
- There exists $R<\infty$ so that for all $\mathbf{t}\in U$, all $i_i\cdots i_l\in\{1,\cdots,m\}^l$, all $l\in\mathbb{N}$,

$$|\tau_{\mathbf{t}}(X_{i_1}\cdots X_{i_l})|\leqslant R^l.$$

Remark : Using (2.3), one can obtain inside the domain of convergence, for all monomial P :

$$\tau_{\mathbf{t}}(P)=\sum_{\overline{k}\in\mathbb{N}^n}\prod_{1\leqslant i\leqslant n}(-s(q_i)t_i)^{k_i}\mathcal{D}_{\mathbf{k}}(P).$$

Proof.

Note that if we assume that $\tau_{\mathbf{t}}$ can be written as a series like in (2.7) then according to Theorem 2.2.2 the $\tau^{\overline{k}}$ are defined uniquely by equation (2.4) so that $\tau^{\overline{k}}(P)=\mathcal{M}_{\overline{k}}(P)$ for all $\overline{k}$, P and reciprocally these equalities imply that $\tau_{\mathbf{t}}$ satisfy $\mathbf{SD[}V_{\mathbf{t}}\mathbf{]}$ inside the domain of convergence. The only point is thus to control the growth of the $\tau^{\overline{k}}(P)$'s to show that

$$\sum_{\overline{k}\in\mathbb{N}^n}\prod_{1\leqslant i\leqslant n}\frac{(-t_i)^{k_i}}{k_i!}\tau^{\overline{k}}(P)$$

has a strictly positive radius of convergence. We write, if $\overline{k}!=\prod k_i!$,

$$\frac{\tau^{\overline{k}}(X_iP)}{\overline{k}!}=\sum_{\substack{0\leqslant p_j\leqslant k_j\\1\leqslant j\leqslant n}}\sum_{P=P_1X_iP_2}\frac{\tau^{\overline{p}}(P_1)}{\overline{p}!}\frac{\tau^{\overline{k}-\overline{p}}(P_2)}{(\overline{k-p})!}+\sum_{\substack{1\leqslant j\leqslant m\\k_j\neq 0}}\frac{\tau^{\overline{k}-\mathbb{1}_j}((D_iq_j)P)}{(\overline{k}-\mathbb{1}_j)!}$$

where the second sum runs over all monomials P_1,P_2 so that P decomposes into $P_1X_iP_2$.

Our induction hypothesis will be that for $\overline{k}$ so that $\sum_i k_i\leqslant M-1$ and all monomial P, as well as for $\sum k_i=M$ and monomials P of degree smaller than L,

$$\left|\frac{\tau^{\overline{k}}(P)}{\overline{k}!}\right|\leqslant A^{\sum k_i}B^{degP}\prod_i C_{k_i}C_{degP}$$

where the C_k are the Catalan's numbers which satisfy

$$C_{k+1} = \sum_{p=0}^{k} C_p C_{k-p}, \quad C_0 = 1, \quad \frac{C_{k+l}}{C_l} \leqslant 4^k \quad \forall l, k \in \mathbb{N}. \tag{2.8}$$

Here, $\deg P$ denotes the degree of the monomial P and we can assume $B \geqslant 2$ without loss of generality. Our induction is trivially true for $\overline{k} = 0$ and all L since $\mathcal{M}_0 = \tau^{\overline{0}} = \sigma^m$ is the law of m free semi-circular variables which are uniformly bounded by 2 so that

$$|\tau^{\overline{0}}(P)| \leqslant 2^{degP}$$

Moreover, it is satisfied for all $\overline{k}$ and $L = 0$ since then $\tau^{\overline{k}}(1) = 1\!\!1_{\overline{k}=\underline{0}}$. Let us assume that it is true for all $\overline{k}$ such that $\sum k_i \leqslant M - 1$ and all monomials, and for $\overline{k}$ such that $\sum k_i = M$ and monomials P of degree less than L for some $L \geqslant 0$. Then

$$\begin{aligned}
\left|\frac{\tau^{\overline{k}}(X_i P)}{k!}\right| &\leqslant \sum_{\substack{0 \leqslant p_j \leqslant k_j \\ 1 \leqslant j \leqslant n}} \sum_{P = P_1 X_i P_2} A^{\sum k_i} B^{degP-1} \prod_{i=1}^{n} C_{p_j} C_{k_j - p_j} C_{degP_1} C_{degP_2} \\
&+ 2 \sum_{1 \leqslant l \leqslant n} A^{\sum k_j - 1} \prod_j C_{k_j} B^{degP + deqq_l - 1} C_{degP + deqq_l - 1} \\
&\leqslant A^{\sum k_i} B^{degP+1} \prod_i C_{k_i} C_{degP+1} \left(\frac{4^n}{B^2} + 2 \frac{\sum_{1 \leqslant j \leqslant n} B^{degq_j - 2} 4^{degq_j - 2}}{A} \right)
\end{aligned}$$

where we used (2.8) in the last line. It is now sufficient to choose A and B such that

$$\frac{4^n}{B^2} + 2 \frac{\sum_{1 \leqslant j \leqslant n} B^{degq_j - 2} 4^{degq_j - 2}}{A} \leqslant 1$$

(for instance $B = 2^{n+1}$ and $A = 4nB^{D-2}4^{D-2}$) to verify the induction hypothesis works for polynomials of all degrees (all L's).

Then

$$\tau_{\mathbf{t}}(P) = \sum_{k \in \mathbb{N}^n} \prod \frac{(-t_i)^{k_i}}{k_i!} \tau^{\overline{k}}(P)$$

is well defined for $|t| < (4A)^{-1}$. Moreover, for all monomial P,

$$|\tau_{\mathbf{t}}(P)| \leqslant \sum_{k \in \mathbb{N}^n} \prod_{i=1}^{n} (4 t_i A)^{k_i} (4B)^{degP} \leqslant \prod_{i=1}^{n} (1 - 4At_i)^{-1} (4B)^{degP}.$$

so that for small t, $\tau_{\mathbf{t}}$ has a uniformly bounded support.

□

Hence, we see that the enumeration of planar maps could be reduced to the study of Schwinger-Dyson's equations **SD[V]**. For instance, the asymptotics of such enumeration can be obtained by studying the optimal domain in which the solutions are analytic. Matrix models can be useful to study also the solution, e.g. we shall deduce from this approach that the solutions to **SD[V]** are tracial states (the positivity condition being unclear a priori).

2.3 Existence of solutions of Schwinger-Dyson's equation from matrix models

Let $V = V_{\mathbf{t}}$ be a polynomial function as before. Consider

$$Z_V^N = \int e^{-N\mathrm{Tr}(V(A_1,\cdots,A_m))}\mu_N(dA_1)\cdots\mu_N(dA_m)$$

and μ_V^N the associated Gibbs measure

$$\mu_V^N(dA_1,\cdots,dA_m) = (Z_V^N)^{-1}e^{-N\mathrm{Tr}(V(A_1,\cdots,A_m))}\mu_N(dA_1)\cdots\mu_N(dA_m).$$

We will always assume in this section that $V = V_{\mathbf{t}}$ is self-adjoint so that the potential is real. This means that if $V_{\mathbf{t}} = \sum_i t_i q_i$ then for all i, there exists j such that $q_j^* = q_i$ and $t_j = \overline{t_i}$. Note that Z_V^N is not necessarily finite. We will see various assumptions in order to make μ_V^N a proper probabilty measure. The empirical distribution of m matrices $\mathbf{A} = (A_1,\cdots,A_m) \in \mathcal{H}_N(\mathbb{C})^m$ is defined as the element of $\mathcal{M}_{ST}^m$ such that

$$\hat{\mu}_{\mathbf{A}}^N(F) := \hat{\mu}_{A_1,\cdots,A_m}^N(F) = \frac{1}{N}\mathrm{Tr}(F(A_1,\cdots,A_m))$$

for all $F \in \mathcal{C}_{st}^m(\mathbb{C})$. Note that the empirical distribution could be defined as well as an element of $\mathcal{M}^m$ but since the random matrices $(A_1,\cdots,A_m)$ under μ_V^N have a priori no uniformly bounded spectral radius, the topology of weak convergence would not be suitable then.

We shall see that if we know that a limit point of $\hat{\mu}_{\mathbf{A}}^N$ under μ_V^N is compactly supported, then it satisfies **SD[V]**. In a second part, we shall give examples of potential V for which this assumption is satisfied. Finally, we discuss localized matrix integrals and show that bounded solutions to **SD[V]** for small potentials can always be constructed by localized matrix integrals.

2.3.1 Limit points of empirical distribution of matrices following matrix models satisfy the SD[V] equations

The integral Z_V^N is well defined provided that the monomials of highest degree in $V_{\mathbf{t}}$ are even and sufficiently large. We shall assume in this paragraph that

$$V_{\mathbf{t}}(\mathbf{X}) = V_{\mathbf{t}}^*(\mathbf{X}) = \sum_{1\leqslant i\leqslant n} t_i q_i(\mathbf{X}) + \overline{t_i}q_i^*(\mathbf{X}) + \sum_{n+1\leqslant i\leqslant n+m} t_i X_{i-n}^D \tag{2.9}$$

with D even, monomial functions q_i of degree less or equal than $D-1$ and $t_i > 0$ for $i \in \{n+1,\cdots,n+m\}$. We shall see in the last paragraph of this section that such assumption can be removed provided a cut-off is added.

For such potentials, we show that we can relate the matrix model to the solutions of **SD[V]**. Assume (2.9). Then

1. There exists $M < \infty$ so that, μ_V^N almost surely for all $i \in \{1, \cdots, m\}$.
$$\limsup_{N\to\infty} \hat{\mu}^N_{A_1,\cdots,A_m}(X_i^D) \leqslant M.$$
2. The limit points of $\hat{\mu}^N_{A_1,\cdots,A_m}$ for the $\mathcal{C}^m_{st}(\mathbb{C})$-topology satisfy the 'weak' Schwinger-Dyson equation
$$\tau \otimes \tau(\partial_i F) = \tau((D_i V + X_i)F) \quad (\mathbf{WSD})[\mathbf{V}]$$
for all $F \in \mathcal{C}^m_{st}(\mathbb{C})$, for all $1 \leqslant i \leqslant m$. Moreover for all $i \in \{1, \cdots, n\}$, $\tau(X_i^D) < +\infty$.

Note here that $(D_i V + X_i)F$ does not belong to $\mathcal{C}^m_{st}(\mathbb{C})$ so that it is not clear what **(WSD)[V]** means a priori. We define it by the following; since $\tau(X_i^D) < +\infty$ and $D_i V$ has degree less than $D-1$, there exists a sequence $V^\delta \in \mathcal{C}^m_{st}(\mathbb{C})$ so that
$$\lim_{\delta\to 0} \max_{1\leqslant i\leqslant m} \sup_{\tau(X_i^D)\leqslant M} \tau(|D_i V^\delta - D_i V - X_i|) = 0$$
from which, since any $F \in \mathcal{C}^m_{st}(\mathbb{C})$ is uniformly bounded,
$$\lim_{\delta\to 0} \max_{1\leqslant i\leqslant m} \sup_{\tau(X_i^D)\leqslant M} |\tau(F D_i V^\delta) - \tau(F(D_i V + X_i))| = 0$$
is well defined.

Proof.

• The first point is trivial since by Jensen's inequality,
$$Z_V^N \geqslant \exp\{-N^2 \int \frac{1}{N}\mathrm{Tr}(V(\mathbf{A})) \prod_{1\leqslant i\leqslant m} d\mu_N(A_i)\} \geqslant \exp\{cN^2\}$$
for some $c > -\infty$, where the last inequality comes from the fact that (see [Voi91])
$$\lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(V(\mathbf{A})) \prod_{1\leqslant i\leqslant m} d\mu_N(A_i) = \sigma^m(V) < \infty$$
where σ^m is the law of m free semi-circular variables.

Now, observe that by Hölder's inequality,
$$|\hat{\mu}^N_{\mathbf{A}}(q_i)| \leqslant \max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(|X_i|^{D-1} + 1)$$
so that we deduce
$$\hat{\mu}^N_{\mathbf{A}}(V) \geqslant \sum_{i=1}^m \left(t_{i+n}\hat{\mu}^N_{\mathbf{A}}(X_i^D) - c(\mathbf{t})\hat{\mu}^N_{\mathbf{A}}(|X_i|^{D-1}) - c(\mathbf{t})\right)$$
with a finite constant $c(\mathbf{t})$. Since $t_{i+n} > 0$, we conclude that $\hat{\mu}^N_{\mathbf{A}}(V) \geqslant m|t|M/2$ when
$$\max_{1\leqslant i\leqslant m} \hat{\mu}^N_{\mathbf{A}}(X_i^D) \geqslant M$$

for M large enough. Thus

$$\mu_V^N \left(\max_{1 \leqslant i \leqslant m} \hat{\mu}_{\mathbf{A}}^N (X_i^D) \geqslant M \right) \leqslant e^{-2^{-1} N^2 M m |t|} e^{-cN^2} \tag{2.10}$$

goes to zero exponentially fast when $M > \frac{2c}{m|t|}$. The claim follows by Borel-Cantelli's lemma.

• We proceed as in [CDG03], following a common idea in physics, which is to make, in Z_V^N, the change of variables $X_i \to X_i + N^{-1} F(\mathbf{X})$ for a given $i \in \{1, \cdots, m\}$ and $F \in \mathcal{C}_{st}^m(\mathbb{R})$. Noticing that the Jacobian for this change of variable is

$$|J| = e^{N \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N (\partial_i F) + O(1)}$$

we get that

$$\int e^{(N \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N (\partial_i F) - N^2 \hat{\mu}_{\mathbf{A}}^N (N^{-1} X_i F(\mathbf{X}) + V(X_i + N^{-1} F(\mathbf{X})) - V(X_i))} \mu_V^N (d\mathbf{A}) = O(1).$$

If we denote

$$E_N = \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N (\partial_i F) - N \hat{\mu}_{\mathbf{A}}^N (N^{-1} X_i F(\mathbf{X}) + V(X_i + N^{-1} F(\mathbf{X})) - V(X_i))$$

we deduce that

$$\int_{\max_{1 \leqslant i \leqslant m} \hat{\mu}_{\mathbf{A}}^N (A_i^D) \leqslant M} e^{N E_N} \mu_V^N (d\mathbf{A}) = O(1).$$

Hence, we conclude by Chebychev inequality and (2.10) that for M big enough, any $\delta > 0$, there exists $\eta > 0$, so that

$$\mu_V^N \left(\{ \max_{1 \leqslant i \leqslant m} \hat{\mu}_{\mathbf{A}}^N (X_i^D) \leqslant M \} \cap \{ |E_N| \leqslant \delta \} \right) \geqslant 1 - e^{-\eta N}.$$

Moreover,

$$\hat{\mu}_{\mathbf{A}}^N (V(X_i + N^{-1} F(\mathbf{X})) - V(X_i))) = N^{-1} \hat{\mu}_{\mathbf{A}}^N ((D_i V) F) + R_N$$

with a remainder R_N of order $N^{-2} \max_{1 \leqslant i \leqslant m} \hat{\mu}_{\mathbf{A}}^N (X_i^{D-2})$ which we can neglect on the event $\max_{1 \leqslant i \leqslant m} \hat{\mu}_{\mathbf{A}}^N (A_i^D) \leqslant M$. This shows, by Borel Cantelli's Lemma, that for all $F \in \mathcal{C}_{st}^m(\mathbb{R})$,

$$\hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N (\partial_i F) - \hat{\mu}_{\mathbf{A}}^N (X_i F + D_i V F)$$

goes to zero almost surely. This result extends to $F \in \mathcal{C}_{st}^m(\mathbb{C})$ since it can always be decomposed into the sum of two elements of $\mathcal{C}_{st}^m(\mathbb{R})$. Moreover, if we let $A_i^\varepsilon = A_i (1 + \varepsilon A_i^2)^{-1} = A_i (\sqrt{-1} + \sqrt{\varepsilon} A_i)^{-1} (-\sqrt{-1} + \sqrt{\varepsilon} A_i)^{-1} \in \mathcal{C}_{st}^m(\mathbb{C})$, then again Hölder's inequality shows that $\tau(|D_i V(A_i) - D_i V(A_i^\varepsilon)|)$ goes to zero uniformly on $\max_{1 \leqslant i \leqslant m} \tau(A_i^D) \leqslant M$. This shows that $\mu \to \mu((D_i V + X_i) F)$ is continuous for the weak $\mathcal{C}_{st}^m(\mathbb{C})$-topology on $\{\mu(A_i^D) \leqslant M\}$ for any $F \in \mathcal{C}_{st}^m(\mathbb{C})$. Therefore, since $\mathcal{M}_{ST}^m$ is compact, we conclude that any limit point of $\hat{\mu}_{\mathbf{A}}^N$ satisfies

$$\tau \otimes \tau(\partial_i F) = \tau((X_i + D_i V) F)$$

□

We therefore have the Assume that there exists a limit point τ_V of $\hat{\mu}^N_{\mathbf{A}}$ under μ^N_V which is compactly supported. Then, it satisfies Schwinger-Dyson's equation **SD[V]**. **Proof.**

The proof is straightforward since if τ_V is compactly supported, it is equivalent to say that τ_V satisfies **WSD[V]** or **SD[V]** since $\mathcal{C}^m_{st}(\mathbb{C})$ is dense in the set of polynomial functions (approximate the A_i's by the A^ε_i's defined in the previous proof).

□

Let us also give the final argument to deduce convergence of the free energy from the previous considerations.

Let $\gamma : [0,1] \to \mathbb{C}^n$ be a continuously differentiable path from 0 to $\mathbf{t}$ such that for all s, $V_{\gamma_s} = \sum_{i=1}^n \gamma_s(i) q_i$ is self-adjoint. Assume that

• $\hat{\mu}^N_{\mathbf{A}}$ converges in $\mathcal{M}^m_{ST}$ almost surely or in expectation under $\mu^N_{V_{\gamma_s}}$ for all s in $[0,1]$.

• $\max_p \mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(|X_p|^l))$ is uniformly bounded for s in $[0,1]$ and N large enough for some l strictly greater than the degree of $V_{\mathbf{t}}$.

Then

1. The free energy
$$F^N_{V_{\mathbf{t}}} = N^{-2} \log \left(Z^N_{V_{\mathbf{t}}} \right)$$
converges as N goes to infinity towards a limit $F_{V_{\mathbf{t}}}$.

2. Moreover, there exists $\varepsilon > 0$, such that, if for all s, γ_s is in $B(0,\varepsilon)$, and if the limit points of $\hat{\mu}^N_{\mathbf{A}}$ under $\mu^N_{V_{\gamma_s}}$ are uniformly compactly supported, then
$$F^N_{V_{\mathbf{t}}} = N^{-2} \log (Z^N_{V_{\mathbf{t}}})$$
converges as N goes to infinity towards
$$F_{V_{\mathbf{t}}} = \sum_{\overline{k} \in \mathbb{N}^n \backslash (0,..,0)} \prod_{1 \leqslant i \leqslant n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}}.$$

Note above that the last series has a positive radius of convergence according to Theorems 2.2.2 and 2.2.3. This emphasizes that the possible divergence of $F^N_{V_{\mathbf{t}}}$ does not survive the large N limit.

Proof.

• First, note that as for all s V_{γ_s} is self adjoint, thus, up to a change of coordinates in $\mathbb{C}^n$, V_{γ_s} can be written as $V_{\gamma_s} = \sum_{i=1}^n (\gamma_s(i) q_i + \overline{\gamma_s(i)} q_i^*)$. By differentiating
$$N^{-2} \log Z^N_{V_{\gamma_s}} = \frac{1}{N^2} \log \int e^{-N \mathrm{Tr} \sum_{i=1}^n (\gamma_s(i) q_i + \overline{\gamma_s(i)} q_i^*)} d\mu^N$$
with respect to s we obtain that
$$\partial_s N^{-2} \log Z^N_{V_{\gamma_s}} = - \sum_{i=1}^n \mu^N_{V_{\gamma_s}} (\hat{\mu}^N_{\mathbf{A}} (q_i \partial_s \gamma_s(i) + q_i^* \overline{\partial_s \gamma_s(i)})).$$

But, under our assumption, $(\hat{\mu}^N_{\mathbf{A}}(q_i\partial_s\gamma_s(i) + q_i^*\overline{\partial_s\gamma_s(i)}))_{N\in\mathbb{N}}$ converges almost surely and is uniformly integrable so that $\mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(q_i\partial_s\gamma_s(i) + q_i^*\overline{\partial_s\gamma_s(i)}))$ is a uniformly bounded sequence which converges as N goes to infinity towards $\tau_{\gamma_s}(q_i\partial_s\gamma_s(i)+q_i^*\overline{\partial_s\gamma_s(i)})$ for $s\in[0,1]$. Integrating with respect to s yields the convergence with $F_{V_{\gamma_s}}$ as above by dominated convergence theorem.

• We can choose $\varepsilon > 0$ such that on $B(0,\varepsilon)$ there is an unique solution τ_{γ_s} of $\mathbf{SD}[V_{\gamma_s}]$ and it satisfy the combinatorial interpretation of Theorem 2.2.3. By Corollary 2.3.2, our hypothesis implies that for s in $[0,1]$ the limit points of $\hat{\mu}^N_{\mathbf{A}}$ are unique and given by τ_{γ_s}. Hence, $\hat{\mu}^N_{\mathbf{A}}$ converges in $\mathcal{M}^m_{ST}$ almost surely towards τ_{γ_s}. Since we assumed our family uniformly integrable, we deduce that $\mu^N_{V_{\gamma_s}}(\hat{\mu}^N_{\mathbf{A}}(q_i\partial_s\gamma_s(i) + q_i^*\overline{\partial_s\gamma_s(i)}))$ converges as N goes to infinity towards $\tau_{\gamma_s}(q_i\partial_s\gamma_s(i) + q_i^*\overline{\partial_s\gamma_s(i)})$ for all $l\in\{1,\cdots,n\}$. We denote $\mathcal{M}_{\overline{k^1},\overline{k^2}}$ (resp. $\mathcal{M}_{\overline{k^1},\overline{k^2}}(P)$) the number of planar maps with k_i^1 vertices of type q_i and k_i^2 of type q_i^* (resp. and with one of type P),

$$\begin{aligned}
\lim_{N\to\infty}\frac{1}{N^2}\log Z^N_{V_{\gamma_s}} &= -\sum_{i=1}^n\int_0^1 \tau_{\gamma_s}(q_i\partial_s\gamma_s(i) + q_i^*\overline{\partial_s\gamma_s(i)})ds\\
&= \sum_{i=1}^n\sum_{k_j^1,k_j^2}\int_0^1\prod_j \frac{(-\gamma_s(j))^{k_j^1}}{k_j^1!}\frac{\left(-\overline{\gamma_s(j)}\right)^{k_j^2}}{k_j^2!}(\mathcal{M}_{\overline{k^1},\overline{k^2}}(q_i)\partial_s\gamma_s(i) + \mathcal{M}_{\overline{k^1},\overline{k^2}}(q_i^*)\partial_s\overline{\gamma_s(i)})\\
&= \sum_{i=1}^n\sum_{k_j^1k_j^2\neq 0}\prod_j\frac{(-t_j)^{k_j^1}}{k_j^1!}\frac{(-\overline{t_j})^{k_j^2}}{k_j^2!}\mathcal{M}_{\overline{k^1},\overline{k^2}}
\end{aligned}$$

where we used in the last line the equality $\mathcal{M}_{\overline{k}}(q_i) = \mathcal{M}_{\overline{k}+\mathbb{1}_i}$.

□

We shall in the next section provide a generic example where the assumption of the second point of Theorem 2.3.3 is satisfied (in fact, a slightly different version since we do not prove that the almost sure limit points of $\hat{\mu}^N_{\mathbf{A}}$ satisfy our compactness assumption, but their average do, which still guarantees the result).

2.3.2 Convex interaction models

Let us assume that we consider a matrix model with potential V such that for all N in $\mathbb{N}$,

$$\varphi^N_{V,a} : (A_k(ij)) \in (\mathbb{R}^{N^2})^m \to \mathrm{Tr}(V(A_1,\cdots,A_m)) + \frac{a}{2}\sum_{k=1}^m \mathrm{Tr}(A_k^2) \tag{2.11}$$

is convex in all dimensions for some $a < 1$, i.e the Hessian of $\varphi^N_{V,a}$ is non negative for all $N\in\mathbb{N}$. An example is V of the form

$$V(A_1,\cdots,A_m) = \sum_{i=1}^n t_iP_i(\sum_{k=1}^m \alpha_k^iA_k) + \sum_{i,j}\beta_{i,j}A_1A_j$$

with non-negative t_i's, convex polynomials of one variable P_i, real α's and β with for all i, $|\sum_j \beta_{i,j}| < 1 - a$. Indeed, by Klein's lemma (c.f. [GZ02]), since $x \to P_i(\sum \alpha_k x_k)$ is convex,

$$\mathbf{A} \to \mathrm{Tr} P_i(\sum \alpha_k^i A_k)$$

is also convex (Here $\mathbf{A}$, by an abuse of notations, denotes the entries of the m-uple of matrices $\mathbf{A} = (A_1, \cdots, A_m)$).

Then, we shall prove that Let V be a self-adjoint polynomial function which satisfies (2.11). Then

- There exists $R_V < \infty$ so that

$$\limsup_{N \to \infty} \mu_V^N(\hat{\mu}_{\mathbf{A}}^N(A_i^{2n})) \leqslant (R_V)^n$$

  for all $n \in \mathbb{N}$ and $i \in \{1, \cdots, m\}$. Here, R_V is uniformly bounded by some R_M when the quantities $(a, V(0,0,..,0), (D_i V(0,0,..,0))_{1 \leqslant i \leqslant m}, \sigma^m(V))$ are bounded by M.
- $\mu_V^N[\hat{\mu}_{\mathbf{A}}^N]$ is tight and its limit points satisfy **SD[V]**.
- Take $V = V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ and let U_a be the set of t_i's for which $V_{\mathbf{t}}$ satisfies (2.11) for a given $a < 1$. For $\varepsilon > 0$ small enough, when $(t_i)_{1 \leqslant i \leqslant n} \in U_a \cap B(0, \varepsilon)$, $\hat{\mu}_{\mathbf{A}}^N$ converges in $L^1(\mu_V^N)$ and almost surely to the unique solution to **SD[V]**.
- Let $a < 1$, there exists $\varepsilon > 0$ such that if there exists a continuously differentiable path $\gamma : [0,1] \to U_a \cap B(0, \varepsilon)$ from 0 to $\mathbf{t}$ then

$$F_{V_{\mathbf{t}}}^N = N^{-2} \log(Z_{V_{\mathbf{t}}}^N)$$

  converges as N goes to infinity towards

$$F_{V_{\mathbf{t}}} = \sum_{\overline{k} \in \mathbb{N}^n \setminus (0,..,0)} \prod_{1 \leqslant i \leqslant n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}}.$$

Remark : Observe that if $V_{\mathbf{t}}$ has only a quadratic interaction term, U_a contains all the range of parameters such that the self potentials of each matrix is convex. For instance, if we look at the Ising model,

$$V_{\mathbf{t}} = \beta AB + \sum_{i=1}^n t_i A^{2i} + \sum_{i=1}^m u_i B^{2i}$$

then $\{|\beta| < a\} \cap \{t_i \in \mathbb{R}^+\} \cap \{u_i \in \mathbb{R}^+\} \subset U_a$.

Proof.

We can assume without loss of generality that $a = 0$ since otherwise we just make a shift on the covariance of the matrices under μ_N. The idea is to use Brascamp-Lieb inequality (c.f. [Har04] for recent improvements) which shows that since

$$f(\mathbf{A}) = e^{-N \mathrm{Tr} V(A_1, \cdots, A_m)}$$

is log-concave, for all convex function g on $(\mathbb{R})^{mN^2}$,

$$\mu_V^N(g(\mathbf{A}-\mathbf{M})) = \int g(\mathbf{A}-\mathbf{M}) \frac{f(\mathbf{A})\prod d\mu_N(A_i)}{\int f(\mathbf{A})\prod d\mu_N(A_i)} \leqslant \int g(\mathbf{A}) \prod d\mu_N(A_i) \tag{2.12}$$

with

$$\mathbf{M} = \int \mathbf{A} d\mu_V^N.$$

Here $\mathbf{A}$ denotes the set of entries of the matrices $(A_1, \cdots, A_m)$. Let us apply (2.12) with $g(\mathbf{A}) = \mathrm{Tr}(A_k^{2p})$ which is convex by Klein's lemma. Hence,

$$\mu_V^N\left(\mathrm{Tr}((A_k - \mathbb{E}[A_k])^{2p})\right) \leqslant \mu_N(\mathrm{Tr}(A^{2p})) \tag{2.13}$$

where $\mathbb{E}[A_k](ij) = \mu_V^N(A_k(ij))$ for $1 \leqslant i, j \leqslant N$. By Theorem 2 p.17 in [Sos99], there exists a finite constant C so that for all $p \leqslant \sqrt{N}$,

$$\mu_N(\mathrm{Tr}(A^{2p})) \leqslant CN4^p.$$

In particular,

$$\limsup_{N\to\infty} \mu_V^N[\frac{1}{N}\mathrm{Tr}((A_k - \mathbb{E}[A_k])^{2p})] \leqslant 4^p. \tag{2.14}$$

Also, by Chebychev's inequality we find that if $||A||_\infty$ denotes the spectral radius of A, for all $k \in \{1, \cdots, m\}$

$$\mu_V^N(||A_k - E[A_k]||_\infty \geqslant 3) \leqslant \mu_V^N(\mathrm{Tr}((A_k - \mathbb{E}[A_k])^{2p} \geqslant 3^{2p}) \leqslant CN\left(\frac{2}{3}\right)^{2p}$$

for all $p \leqslant \sqrt{N}$. Taking $p = \sqrt{N}$, we deduce by Borel Cantelli's lemma that

$$\limsup_{N\to\infty} ||A_k - E[A_k]||_\infty \leqslant 3 \quad \text{a.s.} \tag{2.15}$$

We now control $\mathbb{E}[A_k]$ uniformly. Since the law of A_k is invariant by the action of the unitary group, we deduce that for all unitary matrix U,

$$\mathbb{E}[A_k] = \mathbb{E}[UA_kU^*] = U\mathbb{E}[A_k]U^* \Rightarrow \mathbb{E}[A_k] = \mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))I. \tag{2.16}$$

We now bound $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))$ independently of N. Since V is convex, there are real numbers $(\gamma_i)_{1\leqslant i\leqslant m}$ and $c > -\infty$, $\gamma_i = D_iV(0, \cdots, 0)$ and $c = V(0, \cdots, 0)$ so that for all $N \in \mathbb{N}$ and all matrices $(A_1, \cdots, A_m) \in \mathcal{H}_N(\mathbb{C})^m$,

$$\mathrm{Tr}(V(A_1, \cdots, A_m)) \geqslant \mathrm{Tr}(\sum_{i=1}^m \gamma_i A_i + c).$$

By Jensen's inequality, we know that $Z_V^N \geqslant e^{-dN^2}$ for N sufficiently large, $d = 2\sigma^m(V) < +\infty$ and so Chebychev's inequality implies that for all $y > 0$, all $\lambda > 0$,

$$\begin{aligned}
\mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geqslant y\right) &\leqslant e^{(d-c)N^2-\lambda y N^2}[\int e^{-N\sum_{i=1}^m \gamma_i \mathrm{Tr}(A_i)+N\lambda\mathrm{Tr}(A_k)} \prod_{i=1}^m d\mu_N(A_i) \\
&\quad + \int e^{-N\sum_{i=1}^m \gamma_i \mathrm{Tr}(A_i)-N\lambda\mathrm{Tr}(A_k)} \prod_{i=1}^m d\mu_N(A_i)] \\
&\leqslant 2e^{(d-c)N^2-\lambda y N^2} e^{\frac{N^2}{2}\sum_{i\neq k}\gamma_i^2+\frac{N^2}{2}(\gamma_k+\lambda)^2}
\end{aligned}$$

We now optimize with respect to λ : there exists a constant $A < \infty$ (which depends only on $D_iV(0,\cdots,0)$, $V_i(0,\cdots,0)$ and $\sigma^m(V)$) so that

$$\mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geqslant y\right) \leqslant e^{AN^2-\frac{N^2}{4}y^2}$$

and so

$$\mu_V^N(|\hat{\mu}_{\mathbf{A}}^N(X_k)|) = \int \mu_V^N\left(|\hat{\mu}_{\mathbf{A}}^N(X_k)| \geqslant y\right) dy \leqslant 4\sqrt{A} + \int_{y\geqslant 4\sqrt{A}} e^{-\frac{N^2}{4}(y^2-4A)} dy \leqslant 8\sqrt{A}$$

where we assumed N large enough in the last line. Hence, we have proved that

$$\limsup_N |\mu_V^N(\hat{\mu}_{\mathbf{A}}^N(X_k))| < 8\sqrt{A}. \tag{2.17}$$

Plugging this result in (2.14) and (2.16) we obtain for all $p \geqslant 1$:

$$\begin{aligned}
\limsup_{N\to\infty} \mu_V^N[\hat{\mu}_{\mathbf{A}}^N((A_k)^{2p})] &\leqslant 2^{2p-1}\limsup_{N\to\infty} \mu_V^N[\frac{1}{N}\mathrm{Tr}((A_k-\mu_V^N[A_k])^{2p})] \\
&\quad +2^{2p-1}\limsup_{N\to\infty}(\mu_V^N(\frac{1}{N}\mathrm{Tr}[A_k])^{2p}) \\
&\leqslant 2^{2p-1}4^p + 2^{2p-1}(8\sqrt{A})^{2p} \leqslant R_V^{2p}
\end{aligned}$$

with $R_V = 4(1+8\sqrt{A})$. To prove the convergence of $\mu_N^V[\hat{\mu}_{\mathbf{A}}^N]$, remember that $\mu_N^V[\hat{\mu}_{\mathbf{A}}^N]$ is tight for the $\mathcal{C}_{st}^m(\mathbb{C})$-topology. To study its limit point, recall $\int xe^{-x^2/2}f(x)dx = \int f'(x)e^{-x^2/2}dx$ so that, for $P \in \mathcal{C}_{st}^m(\mathbb{C})$,

$$\begin{aligned}
\int \frac{1}{N}\mathrm{Tr}(A_kP)d\mu_N^V(\mathbf{A}) &= \frac{1}{2N^2}\sum_{ij}\int \partial_{A_k(ij)}(Pe^{-N\mathrm{Tr}(V)})_{ji}\prod d\mu_N(A_i) \\
&= \frac{1}{2N^2}\sum_{ij}\int\left(\sum_{P=QX_kR} 2Q_{ii}R_{jj}\right. \\
&\quad \left. -N\sum_{l=1}^n \sum_{q_l=QX_kR} t_l \sum_{h=1}^N 2P_{ji}Q_{hj}R_{ih}\right) d\mu_V^N(\mathbf{A}) \\
&= \int\left(\frac{1}{N^2}(\mathrm{Tr}\otimes\mathrm{Tr})(\partial_kP) - \frac{1}{N}\mathrm{Tr}(D_kVP)\right) d\mu_V^N(\mathbf{A})
\end{aligned}$$

which yields

$$\int \left(\hat{\mu}_{\mathbf{A}}^N((X_k + D_k V)P) - \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N(\partial_k P)\right) d\mu_N^V(\mathbf{A}) = 0$$

Now, by convexity of V we have concentration of $\hat{\mu}_{\mathbf{A}}^N$ under μ_N^V (since log-Sobolev inequality is satisfied uniformly, according to Bakry-Emery criterion, and that Herbst's argument therefore applies, see [ABC+00], sections 6 and 7) : for all Lipschitz function f on the entries

$$\mu_V^N \left(\mathbf{A} : |f(\mathbf{A}) - \mu_V^N(f)| \geqslant \delta\right) \leqslant e^{-\frac{\delta^2}{2||f||_{\mathcal{L}}^2}} \tag{2.18}$$

where $||f||_{\mathcal{L}}$ is the Lipschitz constant of f. Since for $P \in \mathcal{C}_{st}^m(\mathbb{C})$, $\mathbf{A} \to \hat{\mu}_{\mathbf{A}}^N(P)$ is Lipschitz with constant of order N^{-1} (see [GZ00]), we conclude that since $\partial_i P \in \mathcal{C}_{st}^m(\mathbb{C}) \otimes \mathcal{C}_{st}^m(\mathbb{C})$, for all $P \in \mathcal{C}_{st}^m(\mathbb{C})$,

$$\lim_{N\to\infty} \left| \int \hat{\mu}_{\mathbf{A}}^N \otimes \hat{\mu}_{\mathbf{A}}^N(\partial_k P) d\mu_N^V(\mathbf{A}) - \mu_N^V[\hat{\mu}_{\mathbf{A}}^N] \otimes \mu_N^V[\hat{\mu}_{\mathbf{A}}^N](\partial_k P) \right| = 0.$$

Thus

$$\limsup_{N\to\infty} \left(\mu_V^N(\hat{\mu}_{\mathbf{A}}^N((X_k + D_k V)P)) - \mu_V^N[\hat{\mu}_{\mathbf{A}}^N] \otimes \mu_V^N[\hat{\mu}_{\mathbf{A}}^N](\partial_i P)\right) = 0$$

If τ is a limit point of $\mu_V^N[\hat{\mu}_{\mathbf{A}}^N]$ for the weak $\mathcal{C}_{st}^m(\mathbb{C})$-topology, we can use the previous moment estimates to show that even though $X_k + D_k V$ is a polynomial function, $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N((X_k + D_k V)P))$ converges along subsequences towards $\tau((X_k + D_k V)P))$, and of course $\mu_V^N[\hat{\mu}_{\mathbf{A}}^N] \otimes \mu_V^N[\hat{\mu}_{\mathbf{A}}^N](\partial_k P)$ converges towards $\tau \otimes \tau(\partial_k P)$. Hence, we get that the limit points of $\mu_V^N[\hat{\mu}_{\mathbf{A}}^N]$ satisfy the **WSD[V]**. By the previous moment estimate, these limit points are compactly supported, hence they satisfy **SD[V]**. Similarly, by (2.18), $\hat{\mu}_{\mathbf{A}}^N$ is almost surely tight and its limit points satisfy **SD[V]** according to (2.15) and (2.17).

When $V = V_{\mathbf{t}}$, observe that $R_{\mathbf{t}}$ is uniformly bounded when $|t| \leqslant M$ since $V_{\mathbf{t}}(0, \cdots, 0)$ and $(D_i V_{\mathbf{t}}(0, \cdots, 0))_{1 \leqslant i \leqslant m}$ depends continuously on $\mathbf{t}$. Thus, the first point of the theorem shows that the limit points of $\mu_{V_{\mathbf{t}}}^N[\hat{\mu}_{\mathbf{A}}^N]$ are uniformly compactly supported. Hence, since also we have seen that they satisfy **SD[$V_{\mathbf{t}}$]**, for $\mathbf{t}$ small enough, $\hat{\mu}_{\mathbf{A}}^N$ converges in expectation (and therefore almost surely by concentration), to the unique solution to **SD[$V_{\mathbf{t}}$]**. The last point is now a direct consequence of Theorem 2.3.3.

□

Hence, we see here that convex potentials have uniformly compactly supported limit distributions so that we can apply the whole machinery. We strongly believe that this property extends to much more general potentials. However, we shall see in the next section that we can localize the integral to make sure that all limit points are uniformly compactly supported and still keep the enumerative property, hence bypassing the issue of compactness.

2.3.3 The uses of diverging integrals

In the domain of matrix models, diverging integrals are often considered. For instance, if one wants to consider triangulations, one would like to study the integral

$$Z_N(tx^3) = \int e^{tN \mathrm{Tr}(M^3)} d\mu_N(M)$$

which is clearly infinite if t is real. The same kind of problem arises in many other models (c.f. the dually weighted graph model [KSW96]). However, we shall see below that at least as far as planar maps are concerned, we can localize the integrals to make sense of it, while keeping its enumerative property. Namely, let $V_{\mathbf{t}} = V_{\mathbf{t}}^* = \sum t_i q_i$ and let us consider the localized matrix integrals given, for $L < \infty$, by

$$Z_{V_{\mathbf{t}}}^{N,L} = \int_{||\mathbf{A}||_\infty \leqslant L} e^{-N \mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} \prod d\mu_N(A_i)$$

and the associated Gibbs measure

$$\mu_{V_{\mathbf{t}}}^{N,L}(d\mathbf{A}) = (Z_{V_{\mathbf{t}}}^{N,L})^{-1} \mathbb{1}_{||\mathbf{A}||_\infty \leqslant L} e^{-N \mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} \prod d\mu_N(A_i).$$

Here, $||\mathbf{A}||_\infty = \max_{1 \leqslant i \leqslant m} ||A_i||_\infty$ and $||A_i||_\infty$ denotes the spectral radius of the matrix A_i.

We shall prove There exists $L_0 > 0$ and $\varepsilon_0 > 0$ so that for $\varepsilon < \varepsilon_0$, there exists $L(\varepsilon) > L_0$, $L(\varepsilon)$ going to infinity as ε goes to zero, so that for $\mathbf{t} \in B(0, \varepsilon) \cap \{\mathbf{t} | V_{\mathbf{t}} = V_{\mathbf{t}}^*\}$, for all $L \in [L_0, L(\varepsilon)]$,

$$\lim_{N \to \infty} \frac{1}{N^2} \log Z_{V_{\mathbf{t}}}^{N,L} = \sum_{\overline{k} \in \mathbb{N}^n \setminus (0,..,0)} \prod_{1 \leqslant i \leqslant n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}} \tag{2.19}$$

Moreover, under $\mu_{V_{\mathbf{t}}}^{N,L}$, $\hat{\mu}_{\mathbf{A}}^N$ converges almost surely towards $\tau_{\mathbf{t}}$ described in Theorem 2.2.3. This shows that, up to localization, the first order asymptotics of matrix models gives the right enumeration for any polynomials. The diverging integrals often considered in physics should be therefore thought to be conveniently localized to keep their combinatorial virtue, and are then as good as others. In view of Lemma 2.3.6, this localization procedure should not damage the remainder of the large N expansion neither.

Proof.

Fix $M > 0$ and choose η sufficiently small, so that $V_{\mathbf{t}}(0, \cdots, 0)$, $D_i V_{\mathbf{t}}(0, \cdots, 0)$ and $\sigma^m(V_{\mathbf{t}} V_{\mathbf{t}}^*)$ are uniformly bounded by a constant $M < +\infty$ for $\mathbf{t} \in B(0, \eta)$. We will prove that if L is sufficiently large, there exists $0 < \varepsilon < \eta$ such that Theorem 2.19 is valid for all $\mathbf{t}$ in $B(0, \varepsilon) \cap \{\mathbf{t} | V_{\mathbf{t}} = V_{\mathbf{t}}^*\}$.

We now see our potential as a convex potential in order to find an uniform bound on the support. First we bound the Hessian of

$$\varphi_{V_{\mathbf{t}}}^N : (A_k(ij)) \in (\mathbb{R}^{N^2})^m \cap \{\|\mathbf{A}\|_\infty \leqslant L\} \to \mathrm{Tr}(V(A_1, \cdots, A_m)) \tag{2.20}$$

uniformly in N :

$$Hess \varphi_{V_{\mathbf{t}}}^N (A, A) = \sum_{i=1}^n t_i \sum_{q_i = RXSXT} \mathrm{Tr}(RASAT).$$

Now we use Hölder's inequality :

$$\begin{aligned}|\mathrm{Tr}(RASAT)| = |\mathrm{Tr}(TRASA)| &\leq \sqrt{\mathrm{Tr}((TR)A^*A(TR)^*)}\sqrt{\mathrm{Tr}(SA^*AS^*)}\\ &\leqslant \|TR\|_\infty \|S\|_\infty \mathrm{Tr}(AA^*).\end{aligned}$$

which implies that for $\{\|\mathbf{A}\|_\infty \leqslant L\}$

$$\|Hess\varphi^N_{V_\mathbf{t}}\| \leqslant C|\mathbf{t}|$$

and C depends only on L. Therefore, We can find $\varepsilon > 0$ such that if $\mathbf{t} \in B(0,\varepsilon) \cap \{\mathbf{t}|V_\mathbf{t} = V^*_\mathbf{t}\}$, for all N, $\varphi^N_{V_\mathbf{t}} + \frac{1}{4}\sum_{i=1}^n \mathrm{Tr}(X_i^2)$ is convex on $\{\|\mathbf{A}\|_\infty \leqslant L\}$.

Thus $\tilde{V_\mathbf{t}}(\mathbf{A}) = V_\mathbf{t}(\mathbf{A}) + \infty \mathbb{1}_{||\mathbf{A}||_\infty > L}$ is a convex potential and

$$\mathbb{1}_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{Tr}(V_\mathbf{t}(\mathbf{A}))} = e^{-N\mathrm{Tr}(\tilde{V}(\mathbf{A}))}$$

is log-concave so that we can use the strategy of the proof of the first point in Theorem 2.3.4. The only point to check is that there exists $d < +\infty$ such that $Z^{N,L}_{V_\mathbf{t}} \geqslant e^{-dN^2}$. According to Jensen's inequality,

$$\begin{aligned}Z^{N,L}_{V_\mathbf{t}} &= \int_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{Tr}(V_\mathbf{t}(\mathbf{A}))} \prod d\mu_N(A_i)\\ &\geqslant \mu^N(||\mathbf{A}||_\infty \leqslant L) \exp\left(-\frac{N}{\mu^N(||\mathbf{A}||_\infty \leqslant L)} \int_{||\mathbf{A}||_\infty \leqslant L} \mathrm{Tr}(V_\mathbf{t}(\mathbf{A})) \prod d\mu_N(A_i)\right)\end{aligned}$$

The biggest eigenvalue goes almost surely to 2 and $|\int_{||\mathbf{A}||_\infty \leqslant L} \mathrm{Tr}(V_\mathbf{t}(\mathbf{A})) \prod d\mu_N(A_i)|$ is bounded by $\mu^N(V_\mathbf{t}V_\mathbf{t}^*)^{\frac{1}{2}}$ which goes to $\sigma^m(V_\mathbf{t}V^*_\mathbf{t}) < +\infty$ according to [Voi91]. Thus if $L > 2$, $Z^{N,L}_{V_\mathbf{t}} \geqslant e^{-dN^2}$ for a finite constant d. Thus, we can use the same technique than in Theorem 2.3.4 to show that any limit point of $\hat{\mu}^N_\mathbf{A}$ has a bounded support R_M independent of L.

We choose $L > R_M$. Now the proof is very close to that of Theorem 2.3.1 except that we have to be careful to make perturbations which do not change the constraint $||\mathbf{A}||_\infty \leqslant L$. Let $i \in \{1, \cdots, m\}$ and consider the perturbation $A_i \to A_i + N^{-1}h(A_i)$ and $A_j \to A_j$ for $j \neq i$ with a compactly supported function h which vanishes on $[-R,R]^c$. Then for $L > R$, for sufficiently large N, and $||A_i||_\infty \leqslant L$, $||A_i + N^{-1}h(A_i)||_\infty \leqslant L$ so that we see that the limit points of $\hat{\mu}^N_\mathbf{A}$ under the localized Gibbs measure $\mu^{N,L}_V$ satisfy for $i \in \{1, \cdots, m\}$, for all h of support strictly less than L

$$\mu \otimes \mu(\partial_i h(X_i)) = \mu((D_i V + X_i)h(X_i)). \tag{2.21}$$

These limit points are also laws of operators bounded by $R_M < L$. Thus the limit points satisfy (2.21) for arbitrary polynomials P i.e. they satisfy **SD[V]**. Now according, to Theorem 2.2.3 if $\mathbf{t}$ is sufficiently small, **SD[V]**has an unique solution given by the enumeration of maps. Thus we have shown that for $L > R_M$, there exists $\varepsilon > 0$ such that for $\mathbf{t} \in B(0,\varepsilon))$ $\hat{\mu}^N$ goes almost surely to the soltution of **SD[V]** described in Theorem 2.2.3.

The formula for the free energy is then derived as in Theorem 2.3.3 since L is fixed independently of $\mathbf{t}$ small enough.

□

Let us remark that if we define, following [Voi93], for $\mu \in \mathcal{M}^m$, a microstates $\Gamma(\mu, n, N, \eta)$, $n \in \mathbb{N}$, $N \in \mathbb{N}$, $\eta > 0$, as the set of matrices $A_1, .., A_m$ of $\mathcal{H}_N(\mathbb{C})^m$ such that

$$|\mu(\mathbf{X}_{i_1}..\mathbf{X}_{i_p}) - \frac{1}{N}\mathrm{Tr}(\mathbf{A}_{i_1}..\mathbf{A}_{i_p})| < \eta \tag{2.22}$$

for any $1 \leqslant p \leqslant n$, $i_1, .., i_p \in \{1, .., m\}^p$, then we have For all $\delta > 0$ small enough, for $L \in [L_0(\delta), L(\delta)]$, and $|\mathbf{t}| \leqslant \delta$,

$$\begin{aligned}
&\lim_{N\to\infty} \frac{1}{N^2} \log \int_{||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m) \\
&= \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\cap ||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m) \\
&= \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta)} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m)
\end{aligned}$$

Proof.

The first equality is a direct consequence of the previous theorem since it is equivalent to the fact that $\mu_{V_{\mathbf{t}}}^{N,L}(\Gamma(\tau_V, n, N, \varepsilon))$ goes to one. The second comes from the fact that for n greater than the degree of V,

$$\begin{aligned}
&\lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\cap ||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N(A_1)\cdots d\mu_N(A_m) \\
&= -\tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \cap ||\mathbf{A}||_\infty \leqslant L\right) \\
&= -\tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\eta\to 0, n\to\infty} \lim_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\right)
\end{aligned}$$

where we used in the last equality the result of [BB03], which hold when τ_V is the law of bounded operators with norms strictly smaller than L (see the last remark in [BB03]).

□

As a corollary, we also deduce that for all $V_{\mathbf{t}}$ with $\mathbf{t}$ small enough, the limits of empirical distributions of matrices given by localized matrix models provide solutions of **SD[**$V_{\mathbf{t}}$**]**. Since these limits have to be tracial states, we deduce that when there is a unique solution to **SD[**$V_{\mathbf{t}}$**]** it has to be a tracial state. Thus,

The compactly supported solutions of **SD[**$V_{\mathbf{t}}$**]** are tracial states when $\mathbf{t}$ is sufficiently small. Note that if $(P_i)_{1\leqslant i\leqslant m}$ in $\mathbb{C}\langle X_1, \cdots, X_m\rangle^m$ is the conjugate variable of a tracial state, [Voi02] have shown that $P_i = D_i P$ for $1 \leqslant i \leqslant m$ and some polynomial P. This fact should be compared with our graphical interpretation which works only because P_i is a cyclic derivative.

2.4 Applications to free entropy

Let us recall that Voiculescu's microstates entropy is defined, for $\tau \in \cup_R \mathcal{M}_R^m$, by

$$\chi(\tau) = \lim_{\substack{\eta\to 0, n\to\infty \\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left(\Gamma(\tau, n, N, \eta) \cap ||\mathbf{A}||_\infty \leqslant L\right)$$

with $\Gamma(\tau, n, \eta, N)$ the microstates defined in (2.22). Note that the original definition of Voiculescu is not with respect to the Gaussian measure, but with respect to the Lebesgue measure. However, both definitions only differ by a quadratic term (see [CDG03]). It is an (important) open problem whether in general one can replace the limsup by a liminf in the definition of χ. However, from the previous considerations, we can see the following

Let $n \in \mathbb{N}$ and $(q_i)_{1\leqslant i\leqslant n}$ be monomials in m non-commutative variables $\mathbf{X} = (X_1, \cdots, X_m)$. Let $V_{\mathbf{t}} = V_{\mathbf{t}}^* \sum_{i=1}^n t_i q_i$. By Theorem 2.2.3, we know that there exists $\varepsilon > 0$ so that for $|t| < \varepsilon$, there exists a unique compactly supported solution $\tau_{\mathbf{t}}$ to $\mathbf{SD}[V_{\mathbf{t}}]$. Then, also for $|t| \leqslant \varepsilon$,

$$\chi(\tau_{\mathbf{t}}) = \lim_{\substack{\eta\to 0, n\to\infty \\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \cap \{||\mathbf{A}||_\infty \leqslant L\}\right).$$

Moreover,

$$\chi(\tau_{\mathbf{t}}) = - \sum_{k\in\mathbb{N}^n\setminus(0,\cdots,0)} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \left(\sum_{j=1}^n k_j - 1\right) \mathcal{M}_{\overline{k}}.$$

Proof.

In fact, by Lemma 2.3.6

$$\begin{aligned}\chi(\tau_{\mathbf{t}}) &= \lim_{\substack{\eta\to 0, n\to\infty \\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{\substack{\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \\ \cap\{||\mathbf{A}||_\infty\leqslant L\}}} e^{N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A})) - N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A}) \\ &= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\substack{\eta\to 0, n\to\infty \\ L\to\infty}} \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{\substack{\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \\ \cap\{||\mathbf{A}||_\infty\leqslant L\}}} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A}) \\ &\leqslant \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}}\end{aligned}$$

where the last inequality holds with

$$F_{\mathbf{t}} = \limsup_{N\to\infty} \frac{1}{N^2} \log \int_{||\mathbf{A}||_\infty \leqslant L'} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A})$$

for L' chosen as in Lemma 2.3.6. On the other hand,

$$\begin{aligned}
&\lim_{\substack{\eta\to 0, n\to\infty \\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta) \cap ||\mathbf{A}||_\infty \leqslant L\right) \\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + \lim_{\substack{\eta\to 0, n\to\infty \\ L\to\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \int_{\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\cap ||\mathbf{A}||_\infty \leqslant L} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} d\mu_N^{\otimes m}(\mathbf{A}) \\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}} + \lim_{\eta\to 0, n\to\infty} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_{V_{\mathbf{t}}}^{N,L'}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\right) \\
&= \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}}
\end{aligned}$$

where we used in the last term Theorem 2.3.5 which implies

$$\lim_{N\to\infty} \mu_{V_{\mathbf{t}}}^{N,L'}\left(\Gamma(\tau_{\mathbf{t}}, n, N, \eta)\right) = 1$$

for all $\varepsilon > 0, n \in \mathbb{N}$. Thus, we see that χ is equal to its liminf definition and moreover

$$\chi(\tau_{\mathbf{t}}) = \tau_{\mathbf{t}}(V_{\mathbf{t}}) + F_{\mathbf{t}}.$$

Now, by Theorems 2.3.5 and 2.2.3,

$$F_{\mathbf{t}} = \sum_{\overline{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\leqslant i\leqslant n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}}$$

whereas

$$\tau_{\mathbf{t}}(V_{\mathbf{t}}) = \sum_{i=1}^n t_i \sum_{\substack{k_j\in\mathbb{N}, \\ 1\leqslant j\leqslant n}} \prod_{1\leqslant j\leqslant n} \frac{(-t_j)^{k_j}}{k_j!} \mathcal{M}_{k_1,\cdots,k_{i-1},,k_i+1,k_{i+1},\cdots,k_n}$$

from which the formula for $\chi(\tau_{\mathbf{t}})$ is easily derived.

□

2.5 Applications to the combinatorics of planar maps

For the sake of completeness, we summarize in this last section, the results of a few papers devoted to the enumeration of planar maps, either by a combinatorial approach or by a matrix model approach.

2.5.1 The one matrix case

We now consider the case $m = 1$ where we only have one matrix. Let $V_{\mathbf{t}}(A) = \sum_{i=1}^{2D} t_i A^i$ with $t_{2D} > 0$ a polynomial potential with an even leading power. Then it has been proven in [BAG97] Theorem 5.2 that the empirical measure satisfies a large deviation principle :

Let

$$J(\mu) = \int \left(\frac{x^2}{2} + V_{\mathbf{t}}(x) \right) d\mu(x) - \int \int \log |x-y| d\mu(y) d\mu(x)$$

and

$$I(\mu) = J(\mu) - \inf_{\nu \in P(\mathbb{R})} J(\nu)$$

then the sequence of empirical measure $\hat{\mu}^N$ satisfies a large deviation principle in the scale N^2 with good rate function I. Moreover, the minimum of I is reached at a unique probability measure $\mu_{\mathbf{t}}$ so that

$$\frac{x^2}{2} + V_{\mathbf{t}}(x) - 2 \int \log |y-x| d\mu_{\mathbf{t}}(y) = C_{\mathbf{t}}, \ \ \mu_{\mathbf{t}} a.s.$$

with a finite constant $C_{\mathbf{t}}$, and where the left hand side dominates the right hand side on the whole real line.

We can differentiate in x this last equation to recover Schwinger-Dyson's equation. It is not sufficient in general to determine the solution uniquely ; one need the inequality on the whole real line to fix the support of the solution.

These analysis of $\mu_{\mathbf{t}}$ has also been investigated with the method of orthogonal polynomials which give a rather sharp description of the limit measure and emphasizes a structure similar to the semi-circular law. More precisely Theorem 3.1 in [EM03] gives :

Let $V_{\mathbf{t}}$ be a real polynomial of degree $2D$. There exists $t > 0$ and $\gamma > 0$ such that if for all i, $|t_i| < t$ and $t_{2D} > \gamma \sum_{i<2D} t_i$ then $\mu_{\mathbf{t}}$ is absolutely continuous with density $\Psi_{\mathbf{t}}$ of the form :

$$\Psi_{\mathbf{t}}(x) = \frac{1}{2\pi} \mathbb{1}_{[a,b]}(x) \sqrt{(x-a)(x-b)} h(x)$$

with

$$h(z) = \int_{C(z,R)} \frac{V'_{\mathbf{t}}(s)}{\sqrt{(s-a)(s-b)}} \frac{ds}{s-z}$$

where R is such that $a, b \in C(z, R)$. Besides, the boundaries a and b can be find by the equations :

$$\int_a^b \frac{V'_{\mathbf{t}}(s)}{\sqrt{(s-a)(b-s)}} ds = 0$$

$$\int_a^b \frac{s V'_{\mathbf{t}}(s)}{\sqrt{(s-a)(b-s)}} ds = 2\pi$$

We now look at combinatorics of the Schwinger-Dyson's equation with one variable, for $V_{\mathbf{t}}(x) = \sum_{i=1}^{2D} t_i x^i$. Remember that from Theorem 2.2.3, $\mu_{\mathbf{t}}$ can be seen as the generating function of graphs counted by the numbers of stars of valence i :

$$\mu_{\mathbf{t}}(x^p) = \sum_{k_1, \cdots, k_{2D} \in \mathbb{N}} \prod_{i=1}^{2D} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{\overline{k}}(P).$$

Hence, Theorem 2.5.2 allows to estimate the numbers of one color planar maps. A more direct combinatorial approach can be developed by considering for instance the dual of those graphs. The dual of a graph is simply obtained by replacing each face by a star and each edge by a transverse edge which link the two stars which come from the face adjacent to the edge. In that operation each star is replaced by a face of the same valence. As we work on the sphere we can decide that the face which comes from the star X^p is the external face.

Thus $\mu_{\mathbf{t}}(X^{p+1})$ is also the generating function of connected planar graphs with an external face of valence $p+1$ and enumerated by the number of faces of a given valence. Those objects are classical ones in combinatorics and we can follow [Tut63] to find an equation on these generating functions. The idea is to try to cut the first edge of the external face, then two cases may occur : either the graph is disconnected and we obtain two graphs or it isn't disconnected and the external face has grown. This two cases corresponds in the dual graph to the fact that the first half-edge of the root is a loop or not which is exactly what we use to build our combinatorial interpretation so that we can retrieve the Schwinger-Dyson's equation from this fact. Just by using the equation given by this decomposition and some algebraic tools, combinatorialists have solved some models. For example [BC94] gives an equation on the generating function $M(u,v)$ of maps whose internal faces have degree living in a fixed set $\mathcal{D} \subset \mathbb{N}$ and enumerated by their number of edges and the degree of the external face. To translate this in our framework, one can consider for a finite $\mathcal{D}$ with an even maximal element,

$$V_{\mathbf{t}}(X) = \sum_{d\in\mathcal{D}} t_d X^d$$

Then under this potential, for small t, the limit measure $\mu_{\mathbf{t}}$ will satisfy our combinatorial interpretation. Define

$$M(u,v) = \sum_{p\in\mathbb{N}} \mu_{(-u^{\frac{d}{2}})_{d\in\mathcal{D}}}(X^p)v^p$$

which counts maps according to the degree of the external face and its number of edges. Theorem 1 of [BC94] states :

For a series $F(z) = \sum_i a_i z^i$ we will note $[z^i]F(z)$ the i^{th} coefficient a_i. Then there exists a unique power series R satisfying

$$R = 1 - 4R_1 v - 4R_2 v^2$$

with

$$R_1 = \frac{u}{2}\sum_{i\in D}[v^{i-1}](R^{\frac{1}{2}}) \text{ and } R_2 = \frac{u}{2}\sum_{i\in D}[v^i](R^{\frac{1}{2}}) + u - 3R_1^2.$$

The number m_n of maps with n edges such that every degree of internal face lies in D is then

$$m_n = [u^n]\frac{(R_2(u)+R_1(u)^2)(R_2(u)+9R_1(u)^2)}{(n+1)u^2}$$

The techniques to prove these results are most often purely algebraic. The main difference in nature that we could meet between the approaches by matrix models or by combinatorics to these enumerations is that the first provides for free additional structure ; it shows that these enumerations can be expressed in terms of a probability measure $\mu_{\mathbf{t}}$. This point generalizes to any number of colors where the enumeration can be expressed in terms of tracial states. One may hope that this positivity condition could help in solving these combinatorial problems.

2.5.2 Ising model on random graphs

This model is defined by $m = 2$ and

$$V(A,B) = V_{Ising}(A,B) = -cAB + V_1(A) + V_2(B).$$

In the sequel, we denote in short A for X_1 and B for X_2. It is clear that for $|c| < 1$, V is a convex potential as defined in (2.11) if V_1, V_2 are convex (write $-2AB = (A-B)^2 - A^2 - B^2$ or $2AB = (A+B)^2 - A^2 - B^2$ to see that up to a quadratic term $2^{-1}|c|A^2 + 2^{-1}|c|B^2$, V is convex) Hence we deduce from Theorem 2.3.4

For $c \in \mathbb{R}$ and $V_i(x) = \sum_{j=1}^{D} t_j^i x^{2j}$, $i = 1,2$, set $V_{\mathbf{t},c}(A,B) = -cAB + V_1(A) + V_2(B)$. Let, for $\delta > 0$, $U_\delta = \cap_{i,j}\{0 \leqslant t_j^i \leqslant \delta\} \cap \{|c| < 1-\delta\}$. Then, for $\delta > 0$ small enough and $(\mathbf{t}, c) \in U_\delta$, $\mu_V^N(\hat{\mu}_{\mathbf{A}}^N)$ converges towards the solution $\mu_{\mathbf{t},c}$ of $\mathbf{SD}[V_{\mathbf{t},c}]$ as N goes to infinity. Moreover

$$\mu_{\mathbf{t},c}(P) = \sum_{\substack{\overline{k}\in\mathbb{N}^{2D}\\ r\in\mathbb{N}}} \prod_{i,j} \frac{(-t_j^i)^{k_j^i}}{k_j^i!} \frac{c^r}{r!} \mathcal{M}_{\overline{k^1},\overline{k^2},r}(P)$$

and

$$F(\mathbf{t},c) - F(\mathbf{t},0) = \sum_{\substack{\overline{k}\in\mathbb{N}^{2D}\\ r\geqslant 1}} \prod_{i,j} \frac{(-t_j^i)^{k_j^i}}{k_j^i!} \frac{c^r}{r!} \mathcal{M}_{\overline{k^1},\overline{k^2},r}$$

where $\mathcal{M}_{\overline{k^1},\overline{k^2},r}$ (resp. $\mathcal{M}_{\overline{k^1},\overline{k^2},r}(P)$) is the number of planar maps with k_j^1 vertices of type A^{2j}, k_j^2 of type B^{2j} and r of type AB (resp. and one of type P). **Remark :** Note that we took potentials V_1 and V_2 as polynomials with even powers to guarantee our convexity relation but this condition could easily be relaxed by taking more sophisticated domains than U_δ in which the polynomials would remain convex.

Proof.

This result is a consequence of Theorem 2.2.2, 2.3.4 and 2.3.3. Note here that the control on

$$\mu_V^N(N^{-1}\mathrm{Tr}(AB))$$

assumed in Theorem 2.3.3 is satisfied due to Theorem 2.3.4 which provides a uniform bound when $|c| < \xi$ for $\xi < 1$.

□

According to the graphical interpretation, the limiting measure is linked to planar maps with stars whose type are the monomial of V_1, V_2 and stars of type AB. Those maps are very close from Ising configuration on planar graphs except that two stars of type AB can be linked together. For integers $(k_j^i)_{i\in\{A,B\},1\leqslant i\leqslant D}$, define

$$\begin{aligned}\mathcal{I}_{\{k_j^i\},r}(P) \quad = \sharp\{ \quad & \text{planar maps with } k_j^i \text{ stars of color } i \text{ and degree } 2j,\\ & \text{one star of type } P \text{ (if } P\neq 0) \text{ and } r \text{ stars of type } AB\\ & \text{such that there's no link between any of the } r\ AB\text{-stars. } \}\end{aligned}$$

and its rooted counterpart :

$$\begin{aligned}\mathcal{J}_{\{k_j^i\},r}(P) \quad = \sharp\{ \quad & \text{rooted planar maps with } k_j^i \text{ stars of color } i \text{ and degree } 2j,\\ & \text{one star of type } P \text{ wich is the root and } r \text{ stars of type } AB\\ & \text{such that there's no link between any of the } r\ AB\text{-stars. } \}\end{aligned}$$

There's a relation between these quantities similar to (2.3) :

$$\mathcal{I}_{\{k_j^i\},r}(P) = \mathcal{J}_{\{k_j^i\},r}(P) r! \Pi_{i,j} k_j^i! (2j)^{k_j^i} \tag{2.23}$$

We can now relate these numbers to our limit measure :

Let $\mu_{\mathbf{t},c}$ be as in Corollary 2.5.4, then on its radius of convergence,

$$\mu_{\mathbf{t},c}(P) = \left(\frac{1}{1-c^2}\right)^{\frac{\deg P}{2}} \sum_{\substack{k_j^i\in\mathbb{N}^{2D}\\ r\in\mathbb{N}}} \prod_{i,j} \frac{1}{k_j^i!} \left(\frac{-t_j^i}{(1-c^2)^j}\right)^{k_j^i} \frac{c^r}{r!} \mathcal{I}_{\{k_j^i\},r}(P)$$

and

$$F(\mathbf{t},c) - F(\mathbf{t},0) = \frac{1}{1-c^2} \sum_{\substack{k_j^i\in\mathbb{N}, i\in\{1,2\},\\ j\in\{1,D\}, r\geqslant 1}} \prod_{i,j} \frac{1}{k_j^i!} \left(\frac{-t_j^i}{(1-c^2)^j}\right)^{k_j^i} \frac{c^r}{r!} \mathcal{I}_{\{k_j^i\},r}(0)$$

Proof.

First we define a projection π from rooted maps to rooted Ising graphs such that if M is a map $\pi(M)$ is obtained by deleting pairs of AB stars which are glued. We now apply Corollary 2.5.4, and translate its result in term of rooted diagrams using (2.3) :

$$\mu_{\mathbf{t},c}(P) = \sum_{\substack{\overline{k}\in\mathbb{N}^{2D}\\ r\in\mathbb{N}}} \prod_{i,j} (-2jt_j^i)^{k_j^i} c^r \mathcal{D}_{\{k_j^i\},r}(P)$$

All the maps M appearing in that sum are such that $\pi(M)$ is an Ising graph rooted at a star of type P. For a fixed Ising graph G we must find the contribution in that sum of $\pi^{<-1>}(G)$. But we can construct every graph in that set by adding pairs of stars AB on the edges of G. The numbers of edges of G is $e_G = \frac{degP}{2} + \sum_{i,j} jk_j^i$ so that to get the whole contribution of $\pi^{<-1>}(G)$ we have to multiply the contribution of G by

$$\sum_{a_1,\cdots,a_{e_G}\in\mathbb{N}} c^{2\sum a_i} = \left(\frac{1}{1-c^2}\right)^{\frac{degP}{2}+\sum_{i,j} jk_j^i}.$$

In that sum, a_i stands for the number of pairs of AB stars added on the i^{th} edge. Summing on every graphs, we obtain :

$$\mu_{\mathbf{t},c}(P) = \left(\frac{1}{1-c^2}\right)^{\frac{\deg P}{2}} \sum_{\substack{\overline{k}\in\mathbb{N}^{2D}\\ r\in\mathbb{N}}} \prod_{i,j} \left(\frac{-2jt_j^i}{(1-c^2)^j}\right)^{k_j^i} c^r \mathcal{J}_{\{k_j^i\},r}(P)$$

and the result follows by using (2.23).

The second point can be proven by proceeding in the same way.

□

In the rest of this section, we compare a few different approaches to solve the enumeration problem of the Ising model. In short, let us emphasize that, for the time being, combinatorial and orthogonal polynomials approaches give the more complete and explicit results. However, these techniques are still limited to very few models. The Schwinger-Dyson's equation or the large deviation approaches can be developed for a much wider range of models (such as q-Potts, induced QCD etc). However, it seems to us that these arguments still need some mathematical efforts to provide as transparent and powerful results (namely for the first a mathematical study of the so-called master-loop equations, and for the second a clear understanding of the relations between complex Burgers equations and the master-loop equations).

Orthogonal polynomial approach

Here we take $V_1 = V_2 = (g/4)x^4$. By using orthogonal polynomials techniques, it was proved by [Meh81] that the corresponding free energy $F_{g,c}$ satisfies

$$F_{g,c} - F_{0,c} = \int_0^1 (1-x)[\log f(x) - \log \frac{cx}{2(1-c^2)}]dx$$

with $f(x) = f_{g,c}$ solution to the algebraic equation

$$f(x)\{(1 - 6\frac{g}{c}f(x))^{-2} - c^2\} + 12g^2 f^3(x) - \frac{1}{2}cx = 0$$

and the root to be taken equals $2^{-1}cx(1-c^2)^{-1}$ when $g = 0$.

Starting from there, a simpler expression as been derived in [BK87] (equation (16), (17) with $h = z/g$) :

$$\begin{aligned} F_{z,c} &= \frac{1}{2}\ln h(z) + \frac{h^2(z)}{2}\left(\frac{z-1}{2(3z-1)^3} + c^2\frac{z+1}{3z-1} + \frac{c^4}{2}(3z^4 - 3z^2 + 1)\right) \\ &\quad -h(z)(\frac{1}{3z-1} + c^2(1-z^2)) + \frac{1}{2}\ln(1-z^2) + \frac{3}{4} \end{aligned}$$

with

$$h(z) = \frac{(1-3z)^2}{1 - c^2(1-3z)^2(1-3z^2)} \tag{2.24}$$

Hence, by the preceding, Mehta's result gives a formula for the generating function of $\mathcal{J}$ in the quadrangulation case. However, it does not a priori gives the limiting spectral measures of the matrices. Moreover, this strategy could be only developed completely and rigorously for the Ising model and the matrix coupled in chain model (see [CMM81]).

2.5.3 Direct combinatorial approach

We can also relate this result to the work of [BMS02]. Their approach is purely combinatorial; they use bijection with well labeled trees (whose generating functions are well understood) to obtain algebraic equations for the generating functions of the Ising model. Let $I(X,Y,u)$ be the generating function of the Ising model on quasi-tetravalent graphs, (i.e. tetravalent graphs except for the root which is bivalent and black) where X (resp. Y) counts the black (resp. white) tetravalent stars and u the bicolored edges :

$$I(X,Y,u) = \sum_{m,n,r\in\mathbb{N}} X^m Y^n u^r \sharp \left\{ \begin{array}{c} \text{quasi -tetravalent maps with } m \text{ tetravalent} \\ \text{black stars, } n \text{ tetravalent white stars and} \\ r \text{ bi-colored edges} \end{array} \right\}.$$

If $P(x,y,u)$ is the solution to the algebraic equation :

$$P = 1 + 3xyP^3 + \frac{P(1+3xP)(1+3yP)}{u^2(1-9xyP^2)^2} \tag{2.25}$$

Then, by [BMS02], Proposition 1 p.4, I can be written in function of $P(x,y,u)$ with $x = X(u-\frac{1}{u})^2$ and $y = Y(u-\frac{1}{u})^2$ as

$$I(X,Y,u) = \frac{u^2-1}{u^2}\left(xP^3 + \frac{P(1-3xP-2xP^2-6xyP^3)}{1-9xyP^2} - \frac{yP^3(1+3xP)^3}{u^2(1-9xyP^2)^3}\right).$$

On the other hand, according to Proposition 2.5.5, if $V = tA^4 + uB^4 - cAB$ and $\mu_{t,u,c}$ is the associated limit measure then on its domain of convergence,

$$I(X,Y,u) = (1-u^2)\mu_{X(1-u^2)^{-2},Y(1-u^2)^{-2},u}(A^2).$$

If we make the following change of variable in (2.25) :

$$x = y = \frac{-z}{3c^2h(z/3)}, P = -c^2h(z/3), u = c$$

then we find (2.24). Hence, a combinatorial approach can be developed to solve the problem of the enumeration of planar maps of the Ising model, a strategy which requires some combinatorial insight. The next approach we present, developed in particular by Staudacher, Kazakov and Eynard, is a direct analysis of the **SD[V]** equations. It is a purely analytical and rather robust strategy.

2.5.4 Direct study of the SD[V_{Ising}] equations

Here, the analysis is based on Theorem 2.3.4 which asserts that if V_1, V_2 are convex, for small parameters, $\hat{\mu}^N_{A,B}$ converges almost surely towards the solution $\mu_{\mathbf{t},c}$ of **SD[**$V_{\mathbf{t},c}$**]** which is a generating function for the enumeration of maps. Hereafter we take $c = 1$ up to a rescaling $\bar{x} = \sqrt{c}x$, $\bar{y} = \sqrt{c}x$, $V_1(x) = \bar{V}_1(\bar{x})$, $\mu_{\mathbf{t}}(P(A, X_2)) = \mu_{\mathbf{t},1}(P(\sqrt{c}^{-1}A, \sqrt{c}^{-1}X_2))$. Following [Eyn03b], we shall analyze the solutions of the Schwinger-Dyson's equation. Observe that the following considerations hold for any range of parameters, not only small parameters. For large parameters, we do not know that the Schwinger-Dyson's equation has a unique solution but we still know that any limit point of the empirical measure of the random matrices still satisfies it. In the next section, we shall see that for the Ising model and any range of parameters, there is a unique such limit point, and it will therefore enjoy the properties described below. We here summarize the main result, as found in [Eyn03b]. Let $\mu_{\mathbf{t}}$ be a solution of **SD[**V_{Ising}**]**

$$\mu_{\mathbf{t}}((W_1'(A) - B)P) = \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_A P),$$
$$\mu_{\mathbf{t}}((W_2'(B) - A)P) = \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_B P),$$

with ∂_A (resp. ∂_B) the non-commutative derivative with respect to A (resp. B) μ_A (resp. μ_B) and $W_i(z) = z^2/2 + V_i(z)$. Now, let μ_A (resp. μ_B) be the spectral measure of the matrix A (resp. B) then we shall obtain an algebraic equation for $H\mu_A(x)$ (resp. $H\mu_B(x)$) the Stieljes transform of the limiting measure μ_A (resp. μ_B) given, for $x \in \mathbb{C}\backslash\mathbb{R}$ by :

$$H\mu_A(x) = \mu_{\mathbf{t}}(\frac{1}{x - A}) = \int \frac{1}{x - y} d\mu_A(y)$$

Let for $x, y \in \mathbb{C}\backslash\mathbb{R}$, $Y(x) = W_1'(x) - H\mu_A(x)$ and $X(y) = W_2'(y) - H\mu_B(y)$. Then, there exists a polynomial function $E(x, y)$ so that for all $x, y \in \mathbb{C}\backslash\mathbb{R}$

$$E(X(y), y) = 0 \qquad E(x, Y(x)) = 0.$$

In particular, μ_A and μ_B are absolutely continuous with respect to Lebesgue measure, with Hilbert transform $H\mu_A$ and $H\mu_B$ so that $Y(x) = W_1'(x) - H\mu_A(x)$ satisfies the same algebraic equation with $x \in \mathbb{R}$. **Proof.**

Note that since we know that $\mu_{\mathbf{t}}$ is compactly supported, we can take Stieljes functions in **SD[**V_{Ising}**]** instead of polynomials P since the latest are dense by Weirstrass theorem. We choose $P = P(A) = (x - A)^{-1}$ in the second equation in **SD[**V_{Ising}**]** to obtain :

$$\mu_{\mathbf{t}}\left(\frac{W_2'(B)}{x - A}\right) = -1 + xH\mu_A(x)$$

Then we use this in the first equation written with

$$P(A, B) = \frac{1}{(x - A)} \frac{(W_2'(y) - W_2'(B))}{(y - B)}$$

to get after some calculation

$$U(x, y)(y - Y(x)) = (Y(x) - W_1'(x))(x - W_2'(y)) + 1 - Q(x, y) \qquad (2.26)$$

where

$$U(x,y) = \mu_{\mathbf{t}}\left(\frac{1}{(x-A)}\frac{W_2'(y)-W_2'(B)}{(y-B)}\right),$$

and

$$Q(x,y) = \mu_{\mathbf{t}}\left(\frac{W_1'(x)-W_1'(A)}{(x-A)}\frac{W_2'(y)-W_2'(B)}{(y-B)}\right).$$

To obtain our algebraic equation, we simply define

$$E(x,y) = (y - W_1'(x))(x - W_2'(y)) + 1 - Q(x,y)$$

and we obtain the famous "Master-loop equation"

$$E(x, Y(x)) = 0$$

by taking $y = Y(x)$ in (2.26). In a symmetric way, we can show that if $X(y) = W_2'(y) - H\mu_B(x)$ then we also have $E(X(y), y) = 0$. Note that E is a polynomial function. Hence, this shows that $Y(x)$, $X(y)$ and so the generating functions $H\mu_A(x)$ and $H\mu_B(y)$ are solution to an algebraic equation. However, this equation still contains a certain numbers of unknown ; $\{\mu_{\mathbf{t}}(A^p B^q), p \leqslant \deg(V_1) - 2, q \leqslant \deg(V_2) - 2\}$. It is argued in physics that when $\mathbf{t}$ is small, the supports of μ_A and μ_B should be connected and therefore $(x, Y(x))$ and $(X(y), y)$ should then be genus zero curves. Then, these unknowns should be determined by the asymptotic behavior of $X(y)$ and $Y(x)$ at infinity

$$X(y) \simeq W_2'(y) - \frac{1}{y}(1 + o(1)), \quad Y(x) \simeq W_1'(x) - \frac{1}{x}(1 + o(1)).$$

Note in passing that, as solutions of an algebraic equation, $H\mu_A$ and $H\mu_B$ extends continuously (but in general not differentially) to the real line (as an extended complex number). As a consequence, μ_A and μ_B have densities with respect to the Lebesgue measure, as the limits of the imaginary part of the Stieljes transform on the real line.

□

2.5.5 Large deviations approach

An approach using large deviation was developed in [Gui04], see also [Mat94]. Again, we take $c = 1$ up to rescaling and denote $W_i(x) = x^2/2 + V_i(x)$ for $i = 1, 2$. The main advantage of this strategy is to be valid in the whole range of the parameters. Otherwise, it should provide the same type of information than in the previous paragraph. Namely,

For any polynomials V_1, V_2 going to infinity faster than x^2, $\hat{\mu}^N_{A,B}$ converges almost surely towards $\mu_{\mathbf{t},1} = \mu_{\mathbf{t}}$ which is uniquely defined by the Schwinger-Dyson's equations

$$\mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_A P) = \mu_{\mathbf{t}}((W_1'(A) - B)P), \qquad \mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_B P) = \mu_{\mathbf{t}}((W_2'(B) - A)P) \tag{2.27}$$

and by the fact that $\mu_{\mathbf{t}}|_A$ and $\mu_{\mathbf{t}}|_B$ (which are the limits of $\hat{\mu}_A^N$ and $\hat{\mu}_B^N$ respectively) are the unique minimizers of

$$
\begin{aligned}
S^{V_1,V_2}(\mu) = \mu_A(W_1) + \mu_B(W_2) - 2^{-1} \int\int \log|x-y| d\mu^A(x) d\mu^A(y) \\
- 2^{-1} \int\int \log|x-y| d\mu^B(x) d\mu^B(y) \\
+ \frac{1}{2} \inf_{\rho,m} \{ \int_0^1 \int \frac{m_t(x)^2}{\rho_t(x)} dx dt + \frac{\pi^2}{3} \int_0^1 \int \rho_t(x)^3 dx dt \}
\end{aligned}
$$

where the inf is taken over m, ρ so that $\mu_t(dx) = \rho_t(x)dx \in \mathcal{P}(\mathbb{R})$, $\mu_0(x \in .) = \mu_A(x \in .)$, $\mu_1(x \in .) = \mu_B(x \in .)$, and

$$\partial_t \rho_t(x) + \partial_x m_t(x) = 0.$$

The infimum in $(\rho_., m_.)$ is taken along the solution to a complex Burgers equation ; let $\Omega = \{x \in \mathbb{R}, t \in (0,1) : \rho_t(x) > 0\}$ and define on Ω $u_t(x) = \rho_t(x)^{-1} m_t(x)$ and $f_t(x) = u_t(x) + i\pi\rho_t(x)$. Then on Ω,

$$\partial_t f_t(x) + f_t(x) \partial_x f_t(x) = 0.$$

Moreover, with $\mu_A = \mu_{\mathbf{t}}|_A$ and $\mu_B = \mu_{\mathbf{t}}|_B$, for μ_A-almost all x

$$W_1'(x) - u_0(x) = H\mu_A(x), \quad \mu_A \text{ a.s.}, \quad W_2'(x) + u_1(x) = H\mu_B(x), \quad \mu_B \text{ a.s.} \tag{2.28}$$

In comparison with the previous statements, we note that the above results hold for all c and V_1, V_2, and not only for small parameters.

Proof.

Most of the proof is contained in [Gui04] where the convergence of $\hat{\mu}_A^N$, $\hat{\mu}_B^N$ towards the unique minimizers of S^{V_1,V_2} was proved (see Theorem 3.3 in [Gui04]), as well as the fact that the limit is compactly supported and that $\mu_{\mathbf{t}}$ satisfies (2.27) but for $P \in \mathcal{C}_{st}^m(\mathbb{R})$ (see section 3.2.1, p. 555 and 558, in [Gui04]). It clearly extends to polynomial functions since $\mu_{\mathbf{t}}$ is compactly supported as its marginals are. The only point we stress here is that this imply that $\mu_{\mathbf{t}}$ is also uniquely determined. Indeed, by proceeding by induction over the degree in B of a monomial function P, we see that

$$\tau(BP) = -\tau \otimes \tau(\partial_A P) + \tau(W_1'(A)P)$$

defines uniquely all the moments $\tau(P(A,B))$ from those of $\tau(Q(A))$. Note here that this is specific to the interaction under consideration ; in general the solutions of **SD[V]** is not determined by their restriction to one variable.

□

Using for instance the fact that if we let $g_t(x) = tf_t(x) + x$, the Wronskian of (f,g) is null, we find that on each connected component of Ω, there exists an analytic function F so that

$$tf_t(x) + x = F(f_t(x)).$$

In a small parameter region, it should easily be arguable that Ω is connected, as it is when the parameters are null (where the solution at time t can be seen to be a semi-circular variable with variance $1-t+t^2$). One ca argue that f_t extends continuously to $t=0$ and $t=1$ which yields

$$x = F(f_0(x)) \quad f_1(y) + y = F(f_1(y)) \tag{2.29}$$

for all x in the support of μ_A and all y in the support of μ_B. Noting that $f_0(x) = W_1'(x) - \overline{H\mu_A}(x) = -\overline{Y(x)}$, $f_1(x) = -W_2'(x) + H\mu_B(x) = -X(x)$ it is tempting to hope that (2.29) yields the same result that Property 2.5.6, namely that $(Y(x), x)$ and $(y, X(y))$ satisfy the same algebraic equation. Our knowledge of this field is much too limited to enable us to get this conclusion.

Chapitre 3

Second order asymptotics for matrix models

Ce chapitre est l'article [GMS07] écrit en collaboration avec Alice Guionnet et à paraître dans la revue «Annals of Probability».

Abstract

We study several-matrix models and show that when the potential is convex and a small perturbation of the Gaussian potential, the first order correction to the free energy can be expressed as a generating function for the enumeration of maps of genus one. In order to do that, we prove a central limit theorem for traces of words of the weakly interacting random matrices defined by these matrix models and show that the variance is a generating function for the number of planar maps with two vertices with prescribed colored edges.

3.1 Introduction

In this paper we study the asymptotics of Hermitian random matrices whose distribution is given by a small convex perturbation of the Gaussian Unitary Ensemble (denoted **GUE**). We shall consider m-tuples of random matrices, with an integer number $m \in \mathbb{N}$ fixed throughout this paper. Then, the law μ^N of m independent matrices following the **GUE** is given, for

$N \times N$ Hermitian matrices $\mathbf{A} = (A_1, \ldots, A_m)$, by

$$d\mu^N(\mathbf{A}) = e^{-\frac{N}{2}\mathrm{Tr}(\sum_{i=1}^m A_i^2)} \prod_{i=1}^m \prod_{j=1}^N d(A_i)_{jj} \prod_{1 \leqslant j < k \leqslant N} d\Re e(A_i)_{jk} d\Im m(A_i)_{jk}$$

with Tr the non-normalized trace $\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii}$. In other words, the $\mathbf{A} = (A_1, \cdots A_m)$ are independent Hermitian matrices whose entries are, above the diagonal, independent complex centered Gaussian variables with variance N^{-1}. Let $V(\mathbf{X})$ be a polynomial in m non-commutative indeterminates $\mathbf{X} = (X_1, \cdots, X_m)$ such that $\mathrm{Tr}(V(\mathbf{A}))$ is real for all m-tuple of Hermitian matrices $\mathbf{A} = (A_1, \cdots, A_m)$. Then, we shall study the following probability measure μ_V^N on the set $\mathcal{H}_N(\mathbb{C})^m$ of m-tuple of $N \times N$ Hermitian matrices

$$d\mu_V^N(\mathbf{A}) = \frac{1}{Z_V^N} e^{-N\mathrm{Tr}(V(\mathbf{A}))} d\mu^N(\mathbf{A})$$

where Z_V^N is the normalizing constant so that μ_V^N is a probability measure.

Besides we require that the trace of $W(\mathbf{A}) := V(\mathbf{A}) + \frac{1}{2}\sum_{i=1}^m A_i^2$ is a strictly convex function of the entries of $\mathbf{A} = (A_1, \cdots, A_m) \in \mathcal{H}_N(\mathbb{C})^m$ for any $N \in \mathbb{N}$. In that case Z_V^N is automatically finite. More precisely, for $c > 0$, we say that V is c-convex if for any $N \in \mathbb{N}$, $\mathbf{A} \in \mathcal{H}_N(\mathbb{C})^m \to \mathrm{Tr}(W(\mathbf{A}))$ is real-valued and with Hessian bounded below by cI. An example of c-convex potential is

$$V(X_1, \cdots, X_m) = \sum_j P_j\big(\sum_i \alpha_i^j X_j\big) + \sum_{j,k} \beta_{j,k} X_i X_j$$

with convex polynomials P_j on $\mathbb{R}$, real numbers $\alpha_i^j, \beta_{j,k}$ and $\sum_j |\beta_{j,k}| \leqslant 1 - c$ for all $k \in \{1, \cdots, m\}$ (see section 3.2 for more details).

The central result of this paper can roughly be stated as follows. Let $V = V_{\mathbf{t}}(X_1, \cdots, X_m) = \sum_{j=1}^n t_j q_j(X_1, \cdots, X_m)$ be a polynomial potential with $n \in \mathbb{N}$, $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{C}^n$ and monomials $(q_j)_{1 \leqslant j \leqslant n}$ fixed. For all $c > 0$, there exists $\eta > 0$ so that if $|\mathbf{t}| := \max_{1 \leqslant j \leqslant n} |t_j| \leqslant \eta$ and $V_{\mathbf{t}}$ is c-convex, there exists $F^i(V_{\mathbf{t}}) = F^i(t_1, \cdots, t_n)$ for $i = 0, 1$ so that

$$\log Z_{V_{\mathbf{t}}}^N = N^2 F^0(V_{\mathbf{t}}) + F^1(V_{\mathbf{t}}) + o(1).$$

The first order expansion $F^0(V_{\mathbf{t}})$ was already obtained in [GMS06] and we extend our study here to the second order. The higher order expansions can also be tackled by a refinement of our strategy ; this is the subject of a separate article by E. Maurel-Segala [MS06a]. Moreover, we believe our tools sufficiently robust to tackle other models such as the Gaussian orthogonal ensemble, or the Haar measure on the unitary group for instance. Again, this is the subject of further studies.

We next turn to the combinatorial interpretation of $F^0(V_{\mathbf{t}})$, $F^1(V_{\mathbf{t}})$ as generating functions of maps.

Matrix models have been used intensively in physics in connection with the problem of enumerating maps, see the reviews [DFGZJ95, GDS91]. Let us recall that a map of genus

g is a graph which is embedded into a surface of genus g in such a way that the edges do not intersect and dissecting the surface along the edges decomposes it into faces which are homeomorphic to a disk. We will call a star the couple of a vertex and the half-edges which are glued to this star. A star will have a distinguished half-edge and an orientation and will eventually have colored half-edges when $m \geqslant 2$. When $m = 1$, it is well known that if $V_{\mathbf{t}} = 0$ (i.e $\mathbf{t} = (0, \cdots, 0)$) moments of the random matrices from the **GUE** are related with the enumeration of maps ; for instance, the number $\mathcal{M}_k^g$ of maps with genus g with one star with $2k$ half-edges were computed by Harer and Zagier [HZ86] using the formula

$$\int \frac{1}{N}\mathrm{Tr}(A_1^{2k})d\mu^N(A_1) = \sum_{g=0}^{[\frac{k}{2}]} \frac{1}{N^{2k}}\mathcal{M}_k^g.$$

It was shown in [EM03] (see also [ASM01, ACKM93]) that when $m = 1$, this enumerative property extends to the free energy of matrix models at all orders, as conjectured and widely used in physics (see e.g. [BIZ80]). More precisely, if $V_{\mathbf{t}} = \sum_{i=1}^n t_i x^{n_i}$ with $D = \max n_i = n_p$ even and $t_p/\sum_{i\neq p}|t_i|$ large enough, for all $k \in \mathbb{N}$, there exists $\eta > 0$ so that for $|\mathbf{t}| \leqslant \eta$,

$$\log Z_{V_{\mathbf{t}}}^N = N^2 \sum_{g=0}^{k} \frac{1}{N^{2g}} F^g(V_{\mathbf{t}}) + o(N^{2-2k})$$

with

$$F^g(V_{\mathbf{t}}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}^n\setminus\{0,\cdots,0\}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}^g_{k_1,\cdots,k_n}$$

where $\mathcal{M}^g_{k_1,\cdots,k_n}$ is the number of maps of genus g with k_i vertices of degree n_i, $1 \leqslant i \leqslant n$.

Several-matrices integrals are related with the enumeration of colored (or decorated) maps. To make this statement clear, let us associate to a monomial $q(X) = X_{i_1} \cdots X_{i_p}$ a colored star as follows. We choose m different colors $\{1, \cdots, m\}$. The star associated to q (called a star of type q) is a vertex equipped with colored half-edges such that the first half-edge has color i_1, second has color i_2 till the last half-edge which has color i_p. Because the star has a distinguished half-edge (the one associated with X_{i_1}) and an orientation, this defines a bijection between non-commutative monomials and colored stars. Then, it can be seen [Voi91] that, for any monomial q,

$$\lim_{N\to\infty} \int \frac{1}{N}\mathrm{Tr}(q(\mathbf{A}))d\mu^N(\mathbf{A}) = \mathcal{M}_0(q)$$

with $\mathcal{M}_0(q)$ the number of planar maps with one colored star of type q such that only half-edges of the same color can be glued pair-wise together (then forming a one-colored edge). In [GMS06], we proved that if $V_{\mathbf{t}}$ is c-convex and $\mathbf{t} = (t_1, \cdots, t_n)$ is small enough, the limit $F^0(V_{\mathbf{t}})$ of the free energy given in Theorem 3.1.1 is analytic in the variables t_i in a neighborhood of the origin and its expansion is a generating function for planar maps with prescribed colored stars ;

$$F^0(V_{\mathbf{t}}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}^n \setminus \{0,\cdots,0\}} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{k_1,\cdots,k_n} \tag{3.1}$$

with $\mathcal{M}_{k_1,\cdots,k_n}$ the number of planar maps with k_i colored stars of type q_i, the gluing being allowed only between half-edges of the same color. Note however that we can not retrieve all the numbers $\mathcal{M}_{k_1,\cdots,k_n}$ from the $F^0(V_{\mathbf{t}})$'s because the condition that $\mathrm{Tr}(V_{\mathbf{t}})$ is real requires that the parameters $\mathbf{t}$ satisfy some relations. Namely, if $*$ denotes the involution $(zX_{i_1}\cdots X_{i_k})^* = \bar{z}X_{i_k}\cdots X_{i_1}$, we must have $\mathrm{Tr}(V_{\mathbf{t}}) = \frac{1}{2}\mathrm{Tr}(V_{\mathbf{t}} + V_{\mathbf{t}}^*)$ and therefore if $V_{\mathbf{t}} = \sum t_i q_i$, to each t_i must corresponds a t_j such that $\mathrm{Tr}(q_j) = \mathrm{Tr}(q_i^*)$ and $t_j = \bar{t}_i$. Thus, the $F^0(V_{\mathbf{t}})$'s are generating functions for the number of planar maps with k_i colored stars of type q_i or q_i^*. The convexity assumption also should induce some extra relations between the parameters but it can be removed as shown in Theorem 3.1.4.

In this paper, we shall prove that such a representation also hold for the correction $F^1(V_{\mathbf{t}})$ to the free energy given in Theorem 3.1.1. $F^1(V_{\mathbf{t}})$ is analytic in the parameters t_i in some neighborhood of the origin. Its expansion is a generating function of maps :

$$F^1(V_{\mathbf{t}}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}^n \setminus \{0,\cdots,0\}} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^1_{k_1,\cdots,k_n}$$

with $\mathcal{M}^1_{k_1,\cdots,k_n}$ the number of maps with genus one with k_i colored stars of type q_i. In particular, the above sum converges absolutely for $\max_{1\le i\le n}|t_i|$ small enough. Let us remark that such a representation is commonly assumed to hold in physics since the formal result is always true for finite N. For a few models (namely models similar to the Ising model on random graphs), the analysis has been pushed forward to actually give a rather explicit formula for the generating function $F^1(V_{\mathbf{t}})$ in terms of the limiting spectral measure of one matrix under the Gibbs measure $\mu^N_{V_{\mathbf{t}}}$ (see e.g. B. Eynard et al. [EKK05]). Our strategy is here to study the most general potentials, providing a general formula for $F^1(V_{\mathbf{t}})$ in terms of the limiting empirical measure of all the matrices (see section 3.6).

Our arguments to prove Theorem 3.1.1 are rather different from [EM03] or [ASM01] where orthogonal polynomials were used. In [EM03], the idea was to develop a Riemann-Hilbert approach based on precise asymptotics of orthogonal polynomials. In the case of several-matrices models, the technology of orthogonal polynomials is far to be as much developed (except for the Ising model, see [BEH02]). We shall therefore use different tools ; the first, which is well spread in physics, is the use of Schwinger-Dyson equation, the second, for which we need a convex potential, is the *a priori* concentration inequalities. To sketch our strategy, let us denote $\hat{\mu}^N$ the empirical measure

$$\hat{\mu}^N : P \longrightarrow \frac{1}{N}\mathrm{Tr}(P(\mathbf{A})) = \frac{1}{N}\mathrm{Tr}(P(A_1,\cdots,A_m))$$

where P runs over the set $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ of non-commutative polynomials in m indeterminates. Note that when $m=1$, $\hat{\mu}^N$ is the spectral measure of A_1, and therefore a probability measure on $\mathbb{R}$. When $m \geqslant 2$, $\hat{\mu}^N$ is a tracial state, which generalizes the notion of measures to a non-commutative setting (see e.g. [Voi00]). Observe that, for $1 \leqslant i \leqslant m$,

$$\partial_{t_i} \log Z_{V_{\mathbf{t}}}^N = -N^2 \mu_{V_{\mathbf{t}}}^N \left(\hat{\mu}^N(q_i)\right)$$

so that the second order asymptotics of the free energy will follow from that of $\overline{\mu}^N = \mu_{V_{\mathbf{t}}}^N[\hat{\mu}^N]$ evaluated at the monomials q_i, $1 \le i \le n$. Then, a simple integration by parts shows that, for any $N \in \mathbb{N}$, the following finite N Schwinger-Dyson equation holds

$$\mu_{V_{\mathbf{t}}}^N \left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)\right) = \mu_{V_{\mathbf{t}}}^N \left(\hat{\mu}^N((X_i + D_i V_{\mathbf{t}})P)\right)$$

for any polynomial P and $i \in \{1, \cdots, m\}$. Here, ∂_i, D_i are non-commutative derivatives (see section 3.2 for a definition). Based on this equation and concentration inequalities, it was shown in [GMS06] that for sufficiently small parameters $\mathbf{t} = (t_1, \cdots, t_n)$, $\hat{\mu}^N$ converges almost surely and in expectation (for the weak topology generated by the set $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ of non-commutative polynomials). Its limit $\mu_{\mathbf{t}}$ is solution of the Schwinger-Dyson equation

$$\mu_{\mathbf{t}} \otimes \mu_{\mathbf{t}}(\partial_i P) = \mu_{\mathbf{t}}((X_i + D_i V_{\mathbf{t}})P),\ \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle,\ 1 \leqslant i \leqslant m. \tag{3.2}$$

It is the unique solution which satisfies a bound of the form $|\mu_{\mathbf{t}}(X_i^d)| \le C^d$ for all $d \in \mathbb{N}$ and all $i \in \{1, \cdots, m\}$, when $\mathbf{t} = (t_1, \cdots, t_n)$ is small enough and C finite, independent of $\mathbf{t}$.

In this paper we investigate the correction to this convergence by proving a central limit theorem for $\hat{\mu}^N - \mu_{\mathbf{t}}$. More precisely if we define $\hat{\delta}_{\mathbf{t}}^N(P) := N(\hat{\mu}^N(P) - \mu_{\mathbf{t}}(P))$, then we show

For all $c > 0$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in

$$B_{\eta,c} = B(0, \eta) \cap \{\mathbf{t} | V_{\mathbf{t}} \text{ is } c\text{-convex}\}$$

, for all P in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$, under $\mu_{V_{\mathbf{t}}}^N$, $\hat{\delta}_{\mathbf{t}}^N(P)$ converges in law towards a complex centered Gaussian law γ_P. Moreover, $\{\gamma_P | P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle\}$, equipped with the natural addition $\gamma_P + \gamma_Q = \gamma_{P+Q}$, is a Gaussian space and the covariance function is a generating function for planar maps with two prescribed stars. Such a central limit theorem was proved for more general potentials when $m = 1$ by K. Johansson in [Joh98]. When $m = 1$ but the entries are not Gaussian, we refer the reader to [AZ06]. In the case $m \geqslant 2$ but $V = 0$, the central limit theorem was obtained in [CD01], [MS06b] and [Gui02]. Our proof is rather close to that of [Joh98] and in the physics spirit ; by doing an infinitesimal change of variables, it can be seen that the random variable

$$\hat{\delta}_{\mathbf{t}}^N(\xi_{\mathbf{t}} P) := \sum_{i=1}^m \hat{\delta}_{\mathbf{t}}^N\left((I \otimes \mu_{\mathbf{t}} + \mu_{\mathbf{t}} \otimes I)(\partial_i D_i P) - (X_i + D_i V_{\mathbf{t}}) D_i P\right)$$

converges in law towards a centered Gaussian variable. The main issue is then to show that the $\xi_{\mathbf{t}} P$'s are dense in the set of polynomials. When $m = 1$, K. Johansson could use finite Hilbert transformation to invert the operator $\xi_{\mathbf{t}}$. In our case, we deal with a differential operator acting on non-commutative test functions and we prove by hand that it is invertible for sufficiently small t_i's in section 3.4. Clearly, our analysis is perturbative at this point and does not try to find the optimal domain of validity of the central limit theorem.

To use the central limit theorem to obtain the second order asymptotics of $\mu_{V_{\mathbf{t}}}^N(\hat{\delta}_{\mathbf{t}}^N(P))$ observe that by the finite dimensional Schwinger-Dyson equation, we get

$$N\mu_{V_{\mathbf{t}}}^N(\hat{\delta}_{\mathbf{t}}^N(\xi_{\mathbf{t}}P)) = \mu_{V_{\mathbf{t}}}^N(\hat{\delta}_{\mathbf{t}}^N \otimes \hat{\delta}_{\mathbf{t}}^N(\partial_i D_i P))$$

and the right hand side converges towards the variance of the central limit theorem. So again, to obtain the limit of $N\mu_{V_{\mathbf{t}}}^N(\hat{\delta}_{\mathbf{t}}^N(P))$, we need to invert the operator $\xi_{\mathbf{t}}$ (see section 3.6). The resulting formula for the free energy and the variance are given in terms of differential operators acting on non-commutative polynomial functions. Note that a similar formula for the variance of the central limit theorem governing the fluctuations of words of band matrices was found in [Gui02]. Their interpretation in terms of enumeration of maps can be retrieved from the interpretation of non-commutative derivatives in terms of natural operations on maps (see [GMS06]).

Finally, in the spirit of [GMS06], we study matrix models with a non-necessarily convex potential V. Since in that case Z_V^N has no reason to be finite we need to add a cut-off. For a positive constant L we define

$$\mu_{V,L}^N(d\mathbf{A}) = \frac{1}{Z_{V,L}^N}\mathbb{1}_{\lambda_{\max}(\mathbf{A})<L}e^{-N\mathrm{Tr}(V(\mathbf{A}))}d\mu^N(\mathbf{A}).$$

with $\lambda_{\max}(\mathbf{A})$ the maximum of the spectral radius of the A_i's and $Z_{V,L}^N$ a normalizing constant. The remarkable point that we shall prove is that asymptotically the behavior of this measure is independent of L and gives the same type of expansion than in the convex case. Let $V = V_{\mathbf{t}}(X_1, \cdots, X_m) = \sum_{j=1}^n t_j q_j(X_1, \cdots, X_m)$ be a polynomial potential with $n \in \mathbb{N}$, $\mathbf{t} = (t_1, \cdots, t_n) \in \mathbb{C}^n$ and monomials $(q_j)_{1\leqslant j\leqslant n}$ fixed. Assume that $\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))$ is real for all $\mathbf{A} \in \mathcal{H}_N(\mathbb{C})^m$, all $N \in \mathbb{N}$. There exists $L_0 > 0$ such that for all $L > L_0$, there exists $\eta > 0$ so that if $|\mathbf{t}| := \max_{1\leqslant j\leqslant n}|t_j| \leqslant \eta$ so that

$$\log Z_{V_{\mathbf{t}},L}^N = N^2F^0(V_{\mathbf{t}}) + F^1(V_{\mathbf{t}}) + o(1)$$

with $F^0(V_{\mathbf{t}}), F^1(V_{\mathbf{t}})$ as in (3.1) and Property 3.1.2. Note that in the large N limit, the dependence in L disappears.

In the next section we will describe our hypothesis of convexity and show some useful consequences. In section 3 we give an estimate on the rate of convergence of $\mu_{V_{\mathbf{t}}}^N[\hat{\mu}^N]$ to $\mu_{\mathbf{t}}$. Then, in section 4 we prove a central limit theorem, first only for some specific polynomials and then for arbitrary polynomials. In section 5 and 6, we give an interpretation of the variance and of the free energy in terms of enumeration of maps. Finally, in section 7 we give some hints to generalize our proofs to the setting of Theorem 3.1.4.

3.2 Convex hypothesis and standard consequences

3.2.1 Framework and standard notations

Non-commutative polynomials

We denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ the set of complex polynomials on the non-commutative unknown $X_1,\ldots,X_m$. Let $*$ denotes the linear involution such that for all complex z and all monomials

$$(zX_{i_1}\dots X_{i_p})^* = \overline{z}X_{i_p}\dots X_{i_1}.$$

We will say that a polynomial P is self-adjoint if $P = P^*$ and denote $\mathbb{C}\langle X_1,\cdots,X_m\rangle_{sa}$ the set of self-adjoint elements of $\mathbb{C}\langle X_1,\cdots,X_m\rangle$.

For an integer number N, we denote $\mathcal{H}_N(\mathbb{C})$ the set of $N\times N$ Hermitian matrices. We shall sometimes identify $\mathcal{H}_N(\mathbb{C})$ with the set $\mathbb{R}^{N^2}$ of the corresponding real entries (by the bijection which associates to $A\in\mathcal{H}_N(\mathbb{C})$ the N^2-tuple $((\Re e(A_{ij})_{1\leqslant i\leqslant j\leqslant N}, (\Im m(A_{ij})_{1\leqslant i<j\leqslant N})))$.

Moreover, we shall denote in general by A a random matrix, by X a non-commutative indeterminate (for instance to write polynomials). Bold symbols will in general denotes vectors ; $\mathbf{A}$ (resp. $\mathbf{X}$) will in general denote a m-tuple of matrices (resp. non-commutative indeterminates) whereas $\mathbf{t}$ will denote a vector of complex scalars.

The potential V will be later on assumed to be self-adjoint which guarantees that for all integer N, all $\mathbf{A}=(A_1,\cdots,A_m)\in\mathcal{H}_N(\mathbb{C})^m$, $\mathrm{Tr}(V(\mathbf{A}))$ is real. Note that conversely, if $\mathrm{Tr}(V(\mathbf{A}))$ is real, $\mathrm{Tr}(V(\mathbf{A}))=\mathrm{Tr}((V+V^*)(\mathbf{A})/2)$ and so we can replace V by $(V+V^*)/2$ without loss of generality.

We shall assume also that V satisfies some convexity property in this paper. Namely, we will say that V is convex if for any $N\in\mathbb{N}$,

$$\begin{array}{rcc} \phi_V^N \quad : \mathcal{H}_N(\mathbb{C})^m \simeq (\mathbb{R}^{N^2})^m & \longrightarrow & \mathbb{R} \\ ((A_k)_{ij})_{\substack{1\leqslant i\leqslant j\leqslant N\\ 1\leqslant k\leqslant m}} & \longrightarrow & \mathrm{Tr}(V(A_1,\cdots,A_m)) \end{array}$$

is a convex function of its entries.

Note that as we add a Gaussian potential $\frac{1}{2}\sum_{i=1}^m X_i^2$ to V we can relax the hypothesis a little. We will say that V is c-convex if $c>0$ and $V+\frac{1-c}{2}\sum_1^m X_i^2$ is convex. Then the Hessian of ϕ_W^N with $W=V+\frac{1}{2}\sum_1^m X_i^2$ is symmetric positive with eigenvalues bigger than c.

An example is

$$V_{\mathbf{t}}(X_1,\cdots,X_m)=\sum_{i=1}^n P_i\left(\sum_{k=1}^m \alpha_k^i X_k\right)+\sum_{k,l}\beta_{k,l}X_kX_l$$

with convex real polynomials P_i in one unknown and real $\alpha_k^i,\beta_{k,l}$ such that for all l, $\sum_k|\beta_{k,l}|\leqslant(1-c)$. This is due to Klein's Lemma (see [GZ02]) which states that the trace of a real convex function of a self-adjoint matrix is a convex function of the entries of the matrix.

In the rest of the paper, we shall assume that V is c-convex for some $c>0$ fixed. We will denote $B(0,\eta)=\{\mathbf{t}\in\mathbb{C}^n : \max_{1\leqslant i\leqslant n}|t_i|\leqslant\eta\}$ and $B_{\eta,c}=B(0,\eta)\cap\{\mathbf{t} : V_{\mathbf{t}}$ is c-convex.$\}$.

Non-commutative derivatives

We define for $1\leqslant i\leqslant m$ the non-commutative derivatives ∂_i from $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ to the space $\mathbb{C}\langle X_1,\cdots,X_m\rangle^{\otimes 2}$ by the Leibniz rule

$$\partial_i PQ=\partial_i P\times(1\otimes Q)+(P\otimes 1)\times\partial_i Q$$

and $\partial_i X_j = \mathbb{1}_{i=j} 1 \otimes 1$. So for a monomial P, the following holds

$$\partial_i P = \sum_{P = RX_i S} R \otimes S$$

where the sum runs over all possible monomials R, S so that P decomposes into $RX_i S$. We can iterate the non-commutative derivatives : $\partial_i^2 : \mathbb{C}\langle X_1, \cdots, X_m\rangle \to \mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ is given on monomial functions by

$$\partial_i^2 P = 2 \sum_{P = RX_i S X_i Q} R \otimes S \otimes Q.$$

We denote $\sharp : \mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 2} \times \mathbb{C}\langle X_1, \cdots, X_m\rangle \to \mathbb{C}\langle X_1, \cdots, X_m\rangle$ the map $P \otimes Q \sharp R = PRQ$ and generalize this notation to $P \otimes Q \otimes R \sharp (S, T) = PSQTR$. So $\partial_i P \sharp R$ corresponds to the derivative of P with respect to X_i in the direction R, and similarly $2^{-1}[D_i^2 P \sharp (R, S) + D_i^2 P \sharp (S, R)]$ the second derivative of P with respect to X_i in the directions R, S.

We also define the so-called cyclic derivative D_i. If m is the map $m(A \otimes B) = BA$, let us define $D_i = m \circ \partial_i$. For a monomial P, $D_i P$ can be expressed as

$$D_i P = \sum_{P = RX_i S} SR.$$

We shall denote in short $\mathbf{D}$ the cyclic gradient $(D_1, \cdots, D_m)$.

Non-commutative laws

For $(A_1, \cdots, A_m) \in \mathcal{H}_N(\mathbb{C})^m$, we define the linear form $\hat{\mu}^N_{A_1, \cdots, A_m}$ on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ by

$$\hat{\mu}^N_{A_1, \cdots, A_m}(P) = \frac{1}{N} \mathrm{Tr}\left(P(A_1, \cdots, A_m)\right)$$

with Tr the standard trace $\mathrm{Tr}(A) = \sum_{i=1}^N A_{ii}$. $\hat{\mu}^N_{A_1, \cdots, A_m}$ will sometimes be called the empirical distribution of the matrices $(A_1, \cdots, A_m)$. When there is no ambiguity and the matrices $A_1, \cdots, A_m$ follow the law μ_V^N, we shall drop the subscript $A_1, \cdots, A_m$; $\hat{\mu}^N = \hat{\mu}^N_{A_1, \cdots, A_m}$. In [GMS06], it was shown that if $V_{\mathbf{t}} = \sum_i t_i q_i$ is c-convex, for $|\mathbf{t}| := \max_{1 \leqslant i \leqslant n} |t_i|$ small enough, $\hat{\mu}^N$ converges weakly in expectation and almost surely under μ_V^N towards a limit $\mu_{\mathbf{t}}$ (i.e. for all P in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\hat{\mu}^N(P)$ converges in expectation and almost surely to $\mu_{\mathbf{t}}(P)$). We denote

$$\overline{\mu}^N_{\mathbf{t}}(P) = \mu^N_{V_{\mathbf{t}}}[\hat{\mu}^N(P)].$$

We shall later estimate differences between $\hat{\mu}^N$ and its limit. So, we set

$$\begin{aligned}
\hat{\delta}^N_{\mathbf{t}} &= N(\hat{\mu}^N - \mu_{\mathbf{t}}) \\
\overline{\delta_{\mathbf{t}}}^N &= \int \hat{\delta}^N d\mu_V^N = N(\overline{\mu}^N_{\mathbf{t}} - \mu_{\mathbf{t}}) \\
\underline{\hat{\delta}_{\mathbf{t}}}^N &= N(\hat{\mu}^N - \overline{\mu}^N_{\mathbf{t}}) = \hat{\delta}^N_{\mathbf{t}} - \overline{\delta_{\mathbf{t}}}^N.
\end{aligned}$$

In order to simplify the notations, we will make $\mathbf{t}$ implicit and drop the subscript $\mathbf{t}$ in the rest of this paper so that we will denote $\overline{\mu}^N, \mu, \hat{\delta}^N, \overline{\delta}^N$ and $\underline{\hat{\delta}}^N$ in place of $\overline{\mu}^N_{\mathbf{t}}, \mu_{\mathbf{t}}, \hat{\delta}^N_{\mathbf{t}}, \overline{\delta_{\mathbf{t}}}^N$ and $\underline{\hat{\delta}}^N_{\mathbf{t}}$, as well as V in place of $V_{\mathbf{t}}$.

3.2.2 Brascamp-Lieb inequality and a priori controls

We use here a generalization of Brascamp-Lieb inequality shown by Hargé in [Har04] which implies that if V is c-convex, for all convex function g on $(\mathbb{R})^{mN^2} \simeq \mathcal{H}_N(\mathbb{C})^m$,

$$\int g(\mathbf{A}-\mathbf{M})d\mu_V^N(\mathbf{A}) \leqslant \int g(\mathbf{A})d\mu_c^N(\mathbf{A}) \tag{3.3}$$

where $\mathbf{M} = \int \mathbf{A} d\mu_V^N(\mathbf{A})$ is the m-tuple of deterministic matrices whose entries are $(\mathbf{M}_j)_{k\ell} = \int (A_j)_{k\ell} d\mu_V^N(\mathbf{A})$ for $k,\ell \in \{1,\cdots,N\}$, $j\in\{1,\cdots,m\}$, and μ_c^N is the Gaussian law on $\mathcal{H}_N(\mathbb{C})^m$ with covariance $(Nc)^{-1}$, i.e $\int f(\mathbf{A})d\mu_c^N(\mathbf{A}) = \int f(c^{-\frac{1}{2}}\mathbf{A})d\mu^N(\mathbf{A})$ for all measurable function f on $\mathbb{R}^{mN^2}$.

Recall that $B_{\eta,c}$ is the subset of the complex numbers $\mathbf{t}\in\mathbb{C}^n$ which are bounded by η and so that V is c-convex. Based on Brascamp-Lieb inequality, it was shown in [GMS06] (Theorem 3.4) that

(Compact support) If $c,\eta>0$, then there exists $C_0 = C_0(c,\eta)$ finite such that for all $i\in\{1,\cdots,m\}$, all $n\in\mathbb{N}$, all $\mathbf{t}\in B_{\eta,c}$,

$$\mu(X_i^{2n}) \leqslant \limsup_N \overline{\mu}^N(X_i^{2n}) \leqslant C_0^n.$$

Note that this lemma shows that, for $i\in\{1,\cdots,m\}$, the spectral measure of the matrices is asymptotically contained in the compact set $[-\sqrt{C_0},\sqrt{C_0}]$.

Proof.

Let us recall the proof of this result for completeness. Let $k\in\{1,\cdots,m\}$. As $g:\mathbf{A}\in\mathcal{H}_N(\mathbb{C})^m \to N^{-1}\mathrm{Tr}(A_k^{4d}) = \hat{\mu}^N(X_k^{4d})$ is convex by Klein's lemma, we can use Brascamp-Lieb inequalities (3.3) to see that

$$\overline{\mu}^N((X_k-M_k)^{4d}) \leqslant \mu_c^N(\hat{\mu}^N(X_k^{4d})) \tag{3.4}$$

where $M_k := \mu_V^N(A_k)$ is the deterministic matrix with entries $(M_k)_{ij} = \mu_V^N((A_k)_{ij})$. Thus, since $\mu_c^N(\hat{\mu}^N(X_i^{4d}))$ converges by Wigner theorem [Wig58] towards $c^{-2d}C_{2d} \leqslant (c^{-1}4)^{2d}$ with C_{2d} the Catalan number, we only need to control M_k. First observe that for all k the law of A_k is invariant under the unitary group so that for all unitary matrices U,

$$M_k = \mu_V^N[UA_kU^*] = U\mu_V^N[A_k]U^* \Rightarrow M_k = \mu_V^N(\hat{\mu}^N(X_k))I = \overline{\mu}^N(X_k)I. \tag{3.5}$$

Let us bound $\overline{\mu}^N(X_k)$. Jensen's inequality implies

$$Z_N^V \geqslant e^{-N^2\mu^N(\frac{1}{N}\mathrm{Tr}(V))} = e^{-N^2\mu^N\circ\hat{\mu}^N(V)}.$$

According to [Voi91], $\mu^N \circ \hat{\mu}^N$ converges in moments to the law of m free semicircular operators, which are uniformly bounded. Thus, there exists a finite constant L such that $Z_N^V \geqslant e^{-N^2 L}$. We now use the convexity of V, to find that for all N, all $\mathbf{A} = (A_1, \cdots, A_m) \in \mathcal{H}_N(\mathbb{C})^m$,

$$\operatorname{Tr}\left(V(\mathbf{A}) + \frac{1-c}{2}\sum_{i=1}^m A_i^2\right) \geqslant \operatorname{Tr}\left(V(0) + \sum_{i=1}^m D_i V(0) A_i + (1-c)\sum_{i=1}^m A_i\right).$$

By Chebyshev's exponential inequality, and then using the above bound, we therefore obtain that for any $\lambda \geq 0$,

$$\begin{aligned} & \mu_V^N(\hat{\mu}^N(X_k) \geqslant y) \leqslant e^{-\lambda N^2 y} \mu_V^N(e^{\lambda N^2 \hat{\mu}^N(X_k)}) \\ = \; & e^{-\lambda N^2 y} \frac{Z_c^N}{Z_1^N Z_V^N} \mu_c^N \left(e^{\lambda N^2 \hat{\mu}^N(X_k) - N \operatorname{Tr}\left(V(\mathbf{A}) + \frac{1-c}{2}\sum_{i=1}^m A_i^2\right)}\right) \\ \leqslant \; & e^{N^2(L - V(0) - \lambda y + \frac{m}{2}\log c)} \mu_c^N \left(e^{-N \operatorname{Tr}(\sum_{i=1}^m ((1-c) + D_i V(0)) A_i - \lambda A_k)}\right) \\ = \; & e^{N^2(L - V(0) - \lambda y + \frac{m}{2}\log c)} e^{\frac{N^2}{2c}\sum_{i\neq k}(1-c+D_i V(0))^2 + \frac{N^2}{2c}(1-c+D_k V(0) - \lambda)^2} \end{aligned}$$

where we denoted $V(0) := V(0, \cdots, 0)$ and $D_i V(0) := D_i V(0, \cdots, 0)$. Remark that these constants are uniformly bounded for $\mathbf{t}$ in $B(0, R)$, $R > 0$. Thus, we deduce that

$$\mu_V^N(\hat{\mu}^N(X_k) \geqslant y) \leq e^{N^2[(a-\lambda y) + \frac{1}{2c}(\lambda - b)^2]}$$

with two constants a, b which are uniformly bounded in terms of c, η for $\mathbf{t} \in B_{\eta,c}$. Optimizing with respect to λ shows that there exists $A < +\infty$ so that for $\mathbf{t}$ in $B_{\eta,c}$

$$\begin{aligned} \mu_V^N\left(\hat{\mu}^N(X_k) \geqslant y\right) & \leqslant e^{N^2(a - \frac{c}{2}y^2 - by)} \\ & \leqslant e^{N^2(A - \frac{c}{4}y^2)}. \end{aligned}$$

Replacing X_k by $-X_k$, we bound similarly $\mu_V^N\left(\hat{\mu}^N(X_k) \leqslant -y\right)$ and hence we have proved

$$\mu_V^N\left(|\hat{\mu}^N(X_k)| \geqslant y\right) \leq 2e^{N^2(A - \frac{c}{4}y^2)}.$$

As a consequence,

$$\begin{aligned} \mu_V^N(|\hat{\mu}^N(X_k)|) &= \int_0^\infty \mu_V^N\left(|\hat{\mu}^N(X_k)| \geqslant y\right) dy \\ &\leqslant 2\sqrt{c^{-1}A} + 2\int_{2\sqrt{c^{-1}A}}^\infty e^{-\frac{N^2 c}{4}(y^2 - 4\frac{A}{c})} dy \leqslant 4\sqrt{c^{-1}A} \end{aligned} \tag{3.6}$$

where the last inequality holds for N sufficiently large. Recall that A is a continuous function of the t_i's and therefore our bound on $\sup_N \mu_V^N(|\hat{\mu}^N(X_k)|)$, which controls the spectral radius of M_k in any dimension N, is locally bounded in $\mathbf{t}$. This completes the proof with (3.4).

□

Let us derive some other useful properties due to the convexity hypothesis. Let $\lambda^N_{\max}(A_i)$ be the maximum of the absolute value of the eigenvalues of A_i. We first obtain an estimate on $\lambda^N_{\max}(\mathbf{A})$ the maximum of the $(\lambda^N_{\max}(A_i))_{1\le i\le m}$ under the law μ^N_V.

(Exponential tail of the largest eigenvalue) If $c, \eta > 0$ then there exists $\alpha > 0$ and $M_0 < \infty$ such that for all $\mathbf{t} \in B_{\eta,c}$, all $M \geqslant M_0$ and all integer N

$$\mu^N_V(\lambda^N_{\max}(\mathbf{A}) > M) \leqslant e^{-\alpha MN}.$$

Proof.

Since the largest eigenvalue

$$\lambda^N_{\max}(\mathbf{A}) = \max_{1\leqslant i\leqslant m} \sup_{\|u\|=1} < u, A_i A_i^* u >^{\frac{1}{2}}$$

is a convex function of the entries of the A_i's, we can apply Brascamp-Lieb inequality (3.3) to obtain that for all $s \in [0, \frac{c}{10}]$,

$$\int e^{sN\lambda^N_{\max}(\mathbf{A}-\mathbf{M})} d\mu^N_V(\mathbf{A}) \leqslant \int e^{sN\lambda^N_{\max}(\mathbf{A})} d\mu^N_c(\mathbf{A}) \leqslant C_0^N$$

where the last inequality comes from the bound on the largest eigenvalue of the **GUE** shown for instance in [BADG01]. Now,

$$\lambda^N_{\max}(\mathbf{A}) \leqslant \lambda^N_{\max}(\mathbf{A}-\mathbf{M}) + \lambda^N_{\max}(\mathbf{M}) \leqslant \lambda^N_{\max}(\mathbf{A}-\mathbf{M}) + 4\sqrt{Ac^{-1}}$$

where we used the bound (3.6). Consequently, we deduce that

$$\int e^{sN\lambda^N_{\max}(\mathbf{A})} d\mu^N_V(\mathbf{A}) \leqslant C^N$$

for a positive finite constant C. We conclude by a simple application of Chebyshev's inequality.

$\square$

3.2.3 Concentration inequalities

We next turn to concentration inequalities for traces of polynomials on the subset of $\mathcal{H}_N(\mathbb{C})^m \simeq \mathbb{R}^{N^2 m}$

$$\Lambda^N_M = \{\mathbf{A} \in \mathcal{H}_N(\mathbb{C})^m : \lambda^N_{\max}(\mathbf{A}) = \max_i(\lambda^N_{\max}(A_i)) \leqslant M\}$$

for some fixed $M > 0$. Recall that $\underline{\hat{\delta}}^N = N(\hat{\mu}^N - \bar{\mu}^N)$. We shall prove concentration inequalities for $\underline{\hat{\delta}}^N(P)$ on Λ^N_M for polynomial functions P. However, concentration inequalities should not restrict to polynomial functions but hold more generally for Lipschitz functions (see e.g [GZ02]). We thus define the following Lipschitz semi-norm

$$\|P\|^M_{\mathcal{L}} = \sup_{\mathcal{A}\,C*-\text{algebra}} \sup_{\substack{x_1,\cdots,x_m\in\mathcal{A} \\ \forall i, x_i=x_i^*, \|x_i\|_{\mathcal{A}}\leqslant M}} \left(\sum_{k=1}^m \|D_k P D_k P^*(x_1,\ldots,x_n)\|_{\mathcal{A}}\right)^{\frac{1}{2}}. \tag{3.7}$$

Be aware that this is not a norm since for example $\|1\|_{\mathcal{L}}^M = 0$ or $\|X_1X_2 - X_2X_1\|_{\mathcal{L}}^M = 0$. However, on these particular polynomials, $\hat{\underline{\delta}}^N$ vanishes. This fact can be generalized as follows; if we set

$$m_{M,P}^N := \frac{1}{\mu_V^N(\Lambda_M^N)}\mu_V^N\left(\mathbb{1}_{\Lambda_M^N}\hat{\underline{\delta}}^N(P)\right)$$

we shall see (see the proof below) that on Λ_M^N

$$|\hat{\underline{\delta}}^N(P) - m_{M,P}^N| \leq 2M\sqrt{m}N\|P\|_{\mathcal{L}}^M.$$

Therefore, if we denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle_{\mathcal{L}}^M$ the completion of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ for $\|.\|_{\mathcal{L}}^M$, we can extend $\hat{\underline{\delta}}^N - m_{M,P}^N$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle_{\mathcal{L}}^M$ on Λ_M^N. A similar result will be proved for μ in Lemma 3.4.9 (note however that the arguments of this lemma do not apply here because $\bar{\mu}^N$ is not the law of uniformly bounded matrices).

We shall prove

(Concentration inequality) Let $\mathbf{t}$ be such that V is c-convex. There exists $\alpha, M_0 > 0$ such that for all N in $\mathbb{N}$, all $M > M_0$, all $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle_{\mathcal{L}}^M$, there exists a positive constant $\varepsilon_{P,M}^N$ such that for any $\varepsilon > 0$,

$$\mu_V^N\left(\{|\hat{\underline{\delta}}^N(P) - m_{P,M}^N| \geqslant \varepsilon + \varepsilon_{P,M}^N\} \cap \Lambda_M^N\right) \leqslant 2e^{-\frac{c\varepsilon^2}{2(\|P\|_{\mathcal{L}}^M)^2}}. \tag{3.8}$$

Moreover, there exists a universal constant C such that

$$\varepsilon_{P,M}^N \leq 2CNM\|P\|_{\mathcal{L}}^M e^{-\alpha NM}.$$

If P is a monomial of degree $0 < d < \alpha N$, we have

$$\|P\|_{\mathcal{L}}^M \leqslant dM^{d-1},\ \varepsilon_{P,M}^N \leqslant NCdM^d e^{-\alpha MN},\ |m_{P,M}^N| \leqslant N(3M^d + d^2)e^{-\alpha MN}.$$

Proof.

Since V is c-convex, for all integer number N, the Hessian of

$$\phi_V^N : \mathbf{A} \simeq ((A_k)_{ij})_{\substack{1\leqslant i\leqslant j\leqslant N\\ 1\leqslant k\leqslant m}} \in \mathbb{R}^{mN^2} \longrightarrow \mathrm{Tr}(V(A_1, \cdots, A_m)) \in \mathbb{R}$$

is bounded below by cI. Therefore, since μ_V^N has density $e^{-N\phi_V^N(\mathbf{A})}$ with respect to Lebesgue measure, μ_V^N satisfies a Log-Sobolev inequality with constant $(Nc)^{-1}$ (see e.g. corollaire 5.5.2 p.87 in [ABC+00]). In other words, for any continuously differentiable function f from $\mathbb{R}^{mN^2}$ into $\mathbb{R}$,

$$\int f^2 \log \frac{f^2}{\mu_V^N(f^2)} d\mu_V^N \leq \frac{2}{Nc}\int \|\nabla f\|^2 d\mu_V^N$$

with ∇f the gradient of f and $\|\cdot\|$ the Euclidean norm. Here and in the sequel we identify the measure μ_V^N as a measure on $\mathbb{R}^{N^2m}$. This implies, by the well known Herbst argument (see e.g. [ABC+00], théorème 7.4.1 p. 123), that μ_V^N satisfies concentration inequalities. Let us briefly summarize this argument for completeness. If f is a continuously differentiable function, differentiating $X(\lambda) := \frac{1}{\lambda}\log \mu_V^N[e^{\lambda f}]$ and using Log-Sobolev inequality yields

$$\partial_\lambda X(\lambda) \leq \frac{2}{cN\lambda^2 \mu_V^N(e^{\lambda f})}\mu_V^N(\|\nabla e^{\frac{1}{2}\lambda f}\|^2) \leq \frac{1}{2cN}\|\|\nabla f\|^2\|_\infty.$$

If we assume $\mu_V^N(f) = 0$, we find that $X(0) = 0$ and so integrating with respect to λ yields

$$\mu_V^N(e^{\lambda f}) \leqslant e^{\frac{\lambda^2\|\|\nabla f\|^2\|_\infty}{2cN}}.$$

Using Chebyshev's inequality thus gives, for $\varepsilon > 0$ and $\lambda > 0$,

$$\mu_V^N(f \geq \varepsilon) \leqslant e^{-\lambda\varepsilon} e^{\frac{\lambda^2\|\|\nabla f\|^2\|_\infty}{2cN}}$$

and so optimizing with respect to λ results with

$$\mu_V^N(f \geq \varepsilon) \leqslant e^{-\frac{cN\varepsilon^2}{2\|\|\nabla f\|\|_\infty^2}}.$$

Replacing f by $-f$ gives the well known concentration estimate for any $\varepsilon > 0$

$$\mu_V^N(|f| \geq \varepsilon) \leqslant 2e^{-\frac{cN\varepsilon^2}{2\|\|\nabla f\|\|_\infty^2}}$$

for any continuously differentiable function f such that $\mu_V^N(f) = 0$. This estimate extends, modulo some extra technicalities, to Lipschitz functions and then $\|\|\nabla f\|\|_\infty$ is replaced by the Lipschitz norm

$$\|f\|_{\mathcal{L}} := \sup_{x \neq y} \frac{|f(x) - f(y)|}{\|x - y\|}$$

where x, y belong to $\mathbb{R}^{mN^2}$ and $\|x\|$ denotes the Euclidean norm of x. Then, for all $\varepsilon > 0$, the following estimate holds

$$\mu_V^N(|f - \mu_V^N(f)| > \varepsilon) \leqslant 2e^{-\frac{Nc\varepsilon^2}{2\|f\|_{\mathcal{L}}^2}}. \tag{3.9}$$

We set

$$\begin{aligned} f_P(\mathbf{X}) &:= \underline{\hat{\delta}}^N(P) - m_{P,M}^N \\ &= \mathrm{Tr}(P(\mathbf{X})) - c_{P,M}^N \end{aligned}$$

with $c_{P,M}^N = \frac{1}{\mu_V^N(\Lambda_M^N)}\int \mathbb{1}_{\Lambda_M^N}\mathrm{Tr}(P(\mathbf{A}))d\mu_V^N(\mathbf{A})$. Observing that

$$\partial_{(A_i)_{kl}}\mathrm{Tr}(P(\mathbf{A})) = (D_i P(\mathbf{A}))_{lk},$$

we find that on the closed set Λ_M^N, f_P is a Lipschitz (actually an infinitely differentiable) function of the entries of $\mathbf{A} \in \mathcal{H}_N(\mathbb{C})^m$ with constant

$$\begin{aligned} (\|f_P\|_{\mathcal{L}}^{\Lambda_M^N})^2 &:= \sup_{\mathbf{A}\in\Lambda_M^N} \|\nabla \mathrm{Tr} P(\mathbf{A})\|^2 \\ &= \sup_{\mathbf{A}\in\Lambda_M^N} \sum_{k=1}^m \mathrm{Tr}(D_k P (D_k P)^*) \leq N(\|P\|_{\mathcal{L}}^M)^2 \end{aligned}$$

where we simply used that the set of $N\times N$ matrices is a C^*-algebra. As a consequence, we also find that for $\mathbf{B}\in\Lambda_M^N$,

$$\begin{aligned}|f_P(\mathbf{B})| &= \left|\mathrm{Tr}(P(\mathbf{B})) - \frac{1}{\mu_V^N(\Lambda_M^N)}\int \mathbb{1}_{\Lambda_M^N}\mathrm{Tr}(P(\mathbf{A}))d\mu_V^N(\mathbf{A})\right| \\ &\leqslant \frac{\|\mathrm{Tr}P\|_{\mathcal{L}}^{\Lambda_M^N}}{\mu_V^N(\Lambda_M^N)}\int \mathbb{1}_{\Lambda_M^N}\left(\sum_{i=1}^m \mathrm{Tr}(B_i-A_i)^2\right)^{\frac12} d\mu_V^N(\mathbf{A}) \\ &\leqslant 2\sqrt{m}MN\|P\|_{\mathcal{L}}^M \end{aligned} \tag{3.10}$$

and so on Λ_M^N we can extend f_P to $P\in\mathbb{C}\langle X_1,\cdots,X_m\rangle_{\mathcal{L}}^M$.

We can also extend f_P to the whole space $\mathcal{H}_N(\mathbb{C})^m$ with the same Lipschitz constant by putting

$$\bar f_P(\mathbf{A}) = \sup_{\mathbf{B}\in\Lambda_M^N}\left\{ f_P(\mathbf{B}) - \sqrt{N}\|P\|_{\mathcal{L}}^M\left(\sum_{i=1}^m \mathrm{Tr}(A_i-B_i)^2\right)^{\frac12}\right\}.$$

Then applying (3.9), with

$$\varepsilon_{P,M}^N = |\mu_V^N(1_{(\Lambda_M^N)^c}\bar f_P)| + |1-\mu_V^N(\Lambda_M^N)^{-1}||\mu_V^N(1_{\Lambda_M^N}f_P)|,$$

we obtain

$$\begin{aligned}&\mu_V^N\left(\{|\underline{\hat\delta}^N(P) - m_{P,M}^N| \geqslant \varepsilon + \varepsilon_{P,M}^N\}\cap\Lambda_M^N\right) \\ &= \mu_V^N\left(\{|\bar f_P - \frac{1}{\mu_V^N(\Lambda_M^N)}\mu_V^N(1_{\Lambda_M^N}\bar f_P)| \geqslant \varepsilon + \varepsilon_{P,M}^N\}\cap\Lambda_M^N\right) \\ &\leqslant \mu_V^N\left(|\bar f_P - \mu_V^N(\bar f_P)| \geqslant \varepsilon\right) \\ &\leqslant 2e^{-\frac{Nc\varepsilon^2}{2(\|f_P\|_{\mathcal{L}}^{\Lambda_M^N})^2}} = 2e^{-\frac{c\varepsilon^2}{2(\|P\|_{\mathcal{L}}^M)^2}}.\end{aligned}$$

We now use the exponential decay of the largest eigenvalue to control $\varepsilon_{P,M}^N$. By (3.10) and the definition of $\bar f_P$, note that

$$\bar f_P(\mathbf{A}) \leq 2\sqrt{m}MN\|P\|_{\mathcal{L}}^M + \sqrt{N}\|P\|_{\mathcal{L}}^M\left(\left(\sum_{i=1}^m \mathrm{Tr}(A_i^2)\right)^{\frac12} + \sqrt{mN}M\right).$$

Consequently, if M, N are large enough so that $\mu_V^N((\Lambda_M^N)^c) \leq e^{-\alpha NM} \leq \frac{1}{2}$ by Property 3.2.2,

$$\begin{aligned}
\varepsilon_{P,M}^N &\leqslant \mu_V^N \left(1_{(\Lambda_M^N)^c}(MNm\|P\|_{\mathcal{L}}^M(3+\lambda_{\max}(\mathbf{A}))) \right) + 2MmN\|P\|_{\mathcal{L}}^M e^{-\alpha NM} \\
&\leq MNm\|P\|_{\mathcal{L}}^M (5e^{-\alpha NM} + \int_0^\infty \mu_V^N\left(\{\lambda_{\max}(\mathbf{A}) \geq y \vee M\}\right) dy) \\
&\leq 6m(M^2 + \frac{1}{\alpha N})N\|P\|_{\mathcal{L}}^M e^{-\alpha NM}
\end{aligned}$$

When P is a monomial of degree d,

$$\|P\|_{\mathcal{L}}^M \leq dM^{d-1}.$$

Thus, we only need to control $m_{P,M}^N$;

$$\begin{aligned}
|m_{P,M}^N| &\leqslant \left| \left(\frac{1}{\mu_V^N(\Lambda_M^N)} - 1 \right) \mu_V^N(\mathbb{1}_{\Lambda_M^N} \mathrm{Tr}(P)) \right| + \left| \mu_V^N(\mathbb{1}_{(\Lambda_M^N)^c} \mathrm{Tr}(P)) \right| \\
&\leq 2Ne^{-\alpha MN} M^d + N\mu_V^N(\mathbb{1}_{(\Lambda_M^N)^c} \lambda_{\max}(\mathbf{A})^d) \\
&= 2Ne^{-\alpha MN} M^d + dN \int_0^\infty y^{d-1} \mu_V^N\left(\{\lambda_{\max}(\mathbf{A}) \geq y \vee M\}\right) dy \\
&\leqslant 2Ne^{-\alpha MN} M^d + dN \int_0^\infty y^{d-1} e^{-\alpha N y \vee M} dy \\
&\leqslant (2+1)Ne^{-\alpha MN} M^d + dNe^{-\alpha NM} \sum_{k=1}^d \frac{d-1}{\alpha N} \cdots \frac{d-k}{\alpha N} \\
&\leqslant N(3M^d + d^2)e^{-\alpha NM}
\end{aligned}$$

where we assumed that $d < \alpha N$.

□

For later purposes, we have to find a control on the variance of $\hat{\mu}^N$. Recall that $\underline{\hat{\delta}}^N(P) = N(\hat{\mu}^N(P) - \overline{\mu}^N(P))$. For any $\varepsilon, \eta, c > 0$, there exists $B, C, M_0 > 0$ such that for all $\mathbf{t} \in B_{\eta,c}$, all $M \geqslant M_0$, for all $N \in \mathbb{N}$, and all monomial P of degree less than $\varepsilon N^{\frac{2}{3}}$,

$$\mu_V^N\left((\underline{\hat{\delta}}^N(P))^2 \right) \leqslant B(\|P\|_{\mathcal{L}}^M)^2 + C^d N^2 e^{-\frac{\alpha MN}{2}}.$$

Proof.
If P is a monomial of degree d, we write

$$\mu_V^N((\underline{\hat{\delta}}^N(P))^2) \leqslant \mu_V^N(\mathbb{1}_{\Lambda_M^N}(\underline{\hat{\delta}}^N(P))^2) + \mu_V^N(\mathbb{1}_{(\Lambda_M^N)^c}(\underline{\hat{\delta}}^N(P))^2) = I_1 + I_2. \tag{3.11}$$

For I_1, the previous Lemma implies that

$$\begin{aligned} I_1 &= 2\int_0^\infty x\mu_V^N\left(\{|\mathrm{Tr}(P)-\mu_V^N(\mathrm{Tr}(P))|\geqslant x\}\cap\Lambda_M^N\right)dx \\ &\leqslant (\varepsilon_{P,M}^N+|m_{P,M}^N|)^2+4\int_0^\infty xe^{-\frac{cx^2}{2(\|P\|_{\mathcal{L}}^M)^2}}dx\leqslant Ce^{-\alpha MN}+B(\|P\|_{\mathcal{L}}^M)^2\end{aligned}$$

with a constant $B=\frac{4}{c}$ and a constant C such that $(\varepsilon_{P,M}^N+|m_{P,M}^N|)^2\leq Ce^{-\alpha MN}$ for all $d\leq\varepsilon N^{\frac{2}{3}}$. For the second term, we take $M\geqslant M_0$ with M_0 as in Lemma 3.2.2 (Exponential tail of the largest eigenvalue) to get

$$I_2\leqslant\mu_V^N[(\Lambda_M^N)^c]^{\frac{1}{2}}\mu_V^N((\underline{\hat{\delta}}^N(P))^4)^{\frac{1}{2}}\leqslant e^{-\frac{\alpha MN}{2}}\mu_V^N((\underline{\hat{\delta}}^N(P))^4)^{\frac{1}{2}}.$$

By Cauchy-Schwartz inequality, we obtain the control

$$\mu_V^N[\underline{\hat{\delta}}^N(P)^4]\leqslant 2^4\mu_V^N((N\hat{\mu}^N(P))^4)\leqslant 2^4N^4\mu_V^N((\hat{\mu}^N(PP^*))^2).$$

Now, by non-commutative Hölder's inequality (see for example [PX03]),

$$[\hat{\mu}^N(PP^*)]^2\leqslant\max_{1\leqslant i\leqslant m}\hat{\mu}^N(X_i^{4d})$$

so that we obtain the bound

$$\mu_V^N[\underline{\hat{\delta}}^N(P)^4]\leqslant 2^4N^4\max_{1\leqslant i\leqslant m}\overline{\mu}^N(X_i^{4d}).$$

By (3.5) and (3.6), we obtained a uniform bound $x(=4\sqrt{Ac^{-1}})$ on $\bar{\mu}^N(X_i)$ so that we have proved using (3.4) that

$$\overline{\mu}^N(X_i^{4d})\leqslant 2^{4d}(\mu_c^N(\hat{\mu}^N(X_i^{4d}))+x^{4d}).$$

We can now use the control on the moments as obtained for instance by Soshnikov, Theorem 2 p.17 in [Sos99] to see that there exists $C(\varepsilon)$, $C(\varepsilon)<\infty$ for $\varepsilon>0$, so that

$$\mu_c^N(\hat{\mu}^N(X_i^{4d}))\leqslant C(\varepsilon)^{4d}$$

provided $d\leqslant\varepsilon N^{\frac{2}{3}}$. As a consequence, we get that

$$\overline{\mu}^N(X_i^{4d})\leqslant C(\varepsilon)^{4d} \tag{3.12}$$

for all $d\leqslant\varepsilon N^{\frac{2}{3}}$ and all integer number N. Here $C(\varepsilon)$ denotes a finite constant depending only on ε, η and c which may have changed from line to line. Hence, we conclude that

$$I_2\leqslant 4N^2e^{-\frac{\alpha MN}{2}}C(\varepsilon)^{2d}.$$

Plugging back this estimate into (3.11), we have proved that for N and M sufficiently large , all monomials P of degree $d\leqslant\varepsilon N^{\frac{2}{3}}$, all $\mathbf{t}\in B_{\eta,c}$

$$\mu_V^N\left((\hat{\delta}^N(P))^2\right)\leqslant B(\|P\|_{\mathcal{L}}^M)^2+C^{2d}N^2e^{-\frac{\alpha MN}{2}}$$

with a finite constant C depending only on ε, c, and η.

□

3.3 Bound on the distance between μ and $\overline{\mu}^N$

We here bound, for all monomial P,

$$\overline{\delta}^N(P) = N(\overline{\mu}^N(P) - \mu)(P).$$

For all $c, \varepsilon > 0$, there exists $\eta > 0, C < +\infty$, such that for all integer number N, all $\mathbf{t} \in B_{\eta,c}$, and all monomial functions P of degree less than $\varepsilon N^{\frac{2}{3}}$,

$$|\overline{\delta}^N(P)| \leqslant \frac{C^{\deg\,(P)}}{N}.$$

In particular, $|(\hat{\underline{\delta}}^N - \hat{\delta}^N)(P)| \leq \frac{C^{\deg\,(P)}}{N}$ almost surely.

Proof.

The starting point is the finite dimensional Schwinger-Dyson equation that one gets readily by integration by parts (see [GMS06] proof of theorem 3.4)

$$\mu_V^N(\hat{\mu}^N[(X_i + D_i V)P]) = \mu_V^N\left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)\right) \tag{3.13}$$

Therefore, since μ satisfies the Schwinger-Dyson equation (3.2)

$$\mu[(X_i + D_i V)P] = \mu \otimes \mu(\partial_i P), \tag{3.14}$$

by taking the difference we get that for all polynomial P,

$$\overline{\delta}^N(X_i P) = -\overline{\delta}^N(D_i V P) + \overline{\delta}^N \otimes \overline{\mu}^N(\partial_i P) + \mu \otimes \overline{\delta}^N(\partial_i P) + r(N,P) \tag{3.15}$$

with

$$r(N,P) := N^{-1}\mu_V^N\left(\hat{\underline{\delta}}^N \otimes \hat{\underline{\delta}}^N(\partial_i P)\right).$$

If we take P a monomial of degree $d \leqslant \varepsilon N^{\frac{2}{3}}$ and assume $M \geqslant M_0$ then we see by using Lemma 3.2.4

$$\begin{aligned}
|r(N,P)| &\leqslant \frac{1}{N}\sum_{P=P_1 X_i P_2} \mu_V^N\left(|\hat{\underline{\delta}}^N(P_1)|^2\right)^{\frac{1}{2}} \mu_V^N\left(|\hat{\underline{\delta}}^N(P_2)|^2\right)^{\frac{1}{2}} \\
&\leqslant \frac{C}{N}\sum_{l=0}^{d-1}(Bl^2 M^{2(l-1)} + C^l N^2 e^{-\frac{\alpha M N}{2}})^{\frac{1}{2}} \times \\
&\qquad (B(d-l-1)^2 M^{2(d-l-1)} + C^{(d-l-1)} N^2 e^{-\frac{\alpha M N}{2}})^{\frac{1}{2}} \\
&\leqslant \frac{C}{N} d(B(d-1)^2 M^{2(d-2)} + C^{(d-1)} N^2 e^{-\frac{\alpha M N}{2}}) := r(N,d,M).
\end{aligned}$$

We set

$$\Delta_d^N = \max_{P \text{ monomial of degree } d} |\overline{\delta}^N(P)|.$$

Observe that by (3.12), for any monomial of degree d less than $\varepsilon N^{\frac{2}{3}}$, $|\overline{\mu}^N(P)| \leqslant C(\varepsilon)^d$, $|\mu(P)| \leqslant C_0^d \leqslant C(\varepsilon)^d$. It allows us to obtain the rough bound $\Delta_d^N \leqslant 2NC(\varepsilon)^d$ if $d < \varepsilon N^{\frac{2}{3}}$. By (3.15), writing $D_i V = \sum t_j D_i q_j$, we get that for $d < \varepsilon N^{\frac{2}{3}}$

$$\Delta_{d+1}^N \leqslant \max_{1\leqslant i\leqslant m} \sum_{j=1}^n |t_j| \Delta^N_{d+\deg(D_i q_j)} + 2\sum_{l=0}^{d-1} C(\varepsilon)^{d-l-1}\Delta_l^N + r(N,d,M).$$

We next define, for $\kappa \leqslant 1$,

$$\Delta^N(\kappa,\varepsilon) := \sum_{k=1}^{\varepsilon N^{\frac{2}{3}}} \kappa^k \Delta_k^N.$$

We obtain, if D is the maximal degree of V,

$$\begin{aligned}\Delta^N(\kappa,\varepsilon) \ &\leqslant \ [C'\kappa^{-D}|\mathbf{t}| + 2(1-C(\varepsilon)\kappa)^{-1}\kappa^2]\Delta^N(\kappa,\varepsilon) \\ &\quad + C|\mathbf{t}| \sum_{k=\varepsilon N^{\frac{2}{3}}+1}^{\varepsilon N^{\frac{2}{3}}+D} \kappa^{k-D}\Delta_k^N + \sum_{k=1}^{\varepsilon N^{\frac{2}{3}}} \kappa^{k+1} r(N,k,M) \end{aligned} \tag{3.16}$$

where we choose κ small enough so that $C(\varepsilon)\kappa < 1$. Moreover, since D is finite, using the bound on Δ_k^N, we get

$$\sum_{k=\varepsilon N^{\frac{2}{3}}+1}^{\varepsilon N^{\frac{2}{3}}+D} \kappa^{k-D}\Delta_k^N \leqslant 2DN(\kappa C(\varepsilon))^{\varepsilon N^{\frac{2}{3}}}\kappa^{-D}.$$

Since $\kappa C(\varepsilon) < 1$, as N goes to infinity, this term is negligible with respect to N^{-1} for all $\varepsilon > 0$. The following estimate holds

$$\sum_{k=1}^{\varepsilon N^{\frac{2}{3}}} \kappa^k r(N,k,M) \leqslant \frac{C}{N}\sum_{k=1}^{\varepsilon N^{\frac{2}{3}}} k\kappa^k (B(k-1)^2 M^{2(k-2)} + C^{(k-1)} N^2 e^{-\frac{\alpha NM}{2}}) \leqslant \frac{C''}{N}$$

if κ is small enough so that $M^2\kappa < 1$ and $C\kappa < 1$. We observed here that $N^2 e^{-\frac{\alpha NM}{2}}$ is uniformly bounded independently of $N \in \mathbb{N}$. Now, if $|\mathbf{t}|$ is small, we can choose κ so that

$$\zeta := 1 - [C'\kappa^{-D}|\mathbf{t}| + 2(1-C(\varepsilon)\kappa)^{-1}\kappa^2] > 0.$$

Plugging these controls into (3.16) shows that for all $\varepsilon > 0$, and for $\kappa > 0$ small enough, there exists a finite constant $C(\kappa,\varepsilon)$ so that

$$\Delta^N(\kappa,\varepsilon) \leqslant C(\kappa,\varepsilon)N^{-1}$$

and so for all monomial P of degree $d \leqslant \varepsilon N^{\frac{2}{3}}$,

$$|\overline{\delta}^N(P)| \leqslant C(\kappa,\varepsilon)\kappa^{-d}N^{-1}.$$

□

To get the precise evaluation of $N\overline{\delta}^N(P)$, we shall first obtain a central limit theorem under μ_V^N which in turn will allow us to estimate

$$\lim_{N\to\infty} Nr(N,P).$$

3.4 Central limit theorem

We shall here prove that

$$\hat{\delta}^N(P) = N(\hat{\mu}^N - \mu)(P)$$

satisfies a central limit theorem for all polynomial P. By proposition 3.3.1, it is equivalent to prove a central limit theorem for $\underline{\hat{\delta}}^N(P)$, $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$. We start by giving a weak form of a central limit theorem for Stieltjes-like functions. We then extend the result to polynomial functions in the image of some differential operator. We finally generalize our result to any polynomial functions.

For the rest of the paper, we will always assume the following hypothesis **(H)**.

(H) : Let c be a positive real number. The parameter $\mathbf{t}$ is in $B_{\eta,c}$ with η sufficiently small such that we have the convergence to the solution of (3.2) as well as the control given by Lemma 3.2.1 (Compact support), and Proposition 3.3.1.

Note that **(H)** implies also that the control of Lemma 3.2.1 (Compact support) is uniform, and that we can apply Lemma 3.2.2 (Exponential tail of the largest eigenvalue) and Lemma 3.2.3 (Concentration inequality) with uniform constants.

3.4.1 Central limit theorem for Stieltjes test functions

One of the issue that one needs to address when working with polynomials is that they are not uniformly bounded. For that reason, we will prefer to work in this section with the complex vector space $\mathcal{C}_{st}^m(\mathbb{C})$ generated by the Stieltjes functionals

$$ST^m(\mathbb{C}) = \{ \overrightarrow{\prod_{1\leqslant i\leqslant p}} (z_i - \sum_{k=1}^m \alpha_i^k \mathbf{X}_k)^{-1}; \quad z_i \in \mathbb{C}\backslash\mathbb{R}, \alpha_i^k \in \mathbb{R}, p \in \mathbb{N}\} \tag{3.17}$$

where $\prod^{\rightarrow}$ is the non-commutative product. We can also equip $ST^m(\mathbb{C})$ with an involution

$$(\overrightarrow{\prod_{1\leqslant k\leqslant p}} (z_k - \sum_{i=1}^m \alpha_i^k \mathbf{X}_i)^{-1})^* = \overrightarrow{\prod_{1\leqslant k\leqslant p}} (\overline{z_{p-k}} - \sum_{i=1}^m \alpha_i^{p-k} \mathbf{X}_i)^{-1}.$$

We denote $\mathcal{C}_{st}^m(\mathbb{C})_{sa}$ the set of self-adjoint elements of $\mathcal{C}_{st}^m(\mathbb{C})$. The derivation is defined by the Leibniz rule and

$$\partial_i (z - \sum_{i=1}^m \alpha_i \mathbf{X}_i)^{-1} = \alpha_i (z - \sum_{i=1}^m \alpha_i \mathbf{X}_i)^{-1} \otimes (z - \sum_{i=1}^m \alpha_i \mathbf{X}_i)^{-1}.$$

We recall two notations ; first $\sharp$ is the operator

$$(P \otimes Q)\sharp h = PhQ \text{ and } (P \otimes Q \otimes R)\sharp(g,h) = PgQhR$$

so that for a monomial q

$$\partial_i \circ \partial_j q\#(h_i,h_j) = \sum_{q=q_0X_iq_1X_jq_2} q_0h_iq_1h_jq_2 + \sum_{q=q_0X_jq_1X_iq_2} q_0h_jq_1h_iq_2.$$

Assume **(H)** and let $h_1,\cdots,h_m$ be in $\mathcal{C}^m_{st}(\mathbb{C})_{sa}$. Then the random variable

$$Y_N(h_1,\cdots,h_m) = N\sum_{k=1}^m\{\hat\mu^N \otimes \hat\mu^N(\partial_k h_k) - \hat\mu^N[(X_k + D_kV)h_k]\}$$

converges in law towards a real centered Gaussian variable with covariance

$$C(h_1,\cdots,h_m) = \sum_{k,l=1}^m(\mu\otimes\mu[\partial_k h_l \times \partial_l h_k] + \mu(\partial_l\circ\partial_k V\sharp(h_k,h_l))) + \sum_{k=1}^m \mu(h_k^2).$$

Proof.

Define $W = \frac{1}{2}\sum_i X_i^2 + V$. Notice that $Y_N(h_1,\cdots,h_m)$ is real valued because the $h_k's$ and W are self adjoint. The proof follows from the usual change of variable trick. We take $h_1,\cdots,h_m$ in $\mathcal{C}^m_{st}(\mathbb{C})_{sa}$, $\lambda\in\mathbb{R}$ and perform a change of variable $A_i \to B_i = F(A)_i = A_i + \frac{\lambda}{N}h_i(\mathbf{A})$ in Z^N_V. Note that since the h_i are $\mathcal{C}^\infty$ and uniformly bounded, this defines a bijection on $\mathcal{H}_N(\mathbb{C})^m$ for N big enough. We shall compute the Jacobian of this change of variables up to its second order correction. The Jacobian J may be seen as a matrix $(J_{i,j})_{1\leqslant i,j\leqslant m}$ where the $J_{i,j}$ are in $\mathcal{L}(\mathcal{H}_N(\mathbb{C}))$ the set of endomorphisms of $\mathcal{H}_N(\mathbb{C})$, and we can write $J = I + \frac{\lambda}{N}\overline{J}$ with

$$\begin{array}{rccc} \overline{J}_{i,j}: & \mathcal{H}_N(\mathbb{C}) & \longrightarrow & \mathcal{H}_N(\mathbb{C}) \\ & X & \longrightarrow & \partial_i h_j\#X. \end{array}$$

Now, for $1\leqslant i,j\leqslant m$, $X\longrightarrow \partial_i h_j\#X$ is bounded for the operator norm uniformly in N (since $h_j\in\mathcal{C}_{st}(\mathbb{C}), \partial_i h_j \in \mathcal{C}_{st}(\mathbb{C})\otimes\mathcal{C}_{st}(\mathbb{C})$ is uniformly bounded) so that for sufficiently large N, the operator norm of $\frac{\lambda}{N}\overline{J}$ is less than 1. From this we deduce

$$|\det J| = |\det(I + \frac{\lambda\overline{J}}{N})| = \exp(\mathrm{Tr}\log(I+\frac{\lambda\overline{J}}{N})) = \exp(\sum_{k\geqslant 1}\frac{(-1)^{k+1}\lambda^k}{kN^k}\mathrm{Tr}(\overline{J}^k)).$$

Observe that as $\overline{J}$ is a matrix of size m^2N^2 and of uniformly bounded norm, the k-th term $\frac{(-1)^{k+1}\lambda^k}{N^k}tr(\overline{J}^k)$ is bounded by $\frac{m^2|\lambda|^k}{N^{k-2}}$. Hence, only the two first terms in the expansion will contribute to the order 1 and the sum s_N of the other terms will be of order $\frac{1}{N}$. To compute the two first terms in the expansion, we only have to remark that if ϕ is an endomorphism of $\mathcal{H}_N(\mathbb{C})$ is of the form $\phi(X) = \sum_l A_lXB_l$, with $N\times N$ matrices A_i,B_i then $\mathrm{Tr}\phi = \sum_l \mathrm{Tr}A_l\mathrm{Tr}B_l$ (this can be checked by decomposing ϕ on the canonical basis of $\mathcal{H}_N(\mathbb{C})$). Now,

$$\overline{J}^k_{ij} : X \longrightarrow \sum_{1\leqslant i_1,\cdots,i_{k-1}\leqslant m} \partial_i h_{i_2}\sharp(\partial_{i_2}h_{i_3}\sharp(\cdots(\partial_{i_{k-1}}h_j\sharp X)\cdots)).$$

Thus, we get

$$\mathrm{Tr}(\overline{J}) = \sum_i \mathrm{Tr}\overline{J}_{ii} = \sum_{1 \leqslant i \leqslant m} \mathrm{Tr} \otimes \mathrm{Tr}(\partial_i h_i)$$

and

$$\mathrm{Tr}(\overline{J}^2) = \sum_i \mathrm{Tr}(\overline{J}_{ii}^2) = \sum_{1 \leqslant i,j \leqslant m} \mathrm{Tr} \otimes \mathrm{Tr}(\partial_i h_j \partial_j h_i).$$

We now make the change of variable $A_i \to A_i + \frac{\lambda}{N} h(\mathbf{A})$ to find that

$$\begin{aligned} Z_V^N &= \int e^{-N\mathrm{Tr}(V(\mathbf{A}))} d\mu^N(\mathbf{A}) \\ &= \int e^{-N\mathrm{Tr}(W(A_i + \frac{\lambda}{N} h_i(\mathbf{A})) - W(A_i))} e^{\frac{\lambda}{N} \sum_i \mathrm{Tr}\otimes\mathrm{Tr}(\partial_i h_i)} \times \\ &\qquad e^{-\frac{\lambda^2}{2N^2} \sum_{i,j} \mathrm{Tr}\otimes\mathrm{Tr}(\partial_i h_j \partial_j h_i)} e^{s_N} d\mu_V^N(\mathbf{A}) \end{aligned}$$

with s_N of order $\frac{1}{N}$. The first term can be expanded into

$$W(A_i + \frac{h_i(\mathbf{A})}{N}) - W(A_i) = \frac{1}{N} \sum_i \partial_i W \sharp h_i + \frac{1}{N^2} \sum_{i,j} \partial_i \circ \partial_j W \#(h_i, h_j) + \frac{R_N}{N^3}$$

where R_N is a polynomial in the h_i's and in the X_i's, of degree less than the degree of V minus two in the later. To sum up, the following equality holds

$$\int e^{\lambda Y_N(h_1, \cdots, h_m) - \frac{\lambda^2}{2} C_N(h_1, \cdots h_m) + \frac{1}{N}(\hat{\mu}^N(R_N) + N s_N)} = 1$$

with

$$C_N(h_1, \cdots, h_m) := \hat{\mu}^N \left(\sum_{i,j} \partial_i \circ \partial_j W \#(h_i, h_j) \right) + \hat{\mu}^N \otimes \hat{\mu}^N (\sum_{i,j} \partial_i h_j \partial_j h_i).$$

We can decompose the previous expectation in two terms E_1 and E_2 with

$$E_1 = \mu_V^N \left[\mathbb{1}_{\Lambda_M^N} e^{\lambda Y_N(h_1, \cdots, h_m) - \frac{\lambda^2}{2} C_N(h_1, \cdots, h_m) + \frac{1}{N}(\hat{\mu}^N(R_N) + N s_N)} \right]$$

and

$$E_2 = \mu_V^N \left[\mathbb{1}_{(\Lambda_M^N)^c} e^{\lambda Y_N(h_1, \cdots, h_m) - \frac{\lambda^2}{2} C_N(h_1, \cdots, h_m) + \frac{1}{N}(\hat{\mu}^N(R_N) + N s_N)} \right].$$

We first consider E_1. On $\Lambda_M^N = \{\mathbf{A} : \max_i(\lambda_{\max}^N(A_i)) \leqslant M\}$ the polynomial R_N is uniformly bounded and so $\hat{\mu}^N(R_N) + N s_N$ is of order one, bounded by a constant A_N which goe uniformly to 0 when N goes to infinity. We next show that we can replace $C_N(h_1, \cdots, h_m)$ by its limit $C(h_1, \cdots, h_m)$ in the exponential in E_1. An intermediate step is to replace it by

$$\bar{C}_N(h_1, \cdots, h_m) = \bar{\mu}^N \left(\sum_{i,j} \partial_i \circ \partial_j W \#(h_i, h_j) \right) + \bar{\mu}^N \otimes \bar{\mu}^N (\sum_{i,j} \partial_i h_j \partial_j h_i).$$

In fact, by Lemma 3.2.3 $\hat{\mu}^N(P)$ converges towards its expectation $\overline{\mu}^N(P)$ under $\mu_V^N(1_{\Lambda_M^N}\cdot)$ except on sets with probability of order e^{-N^2} once evaluated at any products of the h_i's and the X_i's (because the Lipschitz constant of finite products of h_i's and X_i's are bounded on Λ_M^N and the error terms $\varepsilon_{P,M}^N$ and $m_{P,M}^N$ can be bounded as we did for polynomials). Hence, we can find a constant $C(M,c) > 0$ such that for N large enough,

$$\begin{aligned}&\mu_V^N\left(\{\left|C_N(h_1,\cdots,h_m) - \bar{C}_N(h_1,\cdots,h_m)\right| > 2\varepsilon\} \cap \Lambda_M^N\right)\\&\leq 2e^{-C(M,c)N^2(\varepsilon_N)^2}.\end{aligned}$$

with $\varepsilon_N = \varepsilon - \varepsilon_{P,M}^N - m_{P,M}^N \sim \varepsilon$. Moreover, $\overline{\mu}^N(P)$ converges to $\mu(P)$ for any polynomial function P (see [GMS06], Theorem 3.1 and 3.4). Since by Weierstrass theorem the h_i's can be uniformly approximated by polynomials on Λ_M^N, uniformly in N, we also know that $\bar{C}_N(h_1,\cdots,h_m)$ converges to $C(h_1,\cdots,h_m)$. Consequently we obtain for some positive constant $C(M,c)$, N large enough,

$$\begin{aligned}&\mu_V^N\left(\{|C_N(h_1,\cdots,h_m) - C(h_1,\cdots,h_m)| > \varepsilon\} \cap \Lambda_M^N\right)\\&\leq 2e^{-C(M,c)N^2(\varepsilon_N)^2}.\end{aligned}$$

Finally, $Y_N(h_1,\cdots,h_m)$ is at most of order N and $C_N(h_1,\cdots,h_m)$ of order one. Hence, the exponential in E_1 is at most of order e^{CN} for some finite constant C. Therefore, if we let

$$E_1' := \mu_V^N\left[\mathbb{1}_{\Lambda_M^N} e^{\lambda Y_N(h_1,\cdots,h_m) - \frac{\lambda^2}{2}C(h_1,\cdots,h_m)}\right],$$

we deduce that

$$\begin{aligned}\left|\log\frac{E_1}{E_1'}\right| &\leq \left|\log e^{A_N}\frac{\mu_V^N\left[\mathbb{1}_{\Lambda_M^N} e^{\lambda Y_N(h_1,\cdots,h_m) - \frac{\lambda^2}{2}(C_N(h_1,\cdots,h_m) - C(h_1,\cdots,h_m))}\right]}{\mu_V^N\left[\mathbb{1}_{\Lambda_M^N} e^{\lambda Y_N(h_1,\cdots,h_m)}\right]}\right|\\&\leq \left|\log(e^{\frac{\lambda^2}{2}\varepsilon_N} + 2e^{CN}e^{-C(M,c)N^2\varepsilon_N^2})\right| + A_N\end{aligned}$$

Letting first N going to infinity and then ε going to zero yields

$$\lim_{N\to\infty}\frac{E_1}{E_1'} = 1$$

Note that this estimate is valid for any M large enough so that Lemma 3.2.3 holds.

Our goal is now to show that for M sufficiently large, E_2 vanishes when N goes to infinity. It would be an easy task if all the quantities where in $\mathcal{C}_{st}^m(\mathbb{C})$ but some derivatives of V appear so that there are polynomials term in the exponential. The idea to pass this difficulty is to make the reverse change of variables. For N bigger than the norm of the h_i's, and with $B_i = A_i + \frac{1}{N}h_i(\mathbf{A})$,

$$\begin{aligned}E_2 &= \mu_V^N\left[\mathbb{1}_{\{\mathbf{A}:\lambda^N(\mathbf{A})\geqslant M\}} e^{\lambda Y_N(h_1,\cdots,h_m) - \frac{\lambda^2}{2}C_N(h_1,\cdots,h_m) + \frac{1}{N}(\hat{\mu}^N(R_N) + Ns_N)}\right]\\&= \mu_V^N\left(\mathbf{B} : \lambda_{\max}^N(\mathbf{A}) \geqslant M\right) \leqslant \mu_V^N\left(\mathbf{B} : \lambda_{\max}^N(\mathbf{B}) \geqslant M - 1\right).\end{aligned}$$

This last quantity goes exponentially fast to 0 for M sufficiently large by Lemma 3.2.2 (Exponential tail of the largest eigenvalue).

Hence, we arrive, for M large enough, at

$$\lim_{N\to\infty}\int \mathbb{1}_{\Lambda_M^N} e^{\lambda Y_N(h_1,\cdots,h_m)} d\mu_V^N \quad = \quad e^{\frac{\lambda^2}{2}C(h_1,\cdots,h_m)}.$$

Since $\mu_V^N(\Lambda_M^N)$ goes to one as N goes to infinity, we find that $Y_N(h_1,\cdots,h_m)$ converges in law under $\mu_V^N(\Lambda_M^N)^{-1}\mu_V^N(.\cap\Lambda_M^N)$ towards a centered Gaussian variable with covariance $C(h_1,\cdots,h_m)$, for any M large enough. For the same reason, we conclude that the same convergence holds under μ_V^N.

□

3.4.2 Central limit theorem for some polynomial functions

We now extend Lemma 3.4.1 to polynomial test functions.

Assume **(H)** and let $P_1,\cdots,P_m$ be in $\mathbb{C}\langle X_1,\cdots,X_m\rangle_{sa}$. Then, the variable

$$Y_N(P_1,\cdots,P_m) = N\sum_{k=1}^m[\hat{\mu}^N\otimes\hat{\mu}^N(\partial_k P_k) - \hat{\mu}^N[(X_k+D_kV)P_k]]$$

converges in law towards a real centered Gaussian variable with covariance

$$C(P_1,\cdots,P_m) = \sum_{k,l=1}^m(\mu\otimes\mu[\partial_k P_l\times\partial_l P_k] + \mu(\partial_l\circ\partial_k V\sharp(P_k,P_l))) + \sum_{k=1}^m\mu(P_k^2).$$

Proof.

Let $P_1,\cdots,P_m$ be self-adjoint polynomials and $h_1^\varepsilon,\cdots,h_m^\varepsilon$ be Stieltjes functionals which approximate $P_1,\cdots P_m$ such as

$$h_i^\varepsilon(\mathbf{X}) = P_i\left(\frac{X_1}{1+\varepsilon X_1^2},\cdots,\frac{X_m}{1+\varepsilon X_m^2}\right).$$

Since $E[Y_N]=0$ by (3.13),

$$Y_N(P_1,\cdots,P_m) = \underline{\hat{\delta}}^N(K_N(P_1,\cdots,P_m))$$

with

$$K_N(P_1,\cdots,P_m) = \sum_{k=1}^m\left(\hat{\mu}^N\otimes I(\partial_k P_k) - (X_k+D_kV)P_k\right)$$

and similarly $Y_N(h_1^\varepsilon,\cdots,h_m^\varepsilon) = \underline{\hat{\delta}}^N(K_N(h_1^\varepsilon,\cdots,h_m^\varepsilon))$. It is not hard to see that

$$\|K_N(h_1^\varepsilon,\cdots,h_m^\varepsilon) - K_N(P_1,\cdots,P_m)\|_{\mathcal{L}}^M \le \varepsilon C(M)$$

for some finite constant $C(M)$ which only depends on M. Hence, we deduce by Lemma 3.2.3 (Concentration inequality) that there exists $m^N_{P,\varepsilon,M}$ and $\varepsilon^N_{P,\varepsilon,M}$ going to zero as N goes to infinity (note here that the control on $m^N_{P,\varepsilon,M}$ and $\varepsilon^N_{P,\varepsilon,M}$ follows exactly the same line as for monomials) such that

$$\mu_V^N \left(\{ |\hat{\delta}^N (K_N(h_1^\varepsilon, \cdots, h_k^\varepsilon) - K_N(P_1, \cdots, P_k)) - m^N_{P,\varepsilon,M}| \geqslant \delta + \varepsilon^N_{P,\varepsilon,M} \right)$$
$$\leqslant e^{-\alpha M N} + e^{-\frac{\delta^2}{2c\varepsilon^2 C(M)^2}}$$

and so for any bounded continuous function $f : \mathbb{R} \to \mathbb{R}$, if ν_{σ^2} is the centered Gaussian law of covariance σ^2,

$$\lim_{N\to\infty} \mu_V^N \Big(f(\underline{\hat{\delta}}^N (K_N(P_1, \cdots, P_k))) \Big) = \lim_{\varepsilon\to 0} \lim_{N\to\infty} \mu_V^N \left(f(\underline{\hat{\delta}}^N (K_N(h_1^\varepsilon, \cdots, h_k^\varepsilon)) \right)$$
$$= \lim_{\varepsilon \to 0} \nu_{C(h_1^\varepsilon, \cdots, h_m^\varepsilon)}(f) = \nu_{C(P_1, \cdots, P_m)}(f)$$

where we used in the second line the version of the lemma for Stieltjes function and in the last line Lemma 3.2.1 (Compact support) to obtain the convergence of $C(h_1^\varepsilon, \cdots, h_m^\varepsilon)$ to $C(P_1, \cdots, P_m)$.

□

Y_N depends on $N\hat{\mu}^N \otimes \hat{\mu}^N$, in which clearly one of the empirical distribution $\hat{\mu}^N$ shall converge to its deterministic limit. This is the content of the next lemma.

Assume **(H)** and let $P_1, \cdots, P_m$ be self-adjoint polynomial functions. Then, the variable

$$Z_N(P_1, \cdots, P_m) = \hat{\delta}^N \left(\sum_{k=1}^m (X_k + D_k V) P_k - \sum_{k=1}^m (\mu \otimes I + I \otimes \mu)(\partial_k P_k) \right)$$

converges in law towards a centered Gaussian variable with covariance

$$C(P_1, \cdots, P_m) = \sum_{k,l=1}^m (\mu \otimes \mu[\partial_k P_l \times \partial_l P_k] + \mu(\partial_l \circ \partial_k V \sharp(P_k, P_l))) + \sum_{k=1}^m \mu(P_k^2).$$

Proof.

The only point is to notice that using (3.2)

$$Y_N(P_1, \cdots, P_m) = \sum_{k=1}^m \left(\hat{\delta}^N \otimes \mu + \mu \otimes \hat{\delta}^N \right) (\partial_k P_k) - \hat{\delta}^N ((X_k + D_k V) P_k) + r_{N,P}$$

with $r_{N,P} = N^{-1} \sum_{k=1}^m \hat{\delta}^N \otimes \hat{\delta}^N (\partial_k P_k)$ of order N^{-1} with probability going to 1 by Lemma 3.2.3 (Concentration inequality) and Property 3.3.1. Thus

$$\begin{aligned} & Y_N(P_1, \cdots, P_m) \\ &= \hat{\delta}^N \left(\sum_{k=1}^m (-(X_k + D_k V) P_k + (I \otimes \mu + \mu \otimes I)(\partial_k P_k)) \right) + r_{N,P} \\ &= -Z_N(P_1, \cdots, P_m) + O(\frac{1}{N}). \end{aligned}$$

This, with the previous lemma, proves the claim.

□

3.4.3 Central limit theorem for all polynomial functions

In the previous part, we have obtained CLT's only for the family of random variables $\hat{\delta}^N(Q)$ with Q in the following subset $\mathcal{F}$ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$

$$\mathcal{F} := \left\{ \sum_{k=1}^{m} (X_k + D_k V) P_k - \sum_{k=1}^{m} (\mu \otimes I + I \otimes \mu)(\partial_k P_k), \forall i, P_i \text{ self-adjoint} \right\}.$$

In this section, we wish to extend it to $\hat{\delta}^N(Q)$ for any self-adjoint polynomial function Q, that is prove Theorem 3.1.3. We have to show a form of density of $\mathcal{F}$ in $\mathbb{C}\langle X_1, \cdots, X_m\rangle$.

The strategy is to see $\mathcal{F}$ as the image of an operator that we will invert. The first operator that comes to mind is

$$\Psi : (P_1, \cdots, P_k) \rightarrow \sum_{k=1}^{m} (X_k + D_k V) P_k - \sum_{k=1}^{m} (\mu \otimes I + I \otimes \mu)(\partial_k P_k)$$

as we immediately have $\mathcal{F} = \Psi(\mathbb{C}\langle X_1, \cdots, X_m\rangle_{sa}, \cdots, \mathbb{C}\langle X_1, \cdots, X_m\rangle_{sa})$.

In order to obtain an operator from $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}$ we will prefer to apply Ψ to $P_k = D_k P$ for all k and for a given P; as we shall see later, $\Psi(D_1 P, \cdots, D_m P)$ is closely related with the projection on functions of the type $\mathrm{Tr}P$ of the operator on the entries $\Delta - \nabla N tr(W).\nabla$ which is symmetric in $L^2(\mu_V^N)$. The resulting operator is a differential operator and hence it would be hard to prove that it is continuous on a fixed space of functions. To avoid this issue and make the argument more readable we have first to divide each monomials of P by its degree.

Then, we define a linear map Σ on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ such that for all monomials q of degree greater or equal to 1

$$\Sigma q = \frac{q}{\deg q}.$$

Moreover, $\Sigma(q) = 0$ if $\deg q = 0$. For later use, we set $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ to be the subset of polynomials P of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}$ such that $P(0, \cdots, 0) = 0$. We let Π be the projection from $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}$ onto $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ (i.e $\Pi(P) = P - P(0, \cdots, 0)$). We now define some operators on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ i.e. from $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ into $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$,

$$\Xi_1 : P \longrightarrow \Pi\left(\sum_{k=1}^{m} \partial_k \Sigma P \sharp D_k V \right)$$

$$\Xi_2 : P \longrightarrow \Pi\left(\sum_{k=1}^{m} (\mu \otimes I + I \otimes \mu)(\partial_k D_k \Sigma P) \right).$$

We denote $\Xi_0 = Id - \Xi_2$ and $\Xi = \Xi_0 + \Xi_1$, where I is the identity on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$. Note that the images Ξ_i's and Ξ are indeed included in $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}$ since V is assumed self-adjoint. With these notations, Lemma 3.4.3, once applied to $P_i = D_i \Sigma P$, $1 \leq i \leq m$, reads

For all P in $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$, $\hat{\delta}^N(\Xi P)$ converges in law to a centered Gaussian variable with covariance

$$\mathcal{C}(P) := C(D_1\Sigma P,\cdots,D_m\Sigma P).$$

Proof.

We have for all tracial state τ, $\tau(\partial_k P\sharp V)=\tau(D_k PV)$ and if P is in $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ (i.e $P(0,\cdots,0)=0$), we have the identity

$$P=\sum_k \partial_k\Sigma P\sharp X_k.$$

Then, as $\hat{\delta}^N$ is tracial and null on constant terms (so that the projection Π can be removed in the definition of Ξ), for all polynomial P,

$$\begin{aligned}\hat{\delta}^N(\Xi P) &= \hat{\delta}^N(P+\sum_{k=1}^m \partial_k\Sigma P\sharp D_kV-\sum_{k=1}^m(\mu\otimes I+I\otimes\mu)(\partial_k D_k\Sigma P))\\ &= \hat{\delta}^N(\sum_{k=1}^m(X_k+D_kV)D_k\Sigma P-\sum_{k=1}^m(\mu\otimes I+I\otimes\mu)(\partial_k D_k\Sigma P))\\ &= Z_N(D_1\Sigma P,\cdots,D_m\Sigma P).\end{aligned}$$

We then use the Lemma 3.4.3 to conclude.

□

To generalize the central limit theorem to all polynomial functions, we need to show that the image of Ξ is dense and to control approximations. If P is a polynomial and q a non-constant monomial we will denote $\lambda_q(P)$ the coefficient of q in the decomposition of P in monomials. We can then define a norm $\|.\|_A$ on $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ for $A>1$ by

$$\|P\|_A=\sum_{\deg q\neq 0}|\lambda_q(P)|A^{\deg q}.$$

In the formula above, the sum is taken on all non-constant monomials. We also define the operator norm given, for T from $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ to $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$, by

$$|||T|||_A=\sup_{\|P\|_A=1}\|T(P)\|_A.$$

Finally, let $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_A$ be the completion of $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ for $\|.\|_A$. We say that T is continuous on $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_A$ if $|||T|||_A$ is finite. We shall prove that Ξ is continuous on $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_A$ with continuous inverse when $\mathbf{t}$ is small. With the previous notations,

1. The operator Ξ_0 is invertible on $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$.
2. There exists $A_0>0$ such that for all $A>A_0$, the operators Ξ_2, Ξ_0 and Ξ_0^{-1} are continuous on $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_A$ and their norm are uniformly bounded for $\mathbf{t}$ in B_η.

3. For all $\varepsilon, A > 0$, there exists $\eta_\varepsilon > 0$ such that for $|\mathbf{t}| < \eta_\varepsilon$, the operator Ξ_1 is continuous on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ and $|||\Xi_1|||_A \leqslant \varepsilon$.
4. For all $A > A_0$, there exists $\eta > 0$ such that for $\mathbf{t} \in B_\eta$, Ξ is continuous, invertible with a continuous inverse on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$. Besides the norms of Ξ and Ξ^{-1} are uniformly bounded for $\mathbf{t}$ in B_η.
5. There exists $C > 0$ such that for all $A > C$, $\mathcal{C}$ is continuous from $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ into $\mathbb{R}$.

Proof.

1. We can write
$$\Xi_0 = I - \Xi_2.$$
Observe that since Ξ_2 reduces the degree of a polynomial by at least 2,
$$P \to \sum_{n \geqslant 0} (\Xi_2)^n(P)$$
is well defined on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ as the sum is finite for any polynomial P. This clearly gives an inverse for Ξ_0.
2. First remark that a linear operator T has a norm less than C with respect to $\|.\|_A$ if and only if for all non-constant monomial q,
$$\|T(q)\|_A \leqslant C A^{\deg q}.$$
Recall that μ is uniformly compactly supported (see Lemma 3.2.1 (Compact support)) and let $C_0 < +\infty$ be such that $|\mu(q)| \leqslant C_0^{\deg q}$ for all monomial q. Take a monomial $q = X_{i_1} \cdots X_{i_p}$, and assume that $A > 2C_0$,
$$\begin{aligned}
\|\Pi\left(\sum_k (I\otimes\mu)\partial_k D_k \Sigma q\right)\|_A &\leqslant p^{-1} \sum_{\substack{k, q = q_1 X_k q_2, \\ q_2 q_1 = r_1 X_k r_2}} \|r_1 \mu(r_2)\|_A \\
\leqslant p^{-1} \sum_{\substack{k, q = q_1 X_k q_2, \\ q_2 q_1 = r_1 X_k r_2}} A^{\deg r_1} C_0^{\deg r_2} &= \frac{1}{p} \sum_{n=0}^{p-1} \sum_{l=0}^{p-2} A^l C_0^{p-l-2} \\
\leqslant A^{p-2} \sum_{l=0}^{p-2} \left(\frac{C_0}{A}\right)^{p-2-l} &\leqslant 2A^{-2}\|q\|_A
\end{aligned}$$
where in the second line, we observed that once $\deg(q_1)$ is fixed, $q_2 q_1$ is uniquely determined and then r_1, r_2 are uniquely determined by the choice of l the degree of r_1. Thus, the factor $\frac{1}{p}$ is compensated by the number of possible decomposition of q i.e. the choice of n the degree of q_1. If $A > 2$, $P \to \Pi\left(\sum_k (I\otimes\mu)\partial_k D_k \Sigma P\right)$ is continuous of norm strictly less than $\frac{1}{2}$. And a similar calculus for $\Pi\left(\sum_k (\mu\otimes I)\partial_k D_k \Sigma\right)$ shows that Ξ_2

is continuous of norm strictly less than 1. It follows immediately that Ξ_0 is continuous. Recall now that

$$\Xi_0^{-1} = \sum_{n\geqslant 0} \Xi_2^n.$$

As Ξ_2 is of norm strictly less than 1, Ξ_0^{-1} is immediately continuous.

3. Let $q = X_{i_1}\cdots X_{i_p}$ be a monomial and let D be the degree of V and $B(\leq Dn)$ the sum of the maximum number of monomials in $D_k V$.

$$\begin{aligned}\|\Xi_1(q)\|_A &\leqslant \frac{1}{p}\sum_{k,q=q_1X_kq_2}\|q_1D_kVq_2\|_A \leqslant \frac{1}{p}\sum_{k,q=q_1X_kq_2}|\mathbf{t}|BA^{p-1+D-1}\\ &= |\mathbf{t}|BA^{D-2}\|q\|_A.\end{aligned}$$

It is now sufficient to take $\eta_\varepsilon < (BA^{D-2})^{-1}\varepsilon$.

4. We choose $\eta < (BA^{D-2})^{-1}|||\Xi_0^{-1}|||_A^{-1}$ so that when $|\mathbf{t}|\leqslant\eta$,

$$|||\Xi_1|||_A|||\Xi_0^{-1}|||_A < 1.$$

By continuity, we can extend Ξ_0, Ξ_1, Ξ_2, Ξ and Ξ_0^{-1} on the space $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_A$. The operator

$$P \to \sum_{n\geqslant 0}(-\Xi_0^{-1}\Xi_1)^n\Xi_0^{-1}$$

is well defined and continuous. And this is clearly an inverse of

$$\Xi = \Xi_0 + \Xi_1 = \Xi_0(I + \Xi_0^{-1}\Xi_1).$$

5. We finally prove that $\mathcal{C}$ is continuous from $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_A$ into $\mathbb{R}$ where we recall that we assumed $A > C_0$. Let us consider the first term

$$\mathcal{C}_1(P) := \sum_{k,l=1}^m \mu\otimes\mu(\partial_k D_l\Sigma P \times \partial_l D_k \Sigma P).$$

Then, we obtain as in the second point of this proof

$$\begin{aligned}|\mathcal{C}_1(P)| &\leqslant 4\sum_{k,l=1}^m\sum_{q,q'}\frac{|\lambda_q(P)||\lambda_{q'}(P)|}{\deg q\deg q'}\sum_{\substack{q=q_1X_kq_2,q'=q'_1X_lq'_2\\ q_2q_1=r_1X_lr_2,q'_2q'_1=r'_1X_kr'_2}} C_0^{\deg q+\deg q'-4}\\ &\leq 4\sum_{q,q'}|\lambda_q(P)||\lambda_{q'}(P)|\deg q\deg q' C_0^{\deg q+\deg q'-4}\\ &\leq 4(\sup_{\ell\geq 0}\ell C_0^{\ell-2}A^{-\ell})^2\|P\|_A^2.\end{aligned}$$

We next turn to show that

$$\mathcal{C}_2(P) := \sum_{k,l=1}^m \mu\left(\partial_k\circ\partial_l V\sharp(D_k\Sigma P, D_l\Sigma P)\right)$$

is also continuous for $\|.\|_A$. In fact, noting that we may assume $V \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ without changing $\mathcal{C}_2$,

$$
\begin{aligned}
|\mathcal{C}_2(P)| &\leqslant \sum_{p,q,q',k,l} |\lambda_p(V)| \sum_{\substack{q,q',p=p_1X_kp_2X_lp_3\\ q=q_1X_kq_2,q'=q'_1X_kq'_2}} \frac{|\lambda_q(P)||\lambda_{q'}(P)|C_0^{\deg p+\deg q+\deg q'-4}}{\deg q \deg q'}\\
&\leq n|\mathbf{t}|D^2 \sum_{q,q'} |\lambda_q(P)||\lambda_{q'}(P)|C_0^{D+\deg q+\deg q'-4}\\
&\leq n|\mathbf{t}|D^2 C_0^{D-4}\|P\|_A^2
\end{aligned}
$$

The continuity of the last term $\mathcal{C}_3(P) = \sum_{i=1}^m \mu\left((D_j\Sigma P)^2\right)$ is obtained similarly.

□

We can compare the norm $\|\cdot\|_A$ to a more intuitive norm, namely $\|\cdot\|_{\mathcal{L}}^M$ defined in (3.7).

We will say that a semi-norm $\mathcal{N}$ is weaker than a semi-norm $\mathcal{N}'$ if and only if there exists $C < +\infty$ such that for all P in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$,

$$\mathcal{N}(P) \leqslant C\mathcal{N}'(P).$$

For $A > M$, the semi-norm $\|.\|_{\mathcal{L}}^M$ restricted to the space $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ is weaker than the norm $\|.\|_A$. **Proof.**

For all P in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$, the following inequalities hold

$$\|P\|_{\mathcal{L}}^M \leqslant \sum_q |\lambda_q(P)|\|q\|_{\mathcal{L}}^M \leqslant \sum_q |\lambda_q(P)| \deg q M^{\deg q} \leqslant (\sup_l l(\frac{M}{A})^l)\|P\|_A.$$

□

To take into account the previous results, we define a new hypothesis **(H')** stronger than **(H)**.

(H') : **(H)** is satisfied, $A - 1 > \max(A_0, M_0, C)$ for the M_0 which appear in the Lemma 3.2.2 (Exponential tail of the largest eigenvalue) and the C which appear in Proposition 3.3.1 Besides, $|\mathbf{t}| \leqslant \eta$ with η as in the fourth point of Lemma 3.4.5 in order that Ξ and Ξ^{-1} are continuous on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ and $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_{A-1}$, and that $\mathcal{C}$ is also continuous for these norms.

The two main additional consequences of this hypothesis are the continuity of Ξ for $\|\cdot\|_A$. The strange condition about the continuity of Ξ on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_{A-1}$ is here for a technical reason which will appear only in the last section on the interpretation of the first order correction to the free energy.

While $(\mathbf{H}')$ is full of conditions, the only important hypothesis is the c-convexity of V. Given such a V, we can always find constants A and η which satisfy the hypothesis. The only restriction will be then that $\mathbf{t}$ is sufficiently small.

We can now prove the general central limit theorem which is up to the identification of the covariance equivalent to Theorem 3.1.3. Assume **(H')**. For all P in $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}$, $\hat{\delta}^N(P)$ converges in law to a centered Gaussian variable γ_P with covariance

$$\sigma^2(P) := \mathcal{C}(\Xi^{-1}\Pi(P)) = C(D_1\Sigma\Xi^{-1}\Pi(P), \cdots, D_m\Sigma\Xi^{-1}\Pi(P)).$$

If $P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle$, $\hat{\delta}^N(P)$ converges to the complex centered Gaussian variable $\gamma_{(P+P^*)/2} + i\gamma_{(P-P^*)/2i}$ (the covariance of $\gamma_{(P+P^*)/2}$ and $\gamma_{(P-P^*)/2i}$ being given by $\sigma^2((P+P^*)/2, (P-P^*)/2i)$ where $\sigma^2(\cdot,\cdot)$ is the bilinear form associated to the quadratic form σ^2).

Proof.

Since $\hat{\delta}^N(P)$ does not depend on constant terms, we can directly take the polynomial $P = \Pi(P)$ in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$. Now, using Lemma 3.4.5 4, we can find an element Q in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ such that $\Xi Q = P$. The space $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ is dense in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ by construction. Thus, there exists a sequence Q_n in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ such that

$$\lim_{n\to\infty} \|Q - Q_n\|_A = 0.$$

Let us define $R_n = P - \Xi Q_n$ in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$.

Now according to Property 3.4.4 for all n, $\hat{\delta}^N(\Xi Q_n)$ converges in law to a Gaussian variable γ_n of variance $\mathcal{C}(Q_n)$ with

$$\mathcal{C}(Q_n) = C(D_1\Sigma Q_n, \cdots, D_m\Sigma Q_n).$$

As $\mathcal{C}$ is continuous by Lemma 3.4.5.4, it can be extended to the space $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ and $\sigma^2(P) = \mathcal{C}(\Xi^{-1}P) = \mathcal{C}(Q) = \lim_n \mathcal{C}(Q_n)$ is well defined. Hence, γ_n converges weakly towards γ_∞, the centered Gaussian law with covariance $\mathcal{C}(Q)$, when n goes to $+\infty$. The last step is to prove the convergence in law of $\hat{\delta}^N(P)$ to γ_∞. We will use the Dudley distance. For $f : \mathbb{R} \to \mathbb{R}$ we define $|f|_{\mathcal{L}} = \|f\|_{\mathcal{L}} + \|f\|_\infty$. The Dudley distance between two measures on $\mathbb{R}$ is

$$\mathcal{D}(\mu, \nu) = \sup_{|f|_{\mathcal{L}} \leqslant 1} |\mu(f) - \nu(f)|.$$

The topology induced by the Dudley metric is the topology of the convergence in law. Below, as a parameter of $\mathcal{D}$ we denote in short $\hat{\delta}^N(P)$ for the law of $\hat{\delta}^N(P)$. We make the following decomposition :

$$\mathcal{D}(\hat{\delta}^N(P), \gamma_\infty) \leqslant \mathcal{D}(\hat{\delta}^N(P), \hat{\delta}^N(\Xi Q_n)) + \mathcal{D}(\hat{\delta}^N(\Xi Q_n), \gamma_n) + \mathcal{D}(\gamma_n, \gamma_\infty). \tag{3.18}$$

By the above remarks, $\mathcal{D}(\hat{\delta}^N(\Xi Q_n), \gamma_n)$ goes to 0 when N goes to $+\infty$ and $\mathcal{D}(\gamma_n, \gamma_\infty)$ goes to 0 when n goes to $+\infty$. We now use the bound on the Dudley distance

$$\mathcal{D}(\hat{\delta}^N(P), \hat{\delta}^N(\Xi Q_n)) \leqslant E[|\hat{\delta}^N(P) - \hat{\delta}^N(\Xi Q_n)| \wedge 1] = E[|\hat{\delta}^N(R_n)| \wedge 1].$$

We control the last term by Lemmas 3.2.3 (Concentration inequality) and 3.2.2 (Exponential tail of the largest eigenvalue) so that for $M \geqslant M_0$,

$$E[|\hat{\delta}^N(R_n)| \wedge 1] \leqslant e^{-\alpha NM} + 2\sqrt{\frac{2\pi}{c}}\|R_n\|_{\mathcal{L}}^M + \varepsilon_{R_n,M}^N + |m_{R_n,M}^N|.$$

But we deduce from Lemma 3.4.6 that since we chose $M < A$, there exists a finite constant C such that

$$\|R_n\|_{\mathcal{L}}^M \leqslant C\|R_n\|_A = C\|\Xi(Q-Q_n)\|_A \leqslant C\||\Xi\||_A\|Q-Q_n\|_A$$

and so $\|R_n\|_{\mathcal{L}}^M$ goes to zero as n goes to infinity. And since $\|R_n\|_{\mathcal{L}}^M$ is finite, $\varepsilon_{R_n,M}^N$ goes to zero. Similarly, using the bound of Lemma 3.2.3 on $m_{P,M}^N$ for P monomial, we find that

$$\begin{aligned} |m_{R_n,M}^N| &\leqslant N\sum_q |\lambda_q(R_n)|\deg(q)(3M^{\deg(q)} + \deg(q)^2)e^{-\alpha MN} \\ &\leq N\sup_{\ell\geq 0}(\ell(3M^\ell+\ell^2)A^{-\ell})\|R_n\|_A e^{-\alpha MN} \end{aligned}$$

goes to zero as N goes to infinity. Thus, $E[|\hat{\delta}^N(R_n)|\wedge 1]$ goes to zero as n and N go to infinity. Putting things together we obtain if we let first N going to $+\infty$ and then n, the desired convergence $\lim_N \mathcal{D}(\hat{\delta}^N(P), \gamma_\infty) = 0$.

□

Note that the convergence in law in Theorem 3.4.7 can be generalized to a convergence *in moments*;

Assume **(H')**. Let P be a self-adjoint polynomial, then $\hat{\delta}^N(P)$ converges in moments to a real centered Gaussian variable with variance $\sigma^2(P)$, i.e for all k in $\mathbb{N}$,

$$\lim_{N\to\infty}\int(\hat{\delta}^N P)^k d\mu_V^N = \frac{1}{\sqrt{2\pi\sigma^2(P)}}\int x^k e^{-\frac{x^2}{2\sigma^2(P)}}dx.$$

Proof.

Indeed, once again we decompose $\int(\hat{\delta}^N P)^k d\mu_V^N$ into $E_1^N + E_2^N$ with

$$E_1^N = \int \mathbb{1}_{\Lambda_M^N}(\hat{\delta}^N P)^k d\mu_V^N \quad E_2^N = \int \mathbb{1}_{(\Lambda_M^N)^c}(\hat{\delta}^N P)^k d\mu_V^N$$

with $M \geqslant M_0$. For E_1, we notice that the law of $\hat{\delta}^N P$ has a sub-Gaussian tail according Lemma 3.2.3 (Concentration inequality). Therefore, we can replace x^k by a bounded continuous function, producing an error independent of N. Applying Theorem 3.4.7 then shows that

$$\lim_{N\to\infty}\int \mathbb{1}_{\Lambda_M^N}(\hat{\delta}^N P)^k d\mu_V^N = \frac{1}{\sqrt{2\pi\sigma^2(P)}}\int x^k e^{-\frac{x^2}{2\sigma^2(P)}}dx.$$

For the second term, we use the trivial bound

$$\begin{aligned} |E_2^N| &\leqslant N^k \int \mathbb{1}_{(\Lambda_M^N)^c}(|\lambda_{max}(\mathbf{A})| + |\mu|(P))^k d\mu_V^N \\ &\leqslant kN^k\int_{\lambda\geqslant M}(\lambda+|\mu|(P))^{k-1}e^{-\alpha\lambda N}d\lambda \end{aligned}$$

which goes to zero as N goes to infinity for all finite k.

□

Another generalization of Theorem 3.4.7 is to extend the set of test functions from polynomials to the completion of $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ for the Lipschitz semi-norm $\|.\|_{\mathcal{L}}^M$. We shall assume that M is strictly greater than C, the constant which bound uniformly the radius of the support of μ according to Lemma 3.2.1 (Compact support), and also greater than M_0, the constant which appear in Lemma 3.2.2 (Exponential tail of the largest eigenvalue) in order to have $\lambda_{\max}(\mathbf{A})$ less than M with high probability. We denote $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_{\mathcal{L}}^M$ the completion of $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ for that norm.

Let us first extend some of the previous quantities to this setting. Recall that for all $N\in\mathbb{N}$, $\sqrt{N}\|P\|_{\mathcal{L}}^M$ is always bigger than $\|\mathrm{Tr}P\|_{\mathcal{L}}^{\Lambda_M^N}$, so that if $\lambda_{max}(\mathbf{A})<M$, $\mathrm{Tr}P(\mathbf{A})$ is well defined. This allows us to define, for P in $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_{\mathcal{L}}^M$, $\hat{\mu}^N(P)=\frac{1}{N}\mathrm{Tr}P(\mathbf{A})$ on Λ_M^N. We can also extend μ to this context by

Let $P\in\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$. Then, with C_0 as in Lemma 3.2.1 (Compact support),

$$|\mu(P)|\leqslant C_0\|P\|_{\mathcal{L}}^{C_0}.$$

Proof.

Let us consider the following norm on $\mathbb{C}\langle X_1,\cdots,X_m\rangle$,

$$\|P\|_\mu:=\limsup_n(\mu((PP^*)^n))^{\frac{1}{2n}}.$$

The completion of $\mathbb{C}\langle X_1,\cdots,X_m\rangle$ for this norm is then a C^*-algebra (see e.g the Gelfand-Neimark-Segal construction). As μ is compactly supported, the norm of the X_i's are bounded by C_0. Besides, for all P,

$$|\mu(P)|\leqslant\|P\|_\mu.$$

Therefore, we can write

$$\begin{aligned}|\mu(P)|=|\mu(P(\mathbf{X}))-\mu(P(0))|&=\left|\mu\left(\int_0^1\sum_{k=1}^m(D_kP)(s\mathbf{X})X_kds\right)\right|\\&\leqslant\int_0^1\left|\mu\left(\sum_{k=1}^m D_kP(s\mathbf{X})X_k\right)\right|ds\\&\leqslant\int_0^1\left(\sum_{k=1}^m\mu(D_kP(s\mathbf{X})D_kP(s\mathbf{X})^*)\right)^{\frac12}\left(\sum_{k=1}^m\mu(X_k^2)\right)^{\frac12}ds\\&\leqslant C_0\sup_{\substack{\mathcal{A}\,C^*-\text{algebra}\\ x_i=x_i^*\|x_i\|\leqslant C_0}}\left(\sum_{k=1}^m\|D_kP(x_1,\cdots,x_m)\|_{\mathcal{A}}^2\right)^{\frac12}=C_0\|P\|_{\mathcal{L}}^{C_0}.\end{aligned}$$

□

Thus, μ extends to $\mathbb{C}\langle X_1,\cdots,X_m\rangle_{\mathcal{L}}^M$. It is a natural question to study the behavior of

$$\hat{\delta}^N(P):=N(\hat{\mu}^N(P)-\mu(P))1_{\Lambda_M^N}$$

for P in $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle_{\mathcal{L}}^M$, the completion of $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$ for $\|\cdot\|_{\mathcal{L}}^M$. Assume **(H')** and let M be bigger than C_0 and M_0.

1. σ^2 is continuous for $\|.\|_{\mathcal{L}}^M$ and so extends to $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_{\mathcal{L}}^M$.
2. For all P in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_{\mathcal{L}}^M$, $\hat{\delta}^N(P)$ converges in law to a Gaussian variable with variance $\sigma^2(P)$.

Proof.

We take a sequence of polynomials S_n which converges to P for the norm $\|.\|_{\mathcal{L}}^M$. Let $R_n = P - S_n$ be the rest. For all n, $\hat{\delta}^N(S_n)$ converges to a centered Gaussian variable γ_n of variance $\sigma^2(S_n)$.

Let us show that σ^2 is continuous for $\|.\|_{\mathcal{L}}^M$. Let P be a polynomial, and M sufficiently large,

$$\sigma^2(P) = \lim_N E[\underline{\hat{\delta}}^N(P)^2] = \lim_N E[\mathbb{1}_{\Lambda_M^N}\underline{\hat{\delta}}^N(P)^2].$$

The first equality comes from the previous corollary about the convergence in moments as well as Lemma 3.3.1 which allows to recenter with respect to the mean rather than the limit, and the second equality comes from Lemma 3.2.2 (Exponential tail of the largest eigenvalue). Now by Lemma 3.2.3 (Concentration inequality), as $\|P\|_{\mathcal{L}}^M$ controls the Lipschitz norm of $\frac{1}{N}\mathrm{Tr}(P)$,

$$\begin{aligned}\lim_{N\to\infty} \mu_V^N[\mathbb{1}_{\Lambda_M^N}(\underline{\hat{\delta}}^N(P))^2] &= 2\lim_{N\to\infty}\int_0^\infty \varepsilon\mu_V^N(\Lambda_M^N \cap \{|\underline{\hat{\delta}}^N(P)| > \varepsilon\})d\varepsilon \\ &\leqslant \int_0^\infty 2\varepsilon e^{-\frac{c\varepsilon^2}{2(\|P\|_{\mathcal{L}}^M)^2}} d\varepsilon = \frac{4}{c}(\|P\|_{\mathcal{L}}^M)^2\end{aligned}$$

where we used that $m_{M,P}^N$ and $\varepsilon_{M,P}^N$ of Lemma 3.2.3 go to zero as N goes to infinity since P is a polynomial. Thus, the quadratic form σ^2 is continuous for $\|.\|_{\mathcal{L}}^M$ and can be extended on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_{\mathcal{L}}^M$. This implies that $\sigma^2(S_n)$ converges to $\sigma^2(P)$. The rest of the proof is exactly as that of Theorem 3.4.7 and we omit it.

□

Note that by Lemma 3.4.5 the norm $\|.\|_A$ is stronger than the norm $\|.\|_{\mathcal{L}}^M$ so that we can use this corollary to extend out central limit theorem on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ and by continuity of σ^2, on this space the formula

$$\sigma^2(P) := \mathcal{C}(\Xi^{-1}P) = C(D_1\Sigma\Xi^{-1}P, \cdots, D_m\Sigma\Xi^{-1}P)$$

remains valid.

3.5 Identification of the variance

3.5.1 Exact formula

We shall provide here a more tractable formula for the variance $\sigma^2(P)$ of the limiting Gaussian distribution found in Theorem 3.4.7. Note that for all polynomials P, Q, $\hat{\delta}^N(P+Q)$

converges to γ_{P+Q}. Thus, $\{\gamma_P | P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^{sa}\} = \{\gamma_P | P \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle\}$ has a natural structure of Gaussian space. In this space all elements are centered and the covariance function is given, for $P, Q \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ by

$$\sigma^2(P,Q) = \mathcal{C}(\Xi^{-1}P, \Xi^{-1}Q) = C(\mathbf{D}\Sigma\Xi^{-1}P, \mathbf{D}\Sigma\Xi^{-1}Q)$$

where $\mathbf{D}$ is the cyclic gradient defined by $\mathbf{D}P = (D_1 P, \cdots, D_m P)$ and

$$\begin{aligned} & C(P_1, \cdots, P_m, Q_1, \cdots, Q_m) \\ = & \sum_{k,l=1}^{m} \left(\mu \otimes \mu[\partial_k P_l \times \partial_l Q_k] + \mu\left(\partial_l \circ \partial_k V \sharp (P_k, Q_l)\right)\right) + \sum_{k=1}^{m} \mu(P_k Q_k). \end{aligned}$$

We now give a more explicit formula for $\sigma^2(P,Q)$. We therefore need to study C and the commutation relations of the cyclic gradient and Ξ.

Let us define the following operators on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$

$$\bar{\Xi}_1 : P \longrightarrow \sum_{k=1}^{m} \partial_k P \sharp D_k V \qquad \bar{\Xi}_2 : P \longrightarrow \sum_{i=1}^{m} (I \otimes \mu) M \circ \partial_i^2 P$$

where $M(A \otimes B \otimes C) = AC \otimes B$. We also define $\bar{\Xi}_0 = \Sigma^{-1} - \bar{\Xi}_2$ and $\bar{\Xi}$ the operator on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ given by $\bar{\Xi}P = \bar{\Xi}_0 P + \bar{\Xi}_1 P$ if $P \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle$. We extend $\bar{\Xi}$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ by setting $\bar{\Xi}1 = 0$. We set, for $i = 0, 1, 2$ or nothing, $\bar{\mathbf{\Xi}}_{\mathbf{i}}$ the operator on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^m$ such that $\bar{\mathbf{\Xi}}_{\mathbf{i}}(P_1, \cdots, P_m) = (\bar{\Xi}_i P_1, \cdots, \bar{\Xi}_i P_m)$.

For all $l \in \{1, \cdots, m\}$, for all $P \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle$, the following equalities hold

$$\begin{aligned} D_l \Sigma^{-1} P &= \Sigma^{-1} D_l P + D_l P, \\ D_l \Xi_1 P &= \bar{\Xi}_1 D_l \Sigma P + \sum_{i=1}^{m} \partial_i D_l V \sharp D_i \Sigma P, \\ D_l \Xi_2 P &= \bar{\Xi}_2 D_l \Sigma P. \end{aligned}$$

Besides, let $\mathrm{Hess}(V) : \mathbb{C}\langle X_1, \cdots, X_m\rangle^m \to \mathbb{C}\langle X_1, \cdots, X_m\rangle^m$ be given by

$$\mathrm{Hess}(V)(v)_l = \sum_{i=1}^{m} \partial_i D_l V \sharp v_i.$$

Then, for any $(P_1, \cdots, P_m) \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^m$, with I the identity on $\mathbb{C}\langle X_1, \cdots, X_m\rangle^m$, the following relation of commutation relation holds

$$\mathbf{D}\Xi = (I + \mathrm{Hess}(V) + \bar{\mathbf{\Xi}})\mathbf{D}\Sigma.$$

Proof.

By linearity, it is sufficient to prove these equalities for a monomial $P = X_{i_1} \cdots X_{i_p}$. Moreover, the projection Π onto $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ is irrelevant in the definition of the operators Ξ_i's since they are followed by derivatives.

$$D_l \Sigma^{-1} P = p D_l P = (p-1) D_l P + D_l P = \Sigma^{-1} D_l P + D_l P.$$

To prove the second equality, write

$$D_l \Xi_1 P = D_l \sum_{i, \Sigma P = q_1 X_i q_2} q_1 D_i V q_2$$

then D_l can differentiate q_1, q_2 or $D_i V$ so that

$$\begin{aligned} D_l \Xi_1 P &= \sum_{i, \Sigma P = r_1 X_l r_2 X_i r_3} r_2 D_i V r_3 r_1 + \sum_{i, \Sigma P = r_1 X_i r_2 X_l r_3} r_3 r_1 D_i V r_2 \\ &+ \sum_{i, \Sigma P = q_1 X_i q_2, D_i V = q_3 X_l q_4} q_4 q_2 q_1 q_3. \end{aligned}$$

The sum of the first two terms gives exactly $\bar{\Xi}_1 D_l \Sigma P$ and the last one is

$$\sum_{i, D_i V = q_3 X_l q_4} q_4 D_i P q_3 = \partial_i D_l V \sharp D_i \Sigma P.$$

Note that if P is a monomial,

$$\Xi_2 P = 2 \sum_{i, \Sigma P = q_1 X_i q_2 X_i q_3} \{\mu[q_1 q_3] q_2 + \mu[q_2] q_1 q_3\}$$

so that we obtain

$$D_l \Xi_2 P = 2 \sum_{i, \Sigma P = q_1 X_l q_1' X_i q_2 X_i q_3} \mu[q_2] q_1' q_3 q_1$$

$$+2 \sum_{i, \Sigma P = q_1 X_i q_2 X_l q_2' X_i q_3} \mu[q_3 q_1] q_2' q_2 + 2 \sum_{i, \Sigma P = q_1 X_i q_2 X_i q_3 X_l q_3'} \mu[q_2] q_3' q_1 q_3.$$

Similar algebra shows that

$$\bar{\Xi}_2 D_l \Sigma P = 2 \sum_{i, D_l \Sigma P = q_1 X_i q_2 X_i q_3} \{\mu(q_2) q_3 q_1\} = D_l \Xi_2 P.$$

Finally, the last point we only have to sum the previous equalities for $P \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ and all $l \in \{1, \cdots, m\}$,

$$\begin{aligned} D_l \Xi \Sigma^{-1} P &= (D_l + \Sigma^{-1} D_l - \bar{\Xi}_2 D_l + \bar{\Xi}_1 D_l)(P) + \sum_{i=1}^{m} \partial_i D_l V \sharp D_i P \\ &= [(I + \mathrm{Hess}(V) + \bar{\mathbf{\Xi}}) \mathbf{D} P]_l \end{aligned}$$

□

Thus we can deduce an expression for $\mathbf{D} \circ \Sigma\Xi^{-1}$. The operator $\bar{\Xi}$ is a symmetric non negative operator in $L^2(\mu)$. Let $\cdot^t$ be the involution on $\mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ defined by $(A \otimes B)^t = B \otimes A$, then for any $(P, Q) \in \mathbb{C}\langle X_1, \cdots, X_m\rangle^m$

$$\mu(P\bar{\Xi}Q) = \sum_{k=1}^{m} \mu \otimes \mu(\partial_k P \times [\partial_k Q]^t).$$

$\bar{\boldsymbol{\Xi}}$ is thus non negative in $L^2(\mu)^m$ equipped with the scalar product $\langle \mathbf{P}, \mathbf{Q}\rangle = \sum_{i=1}^m \mu(P_i Q_i^*)$. $\frac{1-c}{2}I + \text{Hess V}$ is a non negative operator in the sense that for every polynomials $P_1, \cdots, P_m$,

$$\sum_{i=1}^{m} (\text{Hess}(V)P)_i P_i^* \geqslant -(1-c)\sum_{i=1}^{m} P_i P_i^*.$$

Thus, $(I + \text{Hess V} + \bar{\boldsymbol{\Xi}})$ is symmetric definite positive in $L^2(\mu)^m$ and is invertible. If we consider $\mathbf{D}\Sigma\Xi^{-1}$ as a continuous operator from $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ into $L^2(\mu)^m$, the following rule of commutation holds

$$\mathbf{D}\Sigma\Xi^{-1} = (I + \text{Hess V} + \bar{\boldsymbol{\Xi}})^{-1}\mathbf{D}.$$

Proof.

Here, it is easier to come back to the origin of the problem. The idea is that the operator $\bar{\Xi}$ is a projection of the Laplace operator

$$L = \frac{1}{N}\sum_{k=1}^{m}\sum_{i,j=1}^{N} e^{N\text{Tr}(V + 2^{-1}\sum X_l^2)}\partial_{x_{ij}^k} e^{-N\text{Tr}(V+2^{-1}\sum X_l^2)}\partial_{x_{ji}^k}$$

on functions of the matrices. Here $\partial_{x_{ji}^k}$ is a notation and stands for

$$\frac{1}{2}(\partial_{\Re e x_{ji}^k} + \sqrt{-1}\partial_{\Im m x_{ji}^k}).$$

In fact, if we take P a polynomial function,

$$LP = \frac{1}{N}[\sum_{k=1}^{m}\sum_{i,j=1}^{N} N(-D_k V - X_k)_{ji}\partial_k P\sharp\Delta_{ji} + \sum_{k=1}^{m}\sum_{i,j=1}^{N}\partial_k \circ \partial_k P\sharp(\Delta_{ij}, \Delta_{ji})]$$

$$= \sum_{k=1}^{m}\partial_k P\sharp(-D_k V - X_k) + \sum_{k=1}^{m}(I \otimes \hat{\mu}^N)M \circ (\partial_k \circ \partial_k)P$$

with Δ_{ij} the matrix with null entries except in (i, j) where it is equal to 1. As a consequence, we deduce from the convergence of $\hat{\mu}^N$ towards μ that for all polynomials P, Q

$$\lim_{N\to\infty}\int \frac{1}{N}\text{Tr}(QLP)d\mu_N^V = -\mu(Q\bar{\Xi}P).$$

But now, by integration by parts, we obtain

$$\begin{aligned}\int \frac{1}{N}\mathrm{Tr}(QLP)d\mu_N^V &= \int \frac{1}{N^2}\sum_{\alpha,\beta=1}^{N} Q_{\alpha,\beta}(LP)_{\beta,\alpha}d\mu_N^V \\ &= -\int \frac{1}{N^2}\sum_{k=1}^{m}\sum_{i,j,\alpha,\beta=1}^{N} \partial_{x_k^{ij}}Q_{\alpha,\beta}\partial_{x_k^{ji}}P_{\beta,\alpha}d\mu_N^V \qquad (3.19)\\ &= -\int \frac{1}{N^2}\sum_{k=1}^{m}\sum_{i,j,\alpha,\beta=1}^{N} [\partial_k Q\sharp\Delta_{ij}]_{\alpha,\beta}[\partial_k P\sharp\Delta_{ij}]_{\beta,\alpha}d\mu_N^V \\ &= -\sum_{k=1}^{m}\int \hat{\mu}^N\otimes\hat{\mu}^N(\partial_k P\times(\partial_k Q)^t)d\mu_N^V\end{aligned}$$

which converges as N goes to infinity towards

$$\sum_{k=1}^{m}\mu\otimes\mu(\partial_k P\times(\partial_k Q)^t) = \mu(Q\bar{\Xi}P).$$

This shows that $\bar{\Xi}$ is symmetric and non negative (since if $Q = P^*$, the right hand side of (3.19) is clearly non positive for all N). Similarly, remark that

$$(HessVP)_l = \sum_i \partial_i D_l V\sharp P_i.$$

Once estimated at a finite matrix, it is easily seen that

$$\mathrm{Tr}(\partial_i D_l V\sharp P_i P_l^*) = \sum_{\alpha,\beta,\gamma,\delta}(\partial_{x^i_{\alpha\beta}}\partial_{x^l_{\gamma\delta}}\mathrm{Tr}V)(P_i)_{\beta\alpha}(\overline{P_l})_{\delta\gamma}$$

and so the positivity of Hess is deduced at finite N from the convexity of V which, by definition, is the positivity of the Hessian of $\mathrm{Tr}(V)$ in any finite dimension. As a consequence, the operator $I + \mathrm{Hess}(V) + \bar{\mathbf{\Xi}}$ is invertible on $\mathbb{C}\langle X_1,\cdots,X_m\rangle^m \subset (L^2(\mu))^m$. We then obtain the commutation relation by using the third point of the previous Lemma.

□

This gives us an explicit formula for σ^2.

For all P, Q in $\mathbb{C}_0\langle X_1,\cdots,X_m\rangle$, for all $1 \leqslant k, l \leqslant m$, the following identities holds

$$\mu\otimes\mu[\partial_k D_l P\times\partial_l D_k Q] = \mu\otimes\mu[\partial_k D_l P\times[\partial_k D_l Q]^t],$$

$$C(\mathbf{D}P,\mathbf{D}Q) = \sum_{i=1}^{m}\mu(D_i P[(I+\mathrm{Hess}V+\bar{\mathbf{\Xi}})\mathbf{D}Q]_i),$$

$$\sigma^2(\Xi P, Q) = \sum_{i=1}^{m}\mu(D_i\Sigma P D_i Q),$$

$$\sigma^2(P,Q) = \sum_{i=1}^{m}\mu(D_i P(I+\mathrm{Hess}V+\bar{\mathbf{\Xi}})^{-1}D_i Q).$$

Proof.
An elementary computation shows that for all polynomials P,

$$\partial_k D_l P = (\partial_l D_k P)^t.$$

To prove the second equality, recall that

$$C(\mathbf{D}P, \mathbf{D}Q) = \sum_{k,l=1}^{m} \left(\mu \otimes \mu[\partial_k D_l P \times \partial_l D_k Q] + \mu\left(\partial_l \circ \partial_k V \sharp (D_k P, D_l Q)\right)\right) + \sum_{k=1}^{m} \mu(D_k P D_k Q).$$

The third term can be directly written $\sum_{i=1}^{m} \mu(D_i P[\mathbf{D}Q]_i)$. For the second term we use the first equality and Lemma 3.5.2 :

$$\sum_{k,l=1}^{m} \mu \otimes \mu[\partial_k D_l P \times \partial_l D_k Q] = \sum_{i=1}^{m} \mu(D_i P \bar{\Xi} D_i Q).$$

Finally we only need to check if the two terms in the second derivative of V coincide, but this is clear by the trace property :

$$\sum_{k,l=1}^{m} \mu\left(\partial_l \circ \partial_k V \sharp (D_k P, D_l Q)\right) = \sum_{i,j=1}^{m} \mu(D_i P \partial_j D_i V \sharp D_j Q)$$

For the last points we only have to use the commutation rule of Lemma 3.5.2 and the previous point.

$$\begin{aligned}\sigma^2(\Xi P, Q) &= C(\mathbf{D}\Sigma P, \mathbf{D}\Sigma\Xi^{-1}Q) \\ &= \sum_{i=1}^{m} \mu(D_i \Sigma P[(I + \mathrm{Hess}V + \bar{\Xi})\mathbf{D}\Sigma\Xi^{-1}Q]_i) \\ &= \sum_{i=1}^{m} \mu(D_i \Sigma P D_i Q)\end{aligned}$$

The last point is proved with the same technique.

$\square$

3.5.2 Combinatorial interpretation

It was shown in [GMS06] that for small $\mathbf{t}$'s the limit measure μ has a combinatorial interpretation. More precisely let $V = \sum_i t_i q_i$ with some monomials q_i. Note that in order to have a self-adjoint potential, in the decomposition in monomials, the coefficient of a monomial must be the complex conjugate of the coefficient of its adjoint.

We define a set of colors as the set $\{1, \cdots, m\}$ and associate to each monomial $q = X_{i_1} \cdots X_{i_p}$ a star (i.e. a vertex with some half-edges pointing out of it) of p half-edges which are in the clockwise order respectively of color i_1, i_2,. . .i_p. Besides we distinguish the first half-edge so that we clearly obtain a bijection between monomials and stars. We will say that the star is of type q if it comes from a monomial q in that way. Note that a star can equivalently be represented by an annulus with ordered colored dots and a distinguished dot.

Given a set of such stars embedded in the sphere, we can construct some graphs among them simply by gluing pairwise different half-edges of the same color and such that the resulting edges do not cross each other. We call a graph obtained in this way a planar graph. Two planar graphs are said to be equivalent if there is an homeomorphism of the sphere which fix each star and take the first graph on the second. A map is a class of equivalence of connected planar graphs for the relation of homomorphism. We now define

$$\mathcal{M}_{k_1,\cdots,k_n}(P) = \sharp \left\{ \begin{array}{c} \text{maps with } k_i \text{ stars of type } q_i \\ \text{and one of type } P \end{array} \right\}$$

and

$$\mathcal{M}_{k_1,\cdots,k_n}(P,Q) = \sharp \left\{ \begin{array}{c} \text{maps with } k_i \text{ stars of type } q_i \\ \text{one of type } P \text{ and one of type } Q \end{array} \right\}$$

These quantities are only defined for P and Q monomials but we immediately extend them by linearity to arbitrary polynomials P and Q. By convention, the star associated to the monomial 1 is empty so that $\mathcal{M}_{k_1,\cdots,k_n}(P, 1) = 0$.

In [GMS06] section 3.2 there is the following relation between the limit measure and the enumeration of planar graphs. There exists $\eta > 0$ such that for $\mathbf{t} \in B_\eta$, for all polynomial P,

$$\mu(P) = \sum_{k_1,\cdots,k_n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{k_1,\cdots,k_n}(P).$$

We now prove that there is a similar link between the variance $\sigma^2(P)$ which appears in our central limit theorem and the generating function of the $\mathcal{M}_{k_1,\cdots,k_n}(P, Q)$. We define

$$\mathcal{M}(P,Q) = \sum_{k_1,\cdots,k_n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_{k_1,\cdots,k_n}(P,Q).$$

We shall prove that $\sigma^2(P, Q)$ and $\mathcal{M}(P, Q)$ satisfy the same kind of induction relation. For all monomials P, Q and all k,

$$\begin{aligned}
&\mathcal{M}_{k_1,\cdots,k_n}(X_k P, Q)\\
&= \sum_{0\leqslant p_i\leqslant k_i} \sum_{P=RX_kS} \prod_i C_{k_i}^{p_i} \mathcal{M}_{p_1,\cdots,p_n}(R,Q)\mathcal{M}_{k_1-p_1,\cdots,k_n-p_n}(S)\\
&+ \sum_{0\leqslant p_i\leqslant k_i} \sum_{P=RX_kS} \prod_i C_{k_i}^{p_i} \mathcal{M}_{p_1,\cdots,p_n}(S,Q)\mathcal{M}_{k_1-p_1,\cdots,k_n-p_n}(R)\\
&+ \sum_{0\leqslant j\leqslant n} k_j \mathcal{M}_{k_1,\cdots,k_j-1,\cdots,k_n}(D_k V P, Q) + \mathcal{M}_{k_1,\cdots,k_n}(D_k QP)
\end{aligned}$$

and

$$\mathcal{M}(X_k P, Q) = \mathcal{M}((I\otimes\mu + \mu\otimes I)\partial_k P) - \mathcal{M}(D_k VP, Q) + \mu(D_k QP). \tag{3.20}$$

Besides there exists $\eta > 0$ so that there exists $R < +\infty$ such that for all monomials P and Q, all $\mathbf{t} \in B(0,\eta)$,

$$|\mathcal{M}(P,Q)| \leqslant R^{\deg P + \deg Q}.$$

Proof.

The proof is very close to that given of Theorem 2.2 in [GMS06] which explain the decomposition of planar maps with one root. We look at the first half-edge with color k corresponding to X_k in $X_k P$.

1. The first possibility is that the half-edge is glued to another half-edge of $P = RX_kS$. It cuts P in two monomials R and S and it occurs for all decomposition of P into $P = RX_kS$ which is exactly what does D. Then either the component R is linked to Q and to p_i stars of type q_i for each i, this leads to

$$\prod_i C_{k_i}^{p_i} \mathcal{M}_{p_1,\cdots,p_n}(R,Q)\mathcal{M}_{k_1-p_1,\cdots,k_n-p_n}(S)$$

   possibilities or we are in the symmetric case with S linked to Q in place of R.
2. The second case occurs when the half-edge is glued to a star of type q_j for a given j then first we have to choose between the k_j vertices of this type then we contract the edges arising from this gluing to form a star of type $D_i q_j P_1$, there is

$$k_j \mathcal{M}_{k_1,\cdots,k_j-1,\cdots,k_n}(D_k q_j P, Q)$$

   choices.
3. The last case is that the half-edge can be glued with the star associated to $Q = RX_iS$. We contract this half-edge and obtain a star of type $D_k QP$. This leads to

$$\mathcal{M}_{k_1,\cdots,k_n}(D_k QP)$$

   possibilities.

We can now sum on the k's to obtain the relation on $\mathcal{M}$.

Finally, to show the last point of the proposition, we only have to prove that there exists $A > 0, B > 0$ such that for all k's, for all monomials P and Q,

$$\frac{\mathcal{M}_{k_1,\cdots,k_n}(P,Q)}{\prod_i k_i!} \leqslant A^{\sum_i k_i} B^{\deg P + \deg Q}.$$

This follows easily by induction over the degree of P with the previous relation on the $\mathcal{M}$ since we have proved such a control for $\mathcal{M}_{k_1,\cdots,k_n}(Q)$ in [GMS06].

□

We can now relate the variance and the generating function for the enumeration of planar maps with two prescribed vertices.

Assume **(H')** with η small enough. Then, for all polynomials P, Q,

$$\sigma^2(P,Q) = \mathcal{M}(P,Q).$$

Proof.

First we transform the relation on $\mathcal{M}$. We use (3.20) with $P = D_k \Sigma R$ to deduce

$$\mathcal{M}(\Xi R, Q) = \sum_k \mu(D_k Q D_k \Sigma R).$$

Let us define $\Delta = \sigma^2 - \mathcal{M}$. Then according to (3.5.2) and the previous property, Δ is compactly supported and for all polynomials P and Q,

$$\Delta(\Xi P, Q) = 0.$$

Moreover, with $\mathcal{M}(1,Q) = 0 = \sigma^2(1,Q)$,

$$\Delta(1,Q) = 0.$$

To conclude we have to invert one more time the operator Ξ. For a polynomial P we take as in the proof of the central limit theorem, a sequence of polynomial S_n which goes to $S = \Xi^{-1} P$ in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$. Then, write

$$\Delta(P,Q) = \Delta(\Xi(S_n + S - S_n), Q) = \Delta(\Xi(S - S_n), Q)$$

But by continuity of Ξ, $\Xi(S - S_n)$ goes to 0 for the norm $\|.\|_A$. We can always assume $A \geqslant R$ if η is small enough. Moreover, because Δ is compactly supported, Δ is continuous for $\|.\|_A$, and so $\Delta(\Xi(S - S_n), Q)$ goes to zero when n goes to $+\infty$. This proves the theorem.

□

3.6 Second order correction to the free energy

We now deduce from the Central Limit Theorem the precise asymptotics of $N\overline{\delta}^N(P)$ and then compute the second order correction to the free energy.

Let ϕ_0 and ϕ be the linear forms on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$ which are given, if P is a monomial by

$$\phi_0(P) = \sum_{i=1}^{m} \sum_{P=P_1X_iP_2X_iP_3} \sigma^2(P_3P_1, P_2) \tag{3.21}$$

and $\phi = \phi_0 \circ \Sigma$.

Assume $(\mathbf{H}')$. Then, for any P in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$

$$\lim_{N\to\infty} N\overline{\delta}^N(P) = \phi(\Xi^{-1}\Pi(P)).$$

Proof.

Again, we base our proof on the finite dimensional Schwinger-Dyson equation (3.13) which, after centering, and since we can always assume that $P \in \mathbb{C}_0\langle X_1, \cdots, X_m\rangle$, reads for $i \in \{1, \cdots, m\}$,

$$N^2\mu_V^N\big((\hat{\mu}^N - \mu)[(X_i + D_iV)P - (I\otimes\mu + \mu\otimes I)\partial_iP\big) = \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\partial_iP)\right).$$

Taking $P = D_i\Sigma P$ and summing over $i \in \{1, \cdots, m\}$, we thus have

$$N^2\mu_V^N\left((\hat{\mu}^N - \mu)(\Xi P)\right) = \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\sum_{i=1}^{m}\partial_i \circ D_i\Sigma P)\right). \tag{3.22}$$

By Corollary 3.4.8 and Lemma 3.5.1, we see that

$$\lim_{N\to\infty} \mu_V^N\left(\hat{\delta}^N \otimes \hat{\delta}^N(\sum_{i=1}^{m}\partial_i \circ D_i\Sigma P)\right) = \phi(P)$$

which gives the asymptotics of $N\overline{\delta}^N(\Xi P)$ for all P.

To generalize the result to arbitrary P, we proceed as in the proof of the full central limit theorem. We take a sequence of polynomials Q_n which goes to $Q = \Xi^{-1}P$ when n go to ∞ for the norm $\|.\|_A$. We denote $R_n = P - \Xi Q_n = \Xi(Q - Q_n)$. Note that as P and Q_n are polynomials then R_n is also a polynomial. Then we write

$$N\overline{\delta}^N(P) = N\overline{\delta}^N(\Xi Q_n) + N\overline{\delta}^N(R_n)$$

According to Property 3.3.1, for any monomial P of degree less than $\varepsilon N^{\frac{2}{3}}$,

$$|N\overline{\delta}^N(P)| \leqslant C^{\deg(P)}.$$

So if we take the limit in N, for any monomial P,

$$\limsup_N |N\overline{\delta}^N(P)| \leqslant C^{\deg(P)}$$

and if P is a polynomial,

$$\limsup_N |N\overline{\delta}^N(P)| \leqslant \|P\|_C \leqslant \|P\|_A.$$

The last inequality comes from the hypothesis **(H')** which require $C < A$.

We now fix n and let N goes to infinity,

$$\limsup_N |N\overline{\delta}^N(P - \Xi Q_n)| \leqslant \limsup_N |N\overline{\delta}^N(R_n)| \leqslant \|R_n\|_A.$$

If we now let n goes to infinity, the right term vanishes and we are left with

$$\lim_N N\overline{\delta}^N(P) = \lim_n \lim_N N\overline{\delta}^N(Q_n) = \lim_n \phi(Q_n).$$

It is now sufficient to show that ϕ is continuous for the norm $\|.\|_A$. But it can be checked easily that $P \to \sum_{i=1}^m \partial_i \circ D_i P$ is continuous from $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$ to $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_{A-1}$ and σ^2 is continuous for $\|.\|_{A-1}$ due to the technical hypothesis in **(H')**. This proves that ϕ is continuous and then can be extended on $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle_A$. Thus

$$\lim_N N\overline{\delta}^N(P) = \lim_n \phi(Q_n) = \phi(Q).$$

□

This result allows us to estimate the first order correction to the free energy. Assume $(\mathbf{H}')$, then the following asymptotics holds

$$\log Z^N_{V_\mathbf{t}} = N^2 F^0(V_\mathbf{t}) + F^1(V_\mathbf{t}) + o(1)$$

with

$$F^0(V_\mathbf{t}) = -\int_0^1 \mu_{\alpha\mathbf{t}}(\sum_{i=1}^n t_i q_i) d\alpha$$

and

$$F^1(V_\mathbf{t}) = -\int_0^1 \phi_{\alpha\mathbf{t}}(\Xi^{-1}_{\alpha\mathbf{t}} \sum_{i=1}^n t_i q_i) ds$$

with $\Xi_{\alpha\mathbf{t}}$ (resp. $\phi_{\alpha\mathbf{t}}$) the operator Ξ (resp. the linear form ϕ) corresponding to the potential $V_{\alpha\mathbf{t}} = \alpha V_\mathbf{t}$ with parameters $\alpha\mathbf{t}$. **Proof.**

Remark that for $i \in \{1, \cdots, n\}$,

$$\partial_\alpha \log Z^N_{\alpha V_\mathbf{t}} = -N^2 \mu^N_{\alpha V_\mathbf{t}}(\hat{\mu}^N(\sum_{i=1}^n t_i q_i))$$

so that we can write

$$\log Z^N_{V_{\mathbf{t}}} = N^2 F^0(V_{\mathbf{t}}) - \int_0^1 [N\bar{\delta}^N_{\alpha\mathbf{t}}(\sum t_i q_i)] d\alpha \qquad (3.23)$$

Since for all $\alpha \in [0,1]$, $V_{\alpha\mathbf{t}} = \alpha V_{\mathbf{t}}$ is $c \wedge 1$-convex if $V_{\mathbf{t}}$ is c-convex, Proposition 3.6.1 and (3.23) finish the proof of the theorem since by Proposition 3.3.1, all the $N\bar{\delta}^N_{\alpha\mathbf{t}}(q_i)$ can be bounded independently of N, $\alpha \in [0,1]$ and $t \in B_{\eta,c}$ so that dominated convergence theorem applies.

□

As for the combinatorial interpretation of the variance, we relate $F^1(V_{\mathbf{t}})$ to a generating function of maps. This time, we will consider maps on a torus instead of a sphere. Such maps are said to be of genus 1. We define

$$\mathcal{M}^1_{k_1,\cdots,k_n}(P) = \sharp \left\{ \begin{array}{c} \text{maps of genus 1 with } k_i \text{ stars of type } q_i \\ \text{and one of type } P \end{array} \right\}$$

and

$$\mathcal{M}^1_{k_1,\cdots,k_n} = \sharp \left\{ \text{maps with } k_i \text{ stars of type } q_i \right\}.$$

We also define the generating function

$$\mathcal{M}^1(P) = \sum_{k_1,\cdots,k_n} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^1_{k_1,\cdots,k_n}(P).$$

If P is a monomial, we will denote $\mathcal{M}(\partial_i P)$ for $\sum_{P=RX_iS} \mathcal{M}(R,S)$ and we extend this notation to all polynomials by linearity.

For all monomials P and all k,

$$\begin{aligned} &\mathcal{M}^1_{k_1,\cdots,k_n}(X_k P) \\ &= \sum_{0\leqslant p_i \leqslant k_i} \sum_{P=RX_kS} \prod_i C^{p_i}_{k_i} \mathcal{M}^1_{p_1,\cdots,p_n}(R) \mathcal{M}_{k_1-p_1,\cdots,k_n-p_n}(S) \\ &+ \sum_{0\leqslant p_i \leqslant k_i} \sum_{P=RX_kS} \prod_i C^{p_i}_{k_i} \mathcal{M}_{p_1,\cdots,p_n}(R) \mathcal{M}^1_{k_1-p_1,\cdots,k_n-p_n}(S) \\ &+ \sum_{0\leqslant j \leqslant n} k_j \mathcal{M}^1_{k_1,\cdots,k_j-1,\cdots,k_n}(D_k VP, Q) + \sum_{P=RX_kS} \mathcal{M}_{k_1,\cdots,k_n}(R,S) \end{aligned}$$

and

$$\mathcal{M}^1(X_k P) = \mathcal{M}^1((I \otimes \mu + \mu \otimes I)\partial_k P) - \mathcal{M}^1(D_k VP) + \mathcal{M}(\partial_k P). \qquad (3.24)$$

Besides, for η small enough, there exists $R < +\infty$ such that for all monomials P, all $\mathbf{t} \in B(0,\eta)$,

$$|\mathcal{M}^1(P)| \leqslant R^{\deg P}.$$

Proof.

We proceed like for the combinatorial interpretation of the variance. We look at the first half-edge corresponding to X_k, then two cases may occur.

1. The first possibility is that the half-edge is glued to another half-edge of $P = RX_kS$. It forms a loop starting from P. There is two cases.
   (a) The loop can be retractable. It cuts P in two monomials R and S and it occurs for all decomposition of P into $P = RX_kS$ which is exactly what does D. Then either the component R or the component S is of genus 1 and the other component is planar. It produces either
   $$\prod_i C_{k_i}^{p_i} \mathcal{M}^1_{p_1,\cdots,p_n}(R)\mathcal{M}_{k_1-p_1,\cdots,k_n-p_n}(S)$$
   possibilities or the symmetric formula (where we exchange R and S).
   (b) The loop can also be none trivial in the fundamental group of the surface. Then the surface is cut in two. We are left with a planar surface with two fixed stars R and S. This gives
   $$\mathcal{M}_{k_1,\cdots,k_n}(R, S)$$
   possibilities.
2. The second possibility occurs when the half-edge is glued to a half-edge of a star of type q_j for a given j then first we have to choose between the k_j stars of this type then we contract the edges arising from this gluing to form a star of type $D_i q_j P_1$, this create
   $$k_j \mathcal{M}^1_{k_1,\cdots,k_j-1,\cdots,k_n}(D_k q_j P, Q)$$
   possibilities.

We can now sum on the k's to obtain the relation on $\mathcal{M}^1$.

Finally, to show that $\mathcal{M}^1$ is compactly supported we only have to prove that there exists $A > 0, B > 0$ such that for all k's, for all monomials P,
$$\frac{\mathcal{M}^1_{k_1,\cdots,k_n}(P)}{\prod_i k_i!} \leqslant A^{\sum_i k_i} B^{\deg P}.$$
Another time this follows easily by induction with the previous relation on the $\mathcal{M}^1(P)$'s.

□

We now give the combinatorial interpretation for the first order correction to the free energy. Assume **(H')**. There exists $\eta > 0$ small enough so that for $\mathbf{t} \in B_{\eta,c}$, for all non constant monomial P,
$$\phi(\Xi^{-1}P) = \mathcal{M}^1(P)$$
and
$$F^1 = \sum_{k_1,\cdots,k_n \in \mathbb{N}^n - \{0\}} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^1_{k_1,\cdots,k_n}.$$

Proof.

We use the previous property with $P = D_k \Sigma P$ and we sum on k,

$$\mathcal{M}^1(\Xi P) = \mathcal{M}(\sum_k \partial_k D_k \Sigma P) = \sum_k \sigma^2(\partial_k D_k \Sigma P) = \phi(P)$$

where we have used the combinatorial interpretation of the variance (Theorem 3.5.6). As $\mathcal{M}^1$ and ϕ are continuous for $\|.\|_A$ when η is small enough, we can apply this to $\Xi^{-1}P$ and conclude.

Finally, for η sufficiently small the series is absolutely convergent so that we can invert the integral and the sum to obtain

$$\begin{aligned}
F^1(V_{\mathbf{t}}) &= -\int_0^1 \mathcal{M}^1_{\alpha t_1,\cdots,\alpha t_n}(\sum_j t_j q_j) d\alpha \\
&= \int_0^1 \sum_{k_1,\cdots,k_n} \sum_j \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}(-t_j)\mathcal{M}^1_{k_1,\cdots,k_n}(q_j) d\alpha \\
&= \sum_{k_1,\cdots,k_n} \frac{1}{k_1+\cdots+k_n+1} \sum_j \prod_i \frac{(-t_i)^{k_i}}{k_i!}(-t_j)\mathcal{M}^1_{k_1,\cdots,k_j+1,\cdots,k_n} \\
&= \sum_{k_1,\cdots,k_n} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}^1_{k_1,\cdots,k_j,\cdots,k_n}
\end{aligned}$$

□

3.7 Diverging integrals

Physicists often use matrix models in more general settings. We would like to study the case of a potential V for which the integral Z_V^N is not convergent. For example, one may wonder if we can obtain the generating function for planar triangulation. The issue is that for $V = tX^3$, Z_V^N is infinite. The idea to give a meaning to this integral is to add a cut-off, we define for $L > 0$,

$$Z_{V,L}^N = \int_{\mathcal{H}_N(\mathbb{C})^m, \lambda_{\max}(\mathbf{A})<L} e^{-N\mathrm{Tr}(V(A_1,\cdots,A_m))} d\mu^N(A_1,\cdots,A_m).$$

This allow us to define the probability measure

$$\mu_{V,L}^N(dA_1,\cdots,dA_m) = \frac{\mathbb{1}_{\lambda_{\max}(\mathbf{A})<L}}{Z_{V,L}^N} e^{-N\mathrm{Tr}(V(A_1,\cdots,A_m))} d\mu^N(A_1,\cdots,A_m).$$

In [GMS06], we show that for all $L > L_0$ for a well chosen L_0, there exists $\eta > 0$ such that for $|\mathbf{t}| < \eta$, $\hat{\mu}^N$ goes almost surely towards the unique solution to Schynger-Dyson's equation (3.2). This show that the cut-off does not perturb too much the model since the limit does not depend on the choice of the cut-off L and keeps the same interpretation than in case of convex potentials. The aim of this section is to show that we can also extend the central limit theorem to this setting. The key idea is to see this potential as a convex potential. We bound the Hessian of

$$\varphi_{V_{\mathbf{t}}}^N : (A_k(ij)) \in (\mathbb{R}^{N^2})^m \cap \{\lambda_{\max}(\mathbf{A}) \leqslant L\} \to \mathrm{Tr}(V(A_1, \cdots, A_m)) \tag{3.25}$$

uniformly in N :

$$Hess\varphi_{V_{\mathbf{t}}}^N(A, A) = \sum_{i=1}^n t_i \sum_{q_i = RXSXT} \mathrm{Tr}(RASAT).$$

Now, use Hölder's inequality :

$$\begin{aligned}|\mathrm{Tr}(RASAT)| = |\mathrm{Tr}(TRASA)| &\leq \sqrt{\mathrm{Tr}((TR)A^*A(TR)^*)}\sqrt{\mathrm{Tr}(SA^*AS^*)} \\ &\leqslant \|TR\|\|S\|\mathrm{Tr}(AA^*).\end{aligned}$$

which implies that for $\{\lambda_{\max}(\mathbf{A}) \leqslant L\}$

$$\|Hess\varphi_{V_{\mathbf{t}}}^N\| \leqslant C|\mathbf{t}|$$

and C depends only on L. Therefore, We can find $\varepsilon > 0$ such that if $\mathbf{t} \in B(0, \varepsilon) \cap \{\mathbf{t} | V_{\mathbf{t}} = V_{\mathbf{t}}^*\}$, for all N, $\varphi_{V_{\mathbf{t}}}^N + \frac{1}{4}\sum_{i=1}^n \mathrm{Tr}(X_i^2)$ is convex on $\{\lambda_{\max}(\mathbf{A}) \leqslant L\}$.

Thus $\tilde{V_{\mathbf{t}}}(\mathbf{A}) = V_{\mathbf{t}}(\mathbf{A}) + \infty \mathbb{1}_{\lambda_{\max}(\mathbf{A}) > L}$ is a convex potential and

$$\mathbb{1}_{\lambda_{\max}(\mathbf{A}) \leqslant L} e^{-N\mathrm{Tr}(V_{\mathbf{t}}(\mathbf{A}))} = e^{-N\mathrm{Tr}(\tilde{V}(\mathbf{A}))}$$

is log-concave so that most of the step we proved so far can be generalized to this case. Indeed, Brascamp-Lieb and concentration inequalities do not require smoothness for the potential V. In fact, we could have included this case in all of the previous proof but they would have been less readable. We will only sketch the proof in this generalized case and highlight the main differences with the convex case.

First we must control the rate of convergence of the measure to its limit. The important fact is that up to the choice of $\mathbf{t}$ we can obtain bounds independent of L.

There exists non-negative constants L_0, M_0, C, α such that for $L > L_0$ we can find $\eta > 0$ such that for $|\mathbf{t}| < \eta$,

1. $$\mu(X_i^{2n}) \leqslant \limsup_N \overline{\mu}^N(X_i^{2n}) \leqslant C^n$$
2. For all $M > M_0$
   $$\mu_V^N(\lambda_{\max}^N(\mathbf{A}) > M) \leqslant e^{-\alpha MN}$$
3. there exists a finite constant $\varepsilon_{P,M}^N$ such that for any $\varepsilon > 0$,
   $$\mu_V^N\left(\{|\underline{\hat{\delta}}^N(P) - m_{P,M}^N| \geqslant \varepsilon + \varepsilon_{P,M}^N\} \cap \Lambda_M^N\right) \leqslant 2e^{-\frac{c\varepsilon^2}{2\|P\|_{\mathcal{L}}^M}}$$

   and if P is a monomial of degree d, $\varepsilon_{P,M} \leqslant NCdM^d e^{-\alpha MN}$.

Proof.

Since $e^{-N\mathrm{Tr}(\tilde{V}(\mathbf{A}))}$ is log-concave we can still use Brascamp-Lieb inequalities. The only point to check is that we can still find a lower bound for $Z^N_{V,L}$, but this was already done in [GMS06] using Jensen's inequality :

$$Z^{N,L}_{V_\mathbf{t}} = \int_{\lambda_{\max}(\mathbf{A})\leqslant L} e^{-N\mathrm{Tr}(V_\mathbf{t}(\mathbf{A}))} \prod d\mu_N(A_i)$$
$$\geqslant \mu^N(\lambda_{\max}(\mathbf{A}) \leqslant L) \exp\left(-N \int_{\lambda_{\max}(\mathbf{A})\leqslant L} \mathrm{Tr}(V_\mathbf{t}(\mathbf{A})) \frac{\prod d\mu_N(A_i)}{\mu^N(\lambda_{\max}(\mathbf{A}) \leqslant L)}\right)$$

The biggest eigenvalue goes almost surely to 2 and

$$|\int_{\lambda_{\max}(\mathbf{A})\leqslant L} \frac{1}{N}\mathrm{Tr}(V_\mathbf{t}(\mathbf{A})) \prod d\mu_N(A_i)|$$

is bounded by $\mu^N(V_\mathbf{t}V_\mathbf{t}^*)^{\frac{1}{2}}$ which goes to $\sigma^m(V_\mathbf{t}V_\mathbf{t}^*)^{\frac{1}{2}} < +\infty$ according to [Voi91]. Thus if $L > 2$, $Z^{N,L}_{V_\mathbf{t}} \geqslant e^{-dN^2}$ for a finite constant d. Thus we can prove the Property as in section 3.2. The proof of the two last points do not differ from the convex case.

□

The idea, once we have an a priori control on the radius C of the support independently of L, is that we can use it to approximate any polynomial by a compactly supported function with support in $[-L, L]$. We choose $L > L_0 = \max(M_0, C)$ and define for $L_0 < R < L$, ϕ_R the piecewise affine function such that for $|x| < R$, $\phi_R(x) = x$ and ϕ_R has a compact support strictly inside $[-L, L]$. Then we can approximate any polynomial $P(\mathbf{X})$ by $h_R = P(\phi_R(X_1), \cdots, \phi_R(X_m))$. The main improvement in the replacement of P by h_R is that h_R satisfies the finite Schwynger-Dyson's equation (3.13).

If L is bigger than some $L_0 > 0$, and $\varepsilon > 0$, there exists C, η, M_0 such that for $M > M_0$, $|\mathbf{t}| < \eta$ for all polynomial P of degree $d < \varepsilon N^{\frac{2}{3}}$

$$|\overline{\delta}^N(P)| \leqslant C\frac{\|P\|_M}{N}.$$

Proof.

In order to prove the analogue in the convex case (Property 3.3.1) we use the finite Schwynger Dyson's equation which is not always satisfied in this case. In fact, it is only satisfied for compactly supported function h with support in $[-L, L]$ since for such h we can make the infinitesimal change of variable. For a polynomial P,

$$\mu^N_V(\hat{\mu}^N[(X_i + D_iV)P]) - \mu^N_V\left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)\right)$$
$$= \mu^N_V(\hat{\mu}^N[(X_i + D_iV)(P - h_R)]) - \mu^N_V\left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i(P - h_R))\right)$$

Therefore, since μ satisfies the Schwinger-Dyson equation, we get that for all polynomial P,

$$\overline{\delta}^N(X_iP) = -\overline{\delta}^N(D_iVP) + \overline{\delta}^N \otimes \overline{\mu}^N(\partial_i P) + \mu \otimes \overline{\delta}^N(\partial_i P) + r(N, P) \qquad (3.26)$$

with

$$r(N,P) := N^{-1}\mu_V^N\left(\hat{\underline{\delta}}^N\otimes\hat{\underline{\delta}}^N(\partial_i P)\right)$$
$$+ N(\mu_V^N(\hat\mu^N[(X_i+D_iV)(P-h_R)]) - \mu_V^N\left(\hat\mu^N\otimes\hat\mu^N(\partial_i(P-h_R))\right).$$

Thus the only difference with the convex case is the term $N(\mu_V^N(\hat\mu^N[(X_i+D_iV)(P-h_R)]) - \mu_V^N\left(\hat\mu^N\otimes\hat\mu^N(\partial_i(P-h_R))\right))$ but since on Λ_M^N, $P(\mathbf{A}) = h_R(\mathbf{A})$ and $R > M$, this term decreases exponentially fast and this allows to finish the proof exactly as in the proof of Proposition 3.3.1.

□

Since the main tools are available we next turn to the proof of the central limit theorem. Here we have to be careful since the technique of the "infinitesimal change of variable" is no longer true in his full generality. But it still holds if we restrict ourself to compactly supported functional, thus we immediately obtain a weaker version of Lemma 3.4.1 :

If L is bigger than some $L_0 > 0$, there exists η such that for $|\mathbf{t}| < \eta$ if $h_1,\cdots,h_m$ are compactly supported with support in $]-L,L[$ and self-adjoint, the random variable

$$Y_N(h_1,\cdots,h_m) = N\sum_{k=1}^m\{\hat\mu^N\otimes\hat\mu^N(\partial_k h_k) - \hat\mu^N[(X_k+D_kV)h_k]\}$$

converges in law towards a real centered Gaussian variable with variance

$$C(h_1,\cdots,h_m) = \sum_{k,l=1}^m(\mu\otimes\mu[\partial_k h_l\times\partial_l h_k] + \mu(\partial_l\circ\partial_k V\sharp(h_k,h_l))) + \sum_{k=1}^m\mu(h_k^2).$$

The last step is to show that even if we do not have the result for all Stieltjes functions it is sufficient to approach polynomials by compactly supported function with support inside $]-L,L[$. We will another time use the fact that the limit measure has a support bounded independently of L. Using this idea we prove a result similar to Lemma 3.4.2.

If L is bigger than some $L_0 > 0$, there exists η such that for $|\mathbf{t}| < \eta$, if $P_1,\cdots,P_m$ are in $\mathbb{C}\langle X_1,\cdots,X_m\rangle_{sa}$ then the variable

$$Y_N(P_1,\cdots,P_m) = N\sum_{k=1}^m[\hat\mu^N\otimes\hat\mu^N(\partial_k P_k) - \hat\mu^N[(X_k+D_kV)P_k]]$$

converges in law towards a real centered Gaussian variable with variance

$$C(P_1,\cdots,P_m) = \sum_{k,l=1}^m(\mu\otimes\mu[\partial_k P_l\times\partial_l P_k] + \mu(\partial_l\circ\partial_k V\sharp(P_k,P_l))) + \sum_{k=1}^m\mu(P_k^2).$$

Proof.

First choose $L > L_0 = \max(M_0, C)$ and for $L_0 < R < L$ approximate the polynomials $P_i(\mathbf{X})$ by $h^i_R = P_i(\phi_R(X_1), \cdots, \phi_R(X_m))$. Then since C bound the support of μ, observe that $C(P_1, \cdots, P_m) = C(h^1_R, \cdots, h^m_R)$ and we only have to prove that

$$Y_N(P_1, \cdots, P_m) - Y_N(h^1_R, \cdots, h^m_R)$$

goes in law to 0 when N goes to infinity. But, we have the inequality

$$P(|Y_N(P_1, \cdots, P_m) - Y_N(h^1_R, \cdots, h^m_R)| > \varepsilon) \leqslant P(\lambda_{\max}(\mathbf{A}) > M)$$

and the right hand side goes exponentially fast to 0.

□

The other results can be proved as in the convex case with only minor modifications. Following the same way than in the convex case, this allow us to prove the Theorem : If L is bigger than some $L_0 > 0$, there exists η such that for $|\mathbf{t}| < \eta$, for all P in $\mathbb{C}_0\langle X_1, \cdots, X_m\rangle$, $\hat{\delta}^N(P)$ converges in law to a Gaussian variable with variance

$$\sigma^2(P) := \mathcal{C}(\Xi^{-1}P) = C(D_1\Sigma\Xi^{-1}P, \cdots, D_m\Sigma\Xi^{-1}P).$$

Besides the convergence in moments occurs and the covariance keeps its combinatorial interpretation, allowing us to enumerate a larger variety of graphs.

Finally, applying the same strategy than in the convex case we are able to prove the convergence of the free energy. For L is bigger than some $L_0 > 0$, there exists η such that for $|\mathbf{t}| < \eta$, the following asymptotics holds

$$\log Z^N_{V_\mathbf{t},L} = N^2F^0(V_\mathbf{t}) + F^1(V_\mathbf{t}) + o(1)$$

with

$$F^0(V_\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}-\{0\}} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}_{k_1,\cdots,k_n}$$

and

$$F^1(V_\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}-\{0\}} \prod_{i=1}^n \frac{(-t_i)^{k_i}}{k_i!}\mathcal{M}^1_{k_1,\cdots,k_n}.$$

Chapitre 4

High order expansion of matrix models and enumeration of maps

Ce chapitre est l'article [MS06a] qui a été soumis.

Abstract

Perturbation of the **GUE** are known in physics to be related to enumeration of graphs on surfaces. Following [GMS06] and [GMS07], we investigate this idea and show that for a small convex perturbation we can perform a genus expansion : the free energy and the moments of the empirical measure can be developed into a series whose g-th term is a generating function of graphs embedded on a surface of genus g.

4.1 Introduction

Wick's calculus allows to easily compute any moments of Gaussian variables and gives them a combinatorial interpretation since the p-th moment of a Gaussian can be seen as the number of partitions in pairs of $[|1,p|]$. This fact can be used to find moments of the **GUE**, the Gaussian unitary model. Let μ^N be the law on $\mathcal{H}_N(\mathbb{C})^m$ the set of m-tuple $A_1, \cdots, A_m$ of $N \times N$ hermitian matrices such that $\Re e A_i(kl), k < l$, $\Im m A_i(kl), k < l$, $2^{-\frac{1}{2}} A_i(kk)$ is a family of independent real Gaussian variables of variance $(2N)^{-1}$ or more directly

$$\mu^N(d\mathbf{A}) = \frac{1}{Z^N} e^{-\frac{N}{2}\mathrm{Tr}(\sum_{i=1}^m A_i^2)} d^N \mathbf{A}$$

with $d^N\mathbf{A}$ the Lebesgue measure on $\mathcal{H}_N(\mathbb{C})^m = (\mathbb{R}^{N^2})^m$ and Z^N a constant of normalization.

For a edge-colored graph on an orientated surface we say that a vertex is of type $q = X_{i_1}\cdots X_{i_p}$ for a monomial q if this vertex is of degree p and when we look at the edges going out of it, starting from a distinguished one and going in the clockwise order the first edge is of color i_1, the second of color i_2,..., the p-th of color i_p. A graph on a surface is a map if it is connected and its faces are homeomorphic to discs (see section 3 for a precise definition of these notions). Then, a computation (see [HZ86] for the one matrix case and [NS06] for the general case) using Wick's calculus shows that for all non commutative monomials, $X_{i_1}\cdots X_{i_p}$,

$$\mu^N(\frac{1}{N}\mathrm{Tr}(A_{i_1}\cdots A_{i_p})) = \sum_{g\in\mathbb{N}} \frac{1}{N^{2g}}\mathcal{M}_g(X_{i_1}\cdots X_{i_p}) \tag{4.1}$$

where $\mathcal{M}_g(X_{i_1}\cdots X_{i_p})$ is the number up to isomorphism of maps with colored edges on a surface of genus g with one vertex of type $X_{i_1}\cdots X_{i_p}$. The contribution for higher asymptotics is given by graphs of higher genus. This is called a genus expansion or a topological expansion. Besides, one can use Euler's formula to show that the sum in the right hand side is always finite. We see that the first asymptotic of the moments of the **GUE** leads to an enumeration of planar object. The link between limit moments of matrices and combinatorial structure already appeared in the first works on random matrices since [Wig58] proved that the moments of hermitian matrices with i.i.d. entries are Catalan's numbers. In the multi-matrix case this first asymptotic can be described by the notion of freeness, a crucial property in operator algebra, see [Voi00] for an introduction and [Voi91] which proves this type of asymptotic is satisfied not only for the **GUE** but for a far more larger class of matrices. This freeness has also a combinatorial interpretation as a sum over non-crossing partitions (see [Spe97]).

Can such an interpretation be generalized beyond the Gaussian case ? More general genus expansions are of particular interest in physics (see [tH74] which introduce such a concept). The links between them and matrix integrals were discovered in [BIPZ78]. We present them here in a general setting. Take a potential $V(X_1,\cdots,X_m) = \sum_i t_i q_i$ with complex parameters $t_1,\cdots,t_n$ and non-commutative monomials q_i. We are interested in the following perturbation of the **GUE**

$$\mu_V^N(dA_1,\cdots,dA_m) = \frac{1}{Z_V^N}e^{-N\mathrm{Tr}(V(A_1,\cdots,A_m))}d\mu^N(A_1,\cdots,A_m) \tag{4.2}$$

where Z_V^N is the normalizing constant making μ_V^N a probability measure. The derivatives of the moments of this model at $\mathbf{t} = 0$ are exactly moments of the **GUE** and thus can be computed using Wick's calculus and the limit can be formally expressed as a generating function of graphs. For example, for a quartic potential $V = tX^4$, we can obtain the following formal expansion for the free energy (see [BIZ80])

$$\log Z_{tX^4}^N = \sum_{g\in\mathbb{N}} N^{2-2g}\sum_{k\in\mathbb{N}}\frac{(-t)^k}{k!}\mathcal{C}_g^k \tag{4.3}$$

with $\mathcal{C}_g^k$ the number up to isomorphism of connected graphs on a surface of genus g with k vertex of degree 4 and such that faces are homeomorphic to discs (the so-called maps). Note that we have to be careful since the right hand side of (4.3) is divergent for $t \neq 0$. Thus, this equality is purely formal. It can also be stated for general potential $V_{\mathbf{t}}$.

This paper is the sequel of the two articles [GMS06] and [GMS07]. One aim of this series is to investigate what can be said of the previous equality beyond the identification of the formal series. More precisely we would like to know if for some parameters $\mathbf{t}$ the genus expansion is the large N expansion of the free energy :

$$F^N_{V_{\mathbf{t}}} := \frac{1}{N^2} \ln Z^N_{V_{\mathbf{t}}}.$$

First we need to make some assumptions in order for our probability measure to be well defined. We will always assume that :

1. The perturbation is small : we will restrict ourselves to small coefficients t_i in V. Note that we can not get rid of this condition, as the generating functions of combinatorial objects that we consider have arbitrary small radius of convergence.
2. The potential $V + \frac{1}{2}\sum_i X_i^2$ is “uniformly” convex : there exists $c > 0$ such that for all N in $\mathbb{N}$,

$$\varphi^N_V : \begin{array}{ccc} \mathcal{H}_N(\mathbb{C})^m & \longrightarrow & \mathbb{C} \\ (X_1, \cdots, X_m) & \longrightarrow & \mathrm{Tr}(V(X_1, \cdots, X_m) + \frac{1-c}{2}\sum_{i=1}^m X_i^2) \end{array}$$

   is a real and convex function. If V satisfies this condition, we say that V is c-convex.

Thus, for $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$ with $\mathbf{t} = (t_1, \cdots, t_n)$ complex numbers and q_i non-commutative monomials we define

$$B_{\eta,c} = \{\mathbf{t} \in \mathbb{C}^n | |\mathbf{t}| = \max_i |t_i| \leqslant \eta, V_{\mathbf{t}} \text{ is } c\text{-convex}\}.$$

Examples of c-convex potentials can be built using Klein's lemma (see [GZ02]) is

$$V(\mathbf{X}) = \sum_i P_i(\sum_j \alpha_{ij} X_j) + \sum_{k\ell} \beta_{k\ell} X_k X_\ell$$

with real and convex polynomials P_i, real α_{ij}, β_{kl} and for all l, $\sum |\beta_{kl}| < 1 - c$.

We proved in the previous articles the first two terms of the expansion converge : Let $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$, and $c > 0$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, the free energy has the following expansion

$$F^N_{V_{\mathbf{t}}} = F^0(\mathbf{t}) + \frac{1}{N^2} F^1(\mathbf{t}) + o(\frac{1}{N^2}).$$

We also showed that $F^i(\mathbf{t}), i = 0, 1$ enumerates maps of genus i with vertices associated to the monomials of V. More precisely, [GMS06] tackled the first asymptotic for the free energy and for the moments of the measure. In [GMS07], we looked at the second asymptotic for those quantities and in addition we proved a central limit theorem for the moment of the empirical measure which will be crucial in this paper.

Our objective is to generalize this expansion to any genus. Our main concern is the multi-matrix case since the one matrix case is already well understood. For the latter, the first asymptotic of the empirical measure has been studied from a non-perturbative perspective (that is with assumptions only on the growth of V at infinity and not on the size of its coefficients). A large deviation principle was obtained in [BAG97] and a central limit theorem in [Joh98]. An explicit description of the density of the limit measure is given in [EM03]. The next orders in the expansion have also been studied. In [ACKM93], a recursive procedure based on the loop equation is given to compute recursively the asymptotics of observables such as the free energy. Our proofs also rely on this loop equation, called Schwinger-Dyson's equation. More recently, [ASM01] shows an expansion for the expectation of the Stieltjes transform of the empirical measure. Finally, [EM03] gave a genus expansion for the free energy using Riemann-Hilbert methods. This is exactly this expansion that we would like to obtain in the multi-matrix case. Our tools are very different from those of [EM03] but the hypotheses are comparable. (In [EM03] they assume that $V = t_{2m}x^{2m} + \sum_{i<2m} t_i x^i$ with t_{2m} which dominates the other t_i while we assume the convexity of V).

Many techniques used in these articles, such as the use of orthogonal polynomials, can not be generalized to the multi-matrix case. However, there is a huge literature which tackles some specific models, such as the so called two matrix model $V = V_1(A) + V_2(B) + cAB$, whose combinatorics is of crucial importance for models of statistical physics on random graphs. The Ising model on random graphs was solved by physicists in [Meh81] and then by combinatoricians in [BMS02] At a non-perturbative level the first order was studied using large deviation technique in [Gui04]. A recent series of papers ([CEO06], [EO05], [EO07]) introduces tools of algebraic geometry and gives recursive formula to study the asymptotics of these models.

The interested reader should consult the review [DFGZJ95] and [GDS91] and the book [Zvo97]. The last part of [GMS06] also aims to present the many approaches to this problem.

As in [GMS06] and [GMS07], our main tool will be the so-called Schwinger-Dyson's equation and we will try to interpret it as a higher genus Tutte's equation (see [Tut63]). This will lead us to the combinatorial series.

The main result of this paper is that we can go beyond the first two asymptotics, up to any order. Let $V_{\mathbf{t}} = \sum_{i=1}^n t_i q_i$, and $c > 0$, for all $g \in \mathbb{N}$, there exists $\eta_g > 0$ such that for all $\mathbf{t}$ in $B_{\eta_g,c}$, the free energy has the following expansion

$$F_{V_{\mathbf{t}}}^N := \frac{1}{N^2} \log Z_{V_{\mathbf{t}}}^N = F^0(\mathbf{t}) + \frac{1}{N^2} F^1(\mathbf{t}) + \cdots + \frac{1}{N^{2g}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

with F^g the generating function for maps of genus g associated with V :

$$F^g(\mathbf{t}) = \sum_{\mathbf{k} \in \mathbb{N}^n \backslash \{0\}} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{C}_g^{\mathbf{k}}(P)$$

where $\mathbf{k}! = \prod_i k_i!$, $(-\mathbf{t})^{\mathbf{k}} = \prod_i (-t_i)^{k_i}$ and $\mathcal{C}_g^{\mathbf{k}}$ is the number of maps on a surface of genus g with k_i vertices of type q_i. Note that increasing the order of the expansion is done at the cost of reducing the radius of convergence since the full series in power of g is not convergent.

To tackle this problem, we will look at asymptotics of other observables (like in [GMS06] and [GMS07]). In particular we will be interested by the asymptotic of the non-commutative moments of our measure $E_{\mu^N_{V_{\mathbf{t}}}}[\frac{1}{N}\mathrm{Tr}(P)]$ for a non-commutative polynomial P. Such moments appear as derivatives of the free energy since

$$E_{\mu^N_{V_{\mathbf{t}}}}[\frac{1}{N}\mathrm{Tr}(P)] = -\left.\frac{\partial}{\partial u}\right|_{u=0} F^N_{V_{\mathbf{t}}+uP}.$$

With the same hypotheses than in the previous theorem, for all for all $g \in \mathbb{N}$, there exists $\eta > 0$, such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all monomials P

$$E_{\mu^N_{V_{\mathbf{t}}}}[\frac{1}{N}\mathrm{Tr}(P)] = \mathcal{C}^0_{\mathbf{t}}(P) + \cdots + \frac{1}{N^{2g}}\mathcal{C}^g_{\mathbf{t}}(P) + o(\frac{1}{N^{2g}})$$

with $\mathcal{C}_g$ the generating function maps of genus g with some fixed vertices :

$$\mathcal{C}^g_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!}\mathcal{C}^{\mathbf{k}}_g(P)$$

where $\mathcal{C}^{\mathbf{k}}_g(P)$ is the number of maps on a surface of genus g with k_i vertices of type q_i and one of type P.

In fact, we will be able to find the asymptotics of many more observables, such as the higher derivatives of the free energy. Indeed, we show that we can differentiate term by term the expansion of Theorem 4.1.1. Let us introduce for $\mathbf{j} = (j_1, \cdots, j_n) \in \mathbb{N}^n$, the operator of derivation

$$\mathcal{D}_{\mathbf{j}} = \frac{\partial^{\sum_i j_i}}{\partial t_1^{j_1} \cdots \partial t_n^{j_n}}.$$

With the same hypothesis than in the previous theorem, for all non-negative integers $\mathbf{j} = (j_1, \cdots, j_n)$ in $\mathbb{N}^n \backslash \{0\}$, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,

$$\mathcal{D}_{\mathbf{j}} F^N_{V_{\mathbf{t}}} = \mathcal{D}_{\mathbf{j}} F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}).$$

Besides, $\mathcal{D}_{\mathbf{j}} F^g$ is the generating function maps of genus g with some fixed vertices :

$$\mathcal{D}_{\mathbf{j}} F^g(\mathbf{t}) = \sum_{\mathbf{k}\in\mathbb{N}^n} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!}\mathcal{C}^{\mathbf{k}+\mathbf{j}}_g.$$

In the next section, we will define some useful notations from non-commutative probability theory and we will recall the main result of [GMS06]. Next, we will look for recursive relations between the asymptotics of the non-commutative moments of our model. This will lead us to study some combinatorial objects in section 4 whose generating functions satisfy these relations. In the sections 5 and 6, we will prove the equality of these moments and these enumerating functions before proving our main results. Finally the last section will be devoted to the proof of Theorem 4.1.3.

4.2 Notations and reminder

Let us denote by $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ the set of complex polynomials on the non-commutative unknown $X_1,\ldots,X_m$ i.e. the complex linear combination of monomials which are simply the set of finite words on $X_1, \ldots, X_m$. Monomials must be thought as non-commutative moments. Let $*$ denotes the linear involution on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ such that for all complex z and all monomials

$$(zX_{i_1} \ldots X_{i_p})^* = \overline{z} X_{i_p} \ldots X_{i_1}.$$

A polynomial P is self-adjoint if $P = P^*$.

We will denote $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ the dual of $\mathbb{C}\langle X_1, \cdots, X_m\rangle$. If there exists $R > 0$ such that for all monomial $|\tau(X_{i_1} \ldots X_{i_p})| \leqslant R^p$ we will say that τ has a compact support. By analogy with the one variable case, the infimum of the R's which satisfy this inequality for all monomials will be called the radius of the support of τ.

For a polynomial P and a monomial q, we define $\lambda_q(P)$ as the coefficient of q in the decomposition of P. For $M > 0$, we define the norm $\|.\|_M$ on polynomials :

$$||P||_M = \sum_{l\in\mathbb{N}} \sum_{\substack{q \text{ monomial} \\ degq=l}} |\lambda_q(P)| M^l.$$

This norm $\|.\|_M$ is an algebra norm, i.e. for all polynomials P, Q,

$$\|PQ\|_M \leqslant \|P\|_M \|Q\|_M.$$

Note that an element τ of $\mathbb{C}\langle X_1, \cdots, X_m\rangle^*$ has a support of radius less than R if and only if for all polynomials P,

$$|\tau(P)| \leqslant \|P\|_R.$$

We extend $\|.\|_M$ on $\mathbb{C}\langle X_1, \cdots, X_m\rangle \otimes \mathbb{C}\langle X_1, \cdots, X_m\rangle$ by defining this norm on the decomposition in monomials :

$$\| \sum \lambda_{q_1,q_2} q_1 \otimes q_2 \|_M = \sum |\lambda_{q_1,q_2}| M^{\deg q_1 + \deg q_2}$$

with this definition for all polynomials P, Q, $\|P \otimes Q\|_M = \|P\|_M \|Q\|_M$.

For all i in $[1, n]$, we define the non-commutative derivative ∂_i on the space of non-commutative polynomials $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ to $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 2}$ by the Leibniz rule

$$\partial_i(PQ) = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q$$

and $\partial_i X_j = \mathbb{1}_{i=j} 1 \otimes 1$. For a monomial P, we will often use the convenient expression

$$\partial_i P = \sum_{P=RX_iS} R \otimes S$$

where the sum runs over all possible monomials R, S so that P decomposes into RX_iS. We also define another operator of derivation on polynomials, the cyclic derivative D_i which is linear and such that for all monomials :

$$D_i P = \sum_{P=RX_iS} SR.$$

Alternatively, D can be defined as $m \circ \partial$ where $m(A \otimes B) = BA$. We will see that these two operators appear naturally when we differentiate products of matrices and they both possesses a nice combinatorial interpretation. An important fact we will use later is that for all $M' > M$, both ∂_i and D_i are continuous from $(\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_{M'})$ to $(\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_M)$. For example for a monomial q,

$$\frac{\|D_i q\|_M}{\|q\|_{M'}} = \deg q \frac{M^{\deg q - 1}}{M'^{\deg q}} = M^{-1} \deg q \left(\frac{M}{M'}\right)^{\deg q}$$

which is bounded. Note also that due to the particular form of this form, in order to show that an operator θ has a norm bounded by C with respect to this norm, it is sufficient to show that for all monomials q, $\|\theta q\|_M \leqslant C\|q\|_M$

The main object of our study is the law $\mu^N_{V_\mathbf{t}}$ on $\mathcal{H}_N(\mathbb{C})^m$

$$\mu^N_{V_\mathbf{t}}(dA_1, \cdots, dA_m) = \frac{1}{Z^N_{V_\mathbf{t}}} e^{-N\mathrm{Tr}(V_\mathbf{t}(A_1, \cdots, A_m))} d\mu^N(A_1, \cdots, A_m)$$

and we are particularly interested by the behavior of the random variable

$$\hat{\mu}^N : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m\rangle & \longrightarrow & \mathbb{C} \\ P & \longrightarrow & \frac{1}{N}\mathrm{Tr}(P(A_1, \cdots, A_m)) \end{array}$$

and its mean :

$$\overline{\mu}^N_\mathbf{t} : \begin{array}{ccc} \mathbb{C}\langle X_1, \cdots, X_m\rangle & \longrightarrow & \mathbb{C} \\ P & \longrightarrow & E_{\mu^N_{V_\mathbf{t}}}[\frac{1}{N}\mathrm{Tr}(P(A_1, \cdots, A_m))] \end{array}$$

We can now state precisely the main result of [GMS06], we will use it very frequently it in the next sections. For any $c > 0$,$R > 0$ there exists $\eta > 0$ such that for all $\mathbf{t} \in B_{\eta,c}$, for all polynomials P, $\hat{\mu}^N(P)$ goes when N goes to $+\infty$, almost surely and in expectation towards $\mu_\mathbf{t}(P)$ with $\mu_\mathbf{t}$ a solution of the Schwinger-Dyson equation

$$\mu_\mathbf{t} \otimes \mu_\mathbf{t}(\partial_i P) = \mu_\mathbf{t}((X_i + D_i V_\mathbf{t})P) \quad \forall P \in \mathbb{C}\langle X_1, \cdots, X_m\rangle, \forall i \in \{1, \cdots, m\}.$$

Besides, this solution is the unique one which has a support bounded by R : for all monomial $X_{i_1} \cdots X_{i_p}$

$$|\mu_\mathbf{t}(X_{i_1} \cdots X_{i_p})| \leqslant R^p. \tag{4.4}$$

Moreover, on $B_{\eta,c}$, $\mu_\mathbf{t}$ can be seen as a generating function of planar maps : for all polynomials P,

$$\mu_\mathbf{t}(P) = \sum_{k_1, \cdots, k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{k_1, \cdots, k_n}(P)$$

where $\mathcal{M}^{k_1, \cdots, k_n}(P)$ is the number of planar connected graphs with k_i vertices of type q_i and one of type P. For the rest of the paper we will work in this domain $B_{\eta,c}$ where the convergence holds. In order to shorten a little the notations the subscript $\mathbf{t}$ will be most of the time implicit, for example we will often write μ instead of $\mu_\mathbf{t}$, V instead of $V_\mathbf{t}$, $\overline{\mu}^N$ instead of $\overline{\mu}^N_\mathbf{t}$ ...

4.3 First order observable

The starting point is a relation already used in [GMS06] for the matrix model when N is fixed : for all polynomial P, for all i,

$$E[\hat{\mu}^N((X_i + D_iV)P)] = E[(\hat{\mu}^N \otimes \hat{\mu}^N)(\partial_i P)].$$

We will give the proof of a generalization of this equality later. Using this equality and some concentration inequalities we were able to prove that for $\mathbf{t}$ in $B_{\eta,c}$ for a well chosen η, for all polynomial P, $E[\hat{\mu}^N(P)]$ was converging towards $\mu(P)$ with μ the unique solution of the Schwinger-Dyson's equation **SD[V]** :

$$\mu((X_i + D_iV)P)) = (\mu \otimes \mu)(\partial_i P). \tag{4.5}$$

In order to find the next asymptotic, we study the difference between the equation for finite N and the limit equation, if $\nu^N = N^2(\overline{\mu}^N - \mu)$, we obtain by subtracting the two equations :

$$\nu^N((X_i + D_iV)P) - (I \otimes \mu + \mu \otimes I)\partial_i P) = N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(\partial_i P)] \tag{4.6}$$

Here, an important operator shows up in the left hand side. For $P = X_{i_1} \cdots X_{i_p}$ a monomial, define the following operators :

$$\Xi_1 P = \frac{1}{p} \sum_i D_i V D_i P$$

$$\Xi_2 P = \frac{1}{p} \sum_i (I \otimes \mu + \mu \otimes I)\partial_i D_i P.$$

We extend them by linearity on $\mathbb{C}\langle X_1, \cdots, X_m \rangle$ and we define $\Xi_0 = I - \Xi_2$ and $\Xi = \Xi_0 + \Xi_1$. These operators were introduced in [GMS07] to obtain a central limit theorem for the matrix model. We also define the operator of division by the degree i.e. the linear operator $P \to \overline{P}$ such that for all monomial $P = X_{i_1} \cdots X_{i_p}$, $\overline{P} = \frac{1}{p}P$ and $\overline{1} = 0$. These operators allow us to state the relation for the first correction (4.6) in a simpler form. If we apply (4.6) to $D_i\overline{P}$ and then sum on i, we get :

$$\nu^N(\Xi P) = \sum_i N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(\partial_i D_i \overline{P})]. \tag{4.7}$$

Then, the strategy is simple, we only have to understand the asymptotic of $N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(R \otimes S)]$ and then "invert" Ξ. The first order asymptotic of $N^2 E[((\hat{\mu}^N - \mu) \otimes (\hat{\mu}^N - \mu))(R \otimes S)]$ is easy to compute using [GMS07] as it was shown that $N(\hat{\mu}^N - \mu)(Q)$ converges in law towards a Gaussian law when N goes to infinity. The main issue is that when we try to investigate the next asymptotic, terms of type $N^3 E[(\hat{\mu}^N - \mu)^{\otimes 3}(R \otimes S \otimes T)]$ will appear and at their turn they will create terms of greater complexity. That's why we are interested more generally in all the $N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)]$'s and we will eventually find their full asymptotic. First remark that according to [GMS07], for all P, $N(\hat{\mu}^N(P) - \mu(P))$ converges to a Gaussian variable and this convergence occurs in moments (see Corollary 4.8 in [GMS07]). Thus $N^\ell E[(\hat{\mu}^N-\mu)^{\otimes\ell}(P_1\otimes\cdots\otimes P_\ell)]$ has a finite limit when N goes to infinity and this limit is 0 if ℓ is odd. But we need a more precise result which state that this convergence is uniform for all monomials P of reasonable degree. For all $\ell\in\mathbb{N}^*$, $\alpha>0$ there exists $C,\eta,M_0>0$, such that for all $\mathbf{t}\in B_{\eta,c}$, $M>M_0$ and all polynomials $P_1,\ldots,P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,

$$\left|E[N^\ell(\hat{\mu}^N-\mu)^{\otimes\ell}(P_1\otimes\cdots\otimes P_\ell)]\right|\leq C\|P_1\|_M\cdots\|P_\ell\|_M.$$

Proof.

First using Hölder's inequality, write

$$E[N^\ell(\hat{\mu}^N-\mu)^{\otimes\ell}(P_1\otimes\cdots\otimes P_\ell)]\leq\prod_{r=1}^{\ell}E[|N(\hat{\mu}^N-\mu)(P_r)|^\ell]^{\frac{1}{\ell}}.$$

Thus we only have to prove the claim if the P_r are equals. Then we subtract the mean

$$E[|N(\hat{\mu}^N-\mu)(P)|^\ell]\leq 2^\ell\{E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell]+|N(\overline{\mu}^N(P)-\mu(P))|^\ell\}\tag{4.8}$$

Proposition 3.1 in [GMS07] state a rate of convergence of $\overline{\mu}^N(P)$ to $\mu(P)$: there exists $C,M_0>0$ such that for $M>M_0$, for all polynomials P of degree less than $\varepsilon N^{\frac{2}{3}}$,

$$|N(\overline{\mu}^N(P)-\mu(P))|\leqslant C\frac{\|P\|_M}{N}.\tag{4.9}$$

Thus in inequality (4.8) above, we only have to control the first term. We decompose it into the sum

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell]=E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell\mathbb{1}_{\|\mathbf{A}\|\leqslant M}]+E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell\mathbb{1}_{\|\mathbf{A}\|>M}]\tag{4.10}$$

with $\|\mathbf{A}\|$ is the max of the operator norm of the A_i's. The second term can be bounded by

$$(2N)^\ell\overline{\mu}^N((PP*)^\ell)^{\frac{1}{2}}\mathbb{P}(\|\mathbf{A}\|>M)\leqslant(2N)^\ell\|P\|_M^\ell\mathbb{P}(\|\mathbf{A}\|>M)$$

Now according to Lemma 2.2 in [GMS07], we have a control on the decay of the largest eigenvalue : there exists $a>0$ such that if M is sufficiently large, $\mathbb{P}(\|\mathbf{A}\|>M)\leqslant e^{-aMN}$ and since $e^{-aMN}(2N)^\ell$ is uniformly bounded in N we get :

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell\mathbb{1}_{\|\mathbf{A}\|>M}]\leqslant C\|P\|_M^\ell.\tag{4.11}$$

with a constant which may depends on a,ℓ and M.

We are only left with the term

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell\mathbb{1}_{\|\mathbf{A}\|\leqslant M}].$$

We can use the concentration inequality result of Lemma 2.3 in [GMS07]. Borrowing the notations from this lemma we have :

$$\begin{aligned}
&E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|\leqslant M}] \\
&= \int_0^{+\infty} \ell x^{\ell-1}\mathbb{P}(|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|>x, \|\mathbf{A}\|\leqslant M)dx \\
&\leqslant \ell(\varepsilon^N_{P,M}+m^N_{P,M})^\ell + \int_0^{+\infty}\{\ell(x+\varepsilon^N_{P,M}+m^N_{P,M})^{\ell-1} \\
&\times \mathbb{P}(|N(\hat{\mu}^N(P)-\overline{\mu}^N(P))-m^N_{P,M}|>x+\varepsilon^N_{P,M}, \|\mathbf{A}\|\leqslant M)\}dx \\
&\leqslant \ell(\varepsilon^N_{P,M}+m^N_{P,M})^\ell + 2\int_0^{+\infty}\ell(x+\varepsilon^N_{P,M}+m^N_{P,M})^{\ell-1}e^{\frac{-cx^2}{2(\|P\|^M_{\mathcal{L}})^2}}dx
\end{aligned}$$

Now observe that up to a little change in M, our norm $\|.\|_M$ control the lipschitz norm of P :

$$\|P\|^M_{\mathcal{L}} \leqslant (\sum_k \|D_kPD_kP^*\|_M)^{\frac{1}{2}} \leqslant C\|P\|_{M'}$$

for a $M<M'$. This is a direct consequence of the continuity of the derivation D_k from $\mathbb{C}\langle X_1,\cdots,X_m\rangle_{M'}$ to $\mathbb{C}\langle X_1,\cdots,X_m\rangle_M$ for $M<M'$. Thus

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|\leqslant M}] \leqslant C((\varepsilon^N_{P,M})^\ell+(m^N_{P,M})^\ell+\|P\|^\ell_{M'})$$

Note that since $M<M'$, for all polynomial P, $\|P\|_M\leqslant\|P\|_{M'}$ so that if in the end we have a control in terms of these two norms we will be able to convert it into a control with respect to the norm $\|.\|_{M'}$.

The last step is to control $\varepsilon^N_{P,M}$ and $m^N_{P,M}$. The bound computed in Lemma 2.3 in [GMS07] for monomials is easily extended to polynomials : for all P of degree less than αN,

$$\varepsilon^N_{P,M}\leqslant C\|P\|_{M'} \text{ and } m^N_{P,M}\leqslant C\|P\|_{M'}.$$

Thus we get

$$E[|N(\hat{\mu}^N(P)-\overline{\mu}^N(P)|^\ell \mathbb{1}_{\|\mathbf{A}\|\leqslant M}] \leqslant C\|P\|_{M'}$$

which with (4.11) give the control on the left hand side of (4.10). This with (4.9) allow to control (4.8) which conclude the proof.

□

We now try to find some relation between the $N^\ell E[(\hat{\mu}^N-\mu)^{\otimes\ell}(P_1\otimes\cdots\otimes P_\ell)]$'s which generalize (4.7). Remember that, for ℓ odd, those quantities vanish when N goes to infinity. Thus in order to obtain non-trivial limits, we have to distinguish the normalisation according to the parity of ℓ.

For $\ell\geq 1$ we define :

$$\hat{\ell} = \begin{cases} \ell+1 & \text{if } \ell \text{ is odd} \\ \ell & \text{otherwise.}\end{cases}$$

Note that $\hat{\ell}$ is always an even integer. We now define a function from the disjoint union of the $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell}$ for ℓ a non-negative integer to $\mathbb{C}$,

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = E_{\mu_V^N}[N^{\hat{\ell}}(\hat{\mu}^N - \mu)^{\otimes \ell}(P_1 \otimes \cdots \otimes P_\ell)].$$

On $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell}$, this is a ℓ-linear symmetric function which is tracial in each P_r. Our convention will be that for λ in $\mathbb{C} = \mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes 0}$, $\nu^N(\lambda) = \lambda$. The relation that will appear as our main tool are the aim of the next property. In a tensor product $P_1 \otimes \cdots \check{P}_r \cdots \otimes P_\ell$ denotes the tensor product of $P_1, \ldots, P_{r-1}, P_{r+1}, \ldots, P_\ell$ i.e. the term P_r is omitted. For all ℓ, for all polynomial $P_1, \ldots, P_\ell$, for all N, if ℓ is even

$$\begin{aligned}\nu^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \frac{1}{N^2}\sum_{i,r} \nu^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \frac{1}{N^2}\sum_{i} \nu^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

and if ℓ is odd

$$\begin{aligned}\nu^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} \nu^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i} \nu^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

This proposition is the generalization of the equation (4.7). One may wonder why we stress so much the difference between the odd and the even case. The point is to keep in mind which terms are of order 1 and which are negligible. In view of this, the ν_1^N are convenient as they should all be of order 1 and thus the previous equation will lead us to find their limit by induction.

Proof.

To sum up the proposition in a shorter way, we have to prove that for all ℓ for all polynomials $P_1, \ldots, P_\ell$ and for all N,

$$\begin{aligned}&N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell}(\Xi P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)] \\ &= \sum_{i,r} \mu_{\mathbf{t}}(D_i\overline{P_1}D_iP_r)N^{\ell-2}E[(\hat{\mu}^N - \mu)^{\otimes \ell-2}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} N^{\ell-2}E[(\hat{\mu}^N - \mu)^{\otimes \ell-1}(D_i\overline{P_1}.D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)] \\ &+ \sum_{i} N^\ell E[(\hat{\mu}^N - \mu)^{\otimes \ell+1}(\partial_i D_i\overline{P_1} \otimes P_2 \otimes \cdots \otimes P_\ell)].\end{aligned}$$

We will use the integration by part formula :

$$\int xf(x)e^{-x^2/2}dx = \int f'(x)e^{-x^2/2}dx.$$

We generalize this formula into

$$\int \mathrm{Tr}(A_i P)f(\mathrm{Tr}Q)d\mu^N = \frac{1}{N}\sum_{\alpha,\beta}\int (\partial_{A_i(\alpha\beta)}P_{\beta\alpha})f(\mathrm{Tr}Q) + P_{\beta\alpha}\partial_{A_i(\alpha\beta)}\mathrm{Tr}Qf'(\mathrm{Tr}Q)d\mu^N.$$

Two useful computations show the importance of the non-commutative derivatives and their links with the derivation of polynomials of hermitian matrices : if P is a monomial,

$$\sum_{\alpha\beta}\partial_{A_i(\alpha\beta)}P_{\beta\alpha} = \sum_{\alpha\beta}\sum_{P=RX_iS}R_{\beta\beta}S_{\alpha\alpha} = \mathrm{Tr}\otimes\mathrm{Tr}(\partial_i P)$$

and

$$\partial_{A_i(\alpha\beta)}\mathrm{Tr}P = \sum_{P=RX_iS,\gamma}R_{\gamma\beta}S_{\alpha\gamma} = (DP)_{\alpha\beta}.$$

Thus, for $P_1,\ldots,P_\ell$ polynomials :

$$\begin{aligned}
&N^\ell E[\hat{\mu}^N(X_iP_1)(\hat{\mu}^N-\mu)^{\otimes\ell-1}(P_2\otimes\cdots\otimes P_\ell)]\\
&= N^\ell E[(\hat{\mu}^N\otimes\hat{\mu}^N)(\partial_i P_1)(\hat{\mu}^N-\mu)^{\otimes\ell-1}(P_2\otimes\cdots\otimes P_\ell)]\\
&+\sum_r N^{\ell-2}E[\hat{\mu}^N(P_1D_iP_r)(\hat{\mu}^N-\mu)^{\otimes\ell-1}(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)]\\
&- N^\ell E[\hat{\mu}^N(P_1D_iV)(\hat{\mu}^N-\mu)^{\otimes\ell-1}(P_2\otimes\cdots\otimes P_\ell)].
\end{aligned}$$

Now remember that according to Schwinger-Dyson's equation we have :

$$\mu((X_i+D_iV)P_1)-\mu\otimes\mu(\partial_iP_1)=0$$

Then, we subtract the two equalities and use the identity

$$\hat{\mu}^N\otimes\hat{\mu}^N-\mu\otimes\mu = (\hat{\mu}^N-\mu)(I\otimes\mu+\mu\otimes I)+(\hat{\mu}^N-\mu)\otimes(\hat{\mu}^N-\mu)$$

to obtain

$$\begin{aligned}
&N^\ell E[(\hat{\mu}^N-\mu)((X_i+D_iV)P_1-(I\otimes\mu+\mu\otimes I)\partial_iP_1)(\hat{\mu}^N-\mu)^{\otimes\ell-1}(P_2\otimes\cdots\otimes P_\ell)]\\
&= N^\ell E[(\hat{\mu}^N-\mu)^{\otimes\ell+1}(\partial_iP\otimes P_2\otimes\cdots\otimes P_\ell)]\\
&+\sum_r N^{\ell-2}E[(\hat{\mu}^N-\mu)^{\otimes\ell-1}(PD_iP_r\otimes P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)]\\
&+\sum_r N^{\ell-2}\mu(PD_iP_r)E[(\hat{\mu}^N-\mu)^{\otimes\ell-2}\otimes P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)].
\end{aligned}$$

To get the result it is now sufficient to apply this equality with $P_1 = D_i\bar{P}$ and then to sum on i.

□

This proposition gives us some precious hints on the limit ν of the ν^N. It should satisfy the "limit equation", if ℓ is even

$$\nu(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} \mu_{\mathbf{t}}(D_i \overline{P} D_i P_r)\nu(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \tag{4.12}$$

and if ℓ is odd

$$\begin{aligned}\nu(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu_{\mathbf{t}}(D_i \overline{P} D_i P_r)\nu(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} \nu(D_i \overline{P} D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i} \nu(\partial_i D_i \overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell).\end{aligned} \tag{4.13}$$

Hopefully, we will be able to study the solutions ν of these equations. In fact, following [GMS06] we would be able to prove that for $R, L > 0$, there exists $\varepsilon > 0$ such that for $|\mathbf{t}| < \varepsilon$ there exists an unique $\nu : \sqcup_{\ell=0}^{2L} \mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell} \to \mathbb{C}$ linear on each set $\mathbb{C}\langle X_1, \cdots, X_m\rangle^{\otimes \ell}$ with support bounded by R and which satisfy each of the previous equation for $\ell \leq 2L$. But we will proceed in a different way. Looking at these equations, we will try to recognize in them some relations between enumeration of combinatorial objects. This is the aim of the next section.

4.4 Maps of high genus

In this section, we describe the combinatorial objects that appear in the asymptotic of our measure. Remember that it was shown in [GMS06] and [GMS07] that the first two asymptotics can be viewed as a generating function for the enumeration of planar maps with vertices of a given type. Here we follow the same strategy to extend this interpretation.

First, we choose m colors $\{1, \cdots, m\}$, one for each variable X_i. A star must be thought as the neighborhood of a vertex in a plane graph. More precisely, it is a vertex with the half-edges coming out of it. One of these half-edges is distinguished and starting from it the other one are clockwisely ordered. Besides, each of these half-edges is colored.

We say that a star is of type q for a monomial $q = X_{i_1} \cdots X_{i_p}$ if it has p half-edges, the first half-edge is distinguished and of color i_1 and then in the clockwise order the second half-edge is of color i_2, the third of color i_3, ..., the p-th of color i_p. This gives a bijection between monomials and stars.

The combinatorial objects that will appear in the asymptotic of our matrix model are maps. A map is a connected graph on a compact orientated connected surface such that edges do not cross each other and faces are homeomorphic to discs. We will consider edge-colored maps such that each vertex as a distinguished edge going out of it so that we can associate a star and a well defined type to any vertex. The genus of the map is the genus of the surface. We will count maps up to homeomorphism of the surface which preserves the graph.

The typical way to construct a map is to put some stars $q_1, \cdots, q_p$ on a surface of genus g. Then we consider all the half-edges that goes outside the stars and glue them two by two while respecting the following constraints :

- Two half-edges can only be glued if they are of the same color
- The edges created by gluing two half-edges must not cross any other edge.
- At the end of the process faces must be homeomorphic to discs.

For example, one can ask how many maps of genus 1, we can construct above two stars of type $X_1X_2X_1X_2$. The answer as shown in figure 4.1 is 6. Note that as faces are homeomorphic to discs, it is sufficient to know which pairs of half-edges are glued together to build the map. For ℓ in $\mathbb{N}$, P a monomial and integers $\mathbf{k} = (k_1, \cdots, k_n)$, let $\mathcal{M}^{\mathbf{k}}(P)$ be the number of maps of genus 0 (or planar maps) with for all i, k_i stars of type q_i and one of type P where $q_1, \cdots, q_n$ are the monomials which appear in the potential V. We could define the same kind of quantities with the condition of being of genus g for $g > 1$ but then we won't be able to find any closed relation of induction between these quantities. In order to get relation induction on enumeration of maps we follow an idea of Tutte (see [Tut63]). We try to decompose a map in smaller ones by contracting one edge (Note that Tutte used to work on the dual of the graph we are considering, thus his operation is a little different).

Imagine a map of genus 1 with a root of type $P = XRXS$ and that the two half-edges corresponding to the X are glued together. Imagine also that the loop resulting from this operation is not retractable on the surface. How does the contraction of this edge decomposes the map ? Now R and S are separated by that loop, we will have to remind these two monomials. That's why we will introduce maps above a root of type $R \otimes S$. Besides R and S must be linked together, otherwise there would be a face (touched by the loop) which is not a disc, something to avoid for a map.

Thus we define some more complex vertices which will appear when we will try to decompose our maps. Let $P_1,\ldots,P_\ell$ be a family of monomials. We associate to this family a bunch of $\ell - 1$ circles such that outside the circles we put the half-edges of P_1 and in the m-th circle we put the half-edges of a star of type P_{m+1} going out of the central point and in the same order. This object will be called the root and we will name each P_r a vertex of the root (look at figure 4.2, to see a root of type $X_1X_2X_1X_2 \otimes X_1^2 \otimes X_2X_1$).

This corresponds to a star coming from a vertex which have ℓ prescribed loops, the m-th having the germs of edges corresponding to a star of type P_r. Now we construct maps with a root of type $P_1 \otimes \cdots \otimes P_\ell$ and some other vertices of type q_i. We say that such a map is minimal if when we cut the surface along the $\ell - 1$ loops of the root, we do not obtain any component homeomorphic to a disc. This means that for any P_i the component of P_i is not planar i.e. either it is linked to another P_j or it is linked to some other vertices in a way that can't be embedded on a sphere.

For ℓ, g in $\mathbb{N}$, a family of monomials $P_1, \cdots, P_\ell$ and some non-negative integers $\mathbf{k} = (k_1, \cdots, k_n)$, let $\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)$ be the number of minimal maps of genus g with a root of type $P_1 \otimes \cdots \otimes P_\ell$ and for all i, k_i stars of type q_i. For example for $V = tX_1X_2X_1X_2$, the figure 4.1 shows that $\mathcal{M}_1^1(X_1X_2X_1X_2)$ the number of minimal maps of genus 1 with a star of

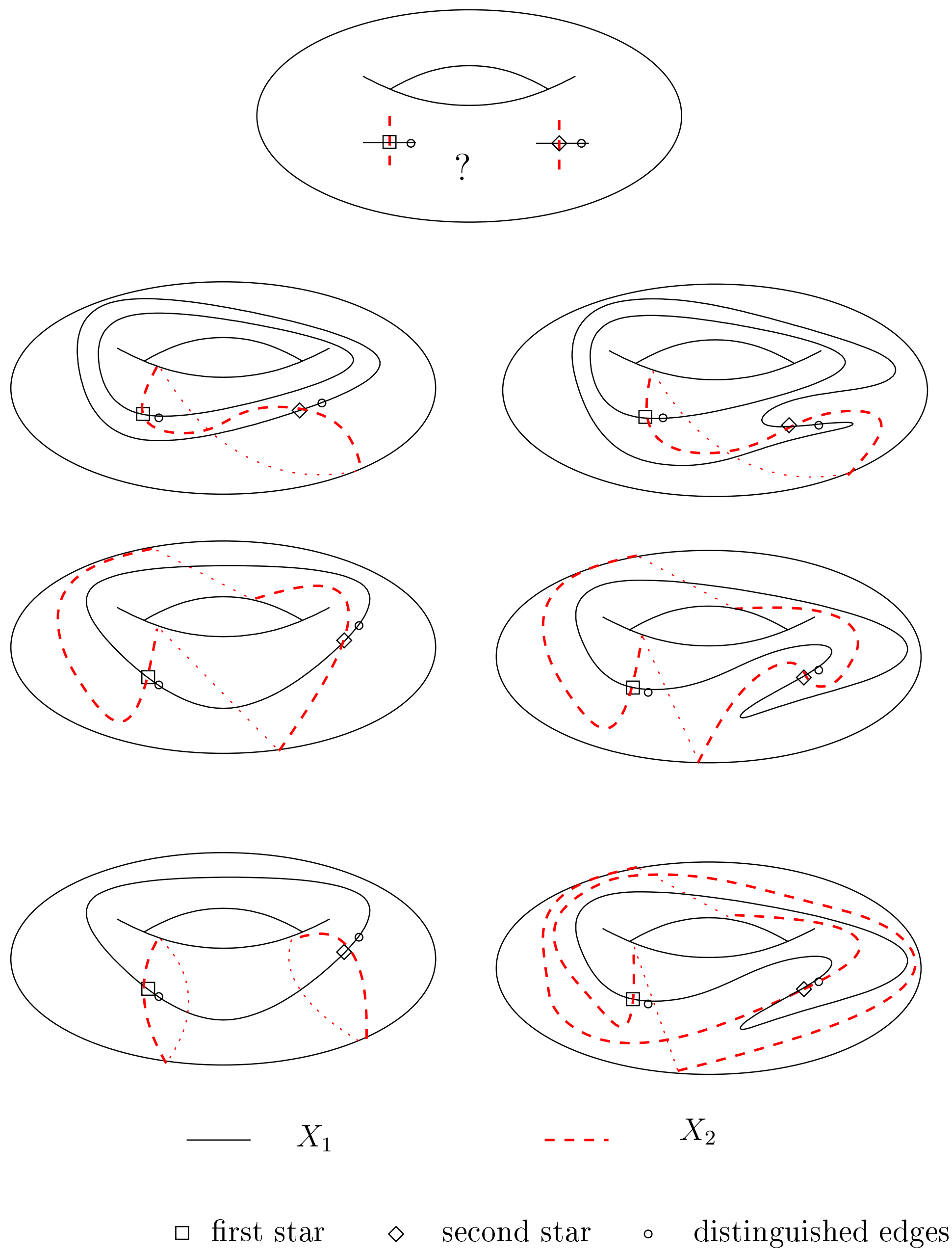

FIG. 4.1 – Maps of genus 1 above two stars of type $X_1X_2X_1X_2$.

type $X_1X_2X_1X_2$ and a root of type $X_1X_2X_1X_2$ is 6.

We extend by linearity $\mathcal{M}_{\mathbf{k}}$ and $\mathcal{M}_g^{\mathbf{k}}$ so that we can compute them on polynomials P_i instead of monomials and we define the power series for these enumerations :

$$\mathcal{I}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}}(P)$$

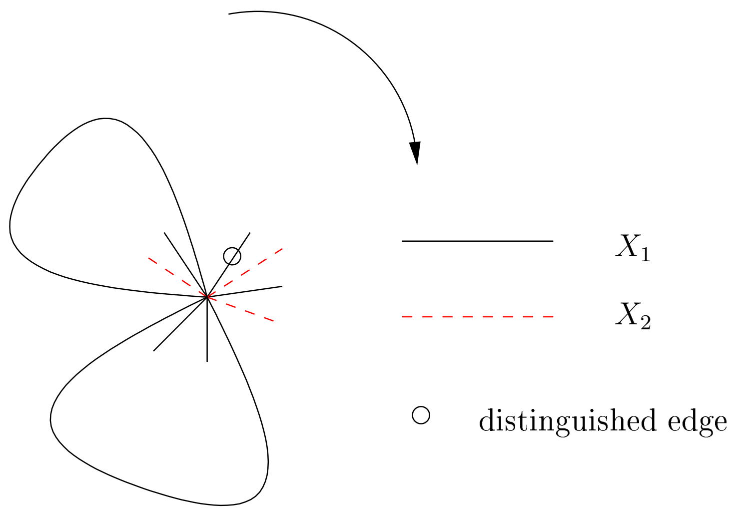

FIG. 4.2 – Root of type $X_1X_2X_1X_2 \otimes X_1^2 \otimes X_2X_1$.

and

$$\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell).$$

By convention we define for λ in $\mathbb{C}$, $\mathcal{I}_g(\lambda) = \lambda \mathbb{1}_{g=0}$ and $\mathcal{I}_g \equiv 0$ ig $g < 0$.

Recall that it was proved in [GMS06] that for $\mathbf{t}$ sufficiently small $\mathcal{I}(P) = \mu(P)$ for all P. This was proved using the fact that these two quantities satisfy the same induction relation. The induction relation for the enumeration of maps where given by a decomposition of maps following the strategy of Tutte. We now try to generalize this fact and we begin by looking at the relation given by decomposing maps. First, some values can be directly computed

$$\mathcal{M}_g^{\mathbf{k}}(1 \otimes P_2 \otimes \cdots \otimes P_\ell) = \mathbb{1}_{g=\mathbf{k}=\ell=0}$$

because the component of 1 is automatically planar.

We now want to count maps that contribute to $\mathcal{M}_g^{\mathbf{k}}(X_iP_1 \otimes \cdots \otimes P_\ell)$ with P_i monomials. We look at the first half-edge of the root $X_iP_1 \otimes \cdots \otimes P_\ell$ and see where it is glued. Remember that it must not be planar.

Then three cases may occur (see figure 4.3) :

1. Either (upper right picture in fig 4.3) the half-edge is glued to a vertex of type $q_j = RX_iS$ for a given j. First we have to choose between the k_j vertices of this type, then we contract the edge coming from this gluing to form a vertex of type SRP_1. This creates

$$\sum_{1\le j\le n, k_j\neq 0} k_j \mathcal{M}_g^{k_1,\cdots,k_j-1,\cdots,k_n}(D_iq_jP_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

possibilities.

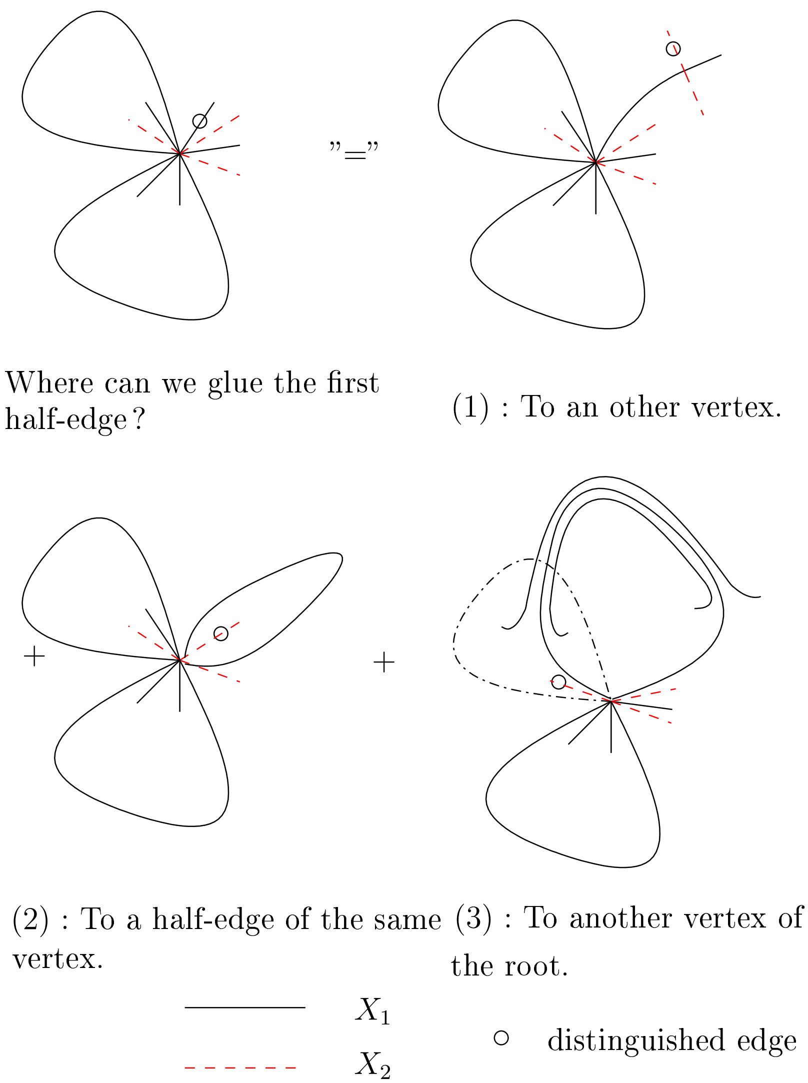

FIG. 4.3 – The decomposition process for maps.

2. The second case (bottom left picture in fig 4.3) occurs if the half-edge is glued to another half-edge of $P_1 = RX_iS$. It cuts P_1 in two : R and S. It occurs for all decomposition of P_1 into $P_1 = RX_iS$. To write the expression that will arise in a more convenient way, we will use the non-commutative derivative ∂ which satisfy for P a monomial

$$\partial_i P = \sum_{P=RX_iS} R \otimes S.$$

We are now left with two separate circles, one for R and one for S. Either both are non-planar which leads to

$$\sum_{P_1=RX_iS} \mathcal{M}_g^{\mathbf{k}}(R \otimes S \otimes P_2 \otimes \cdots \otimes P_\ell) = \mathcal{M}_g^{\mathbf{k}}(\partial_i P_1 \otimes P_2 \otimes \cdots \otimes P_\ell)$$

possibilities or one of the component is planar then the two components can not be linked thus we have to share the vertices of type q_i between them. They are $\binom{\mathbf{k}}{\mathbf{k}'} = \prod_j \binom{k_j}{k'_j}$ ways of choosing for all j, k'_j vertices of type q_j for the component of R.
If the component of R is planar and the one of S is not this leads to

$$\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}_g^{\mathbf{k}''}((\mathcal{M}^{\mathbf{k}'} \otimes I)(\partial_i P_1) \otimes \cdots \otimes P_\ell)$$

possibilities or S is planar and R is not,

$$\sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}_g^{\mathbf{k}'}((I \otimes \mathcal{M}^{\mathbf{k}''})(\partial_i P_1) \otimes \cdots \otimes P_\ell)$$

possibilities.

3. The last case occurs if the half-edge is glued with another vertex $P_r = RX_iS$ of the root. This create a vertex of type $D_iP_rP_1$. Note that the edge can not cross the circles of the root so it must go through a handle of the surface thus it changes the genus by one. But this vertex is now free from the condition of non planarity so it can either be planar and thus be separate from the other vertices of the root :

$$\sum_{2\leq m\leq l, \mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}^{\mathbf{k}'}(D_iP_rP_1)\mathcal{M}_{g-1}^{\mathbf{k}''}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

possibilities or it may still be non-planar :

$$\sum_{2\leq m\leq \ell} \mathcal{M}_{g-1}^{\mathbf{k}}(D_iP_rP_1 \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)$$

possibilities.

This decomposition gives us the relation

$$\begin{aligned}
&\mathcal{M}_g^{\mathbf{k}}(X_iP_1 \otimes \cdots \otimes P_\ell) \\
&= \sum_{1\leq j\leq n, k_j\neq 0} k_j \mathcal{M}_g^{k_1,\cdots,k_j-1,\cdots,k_n}(D_iq_jP_1 \otimes P_2 \otimes \cdots \otimes P_\ell) \\
&+ \mathcal{M}_g^{\mathbf{k}}(\partial_i P_1 \otimes P_2 \otimes \cdots \otimes P_\ell) \\
&+ \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}_g^{\mathbf{k}''}((\mathcal{M}^{\mathbf{k}'} \otimes I)(\partial_i P_1) \otimes \cdots \otimes P_\ell) \\
&+ \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mathcal{M}_g^{\mathbf{k}'}((I \otimes \mathcal{M}^{\mathbf{k}''})(\partial_i P_1) \otimes \cdots \otimes P_\ell) \\
&+ \sum_{2\leq m\leq \ell} \mathcal{M}_{g-1}^{\mathbf{k}}(D_iP_rP_1 \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).
\end{aligned}$$

We can sum these identities and then sum on the k_i's to obtain the following equality, for all g, for all $P_1\ldots,P_\ell$,

$$\begin{aligned}\mathcal{I}_g(X_iP_1\otimes\cdots\otimes P_\ell) &= \sum_j(-t_j)\mathcal{I}_g(D_iq_jP_1\otimes\cdots\otimes P_\ell)\\ &+\mathcal{I}_g((I\otimes I+\mathcal{I}\otimes I+I\otimes\mathcal{I})\partial_iP_1\otimes\cdots\otimes P_\ell)\\ &+\sum_{m\geq 2}\mathcal{I}_{g-1}((I+\mathcal{I})D_iP_rP_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell)\end{aligned}$$

We can reformulate this by applying it to $P_1=D_i\bar{P}$ and then summing on i :

$$\begin{aligned}\mathcal{I}_g(\Xi P\otimes\cdots\otimes P_\ell) &= \sum_{m\geq 2,i}\mu(D_iP_rD_i\bar{P})\mathcal{I}_{g-1}(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)\\ &+\sum_{m\geq 2,i}\mathcal{I}_{g-1}(D_iP_rD_i\bar{P}\otimes\cdots\check{P}_r\cdots\otimes P_\ell)\\ &+\sum_i\mathcal{I}_g(\partial_iD_i\bar{P}\otimes\cdots\otimes P_\ell).\end{aligned}\tag{4.14}$$

where we used the identity $\mathcal{I}=\mu$. Note that maps that appear in the enumeration must satisfy the condition of non-planarity, this imposes a high genus. We have to break the "planarity" of ℓ components, this can't be done without at least $[\frac{\ell+1}{2}]$ handles on the surface (each handle allow one edge to cross from one vertex of the root to another one, breaking the planarity of two components at most). Thus if $g<Ent(\frac{\ell+1}{2})$, $\mathcal{I}_g(P_1\otimes\cdots\otimes P_\ell)=0$. This allow us to write the previous equation in a special case which will appear to be useful, if ℓ is even,

$$\mathcal{I}_{\frac{\ell}{2}}(\Xi P\otimes\cdots\otimes P_\ell)=\sum_{m\geq 2,i}\mu(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{\ell-2}{2}}(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell).\tag{4.15}$$

Thus for ℓ even, $\mathcal{I}_{\frac{\ell}{2}}$ satisfy the limit equation (4.12) of the matrix model. One can easily check that for ℓ odd $\mathcal{I}_{\frac{\hat{\ell}}{2}}=\mathcal{I}_{\frac{\ell+1}{2}}$ satisfy also the limit equation (4.13) :

$$\begin{aligned}\mathcal{I}_{\frac{\ell+1}{2}}(\Xi P\otimes\cdots\otimes P_\ell) &= \sum_{m\geq 2,i}\mathcal{I}(D_iP_rD_i\overline{P})\mathcal{I}_{\frac{(\ell-2)+1}{2}}(P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)\\ &+\sum_{m\geq 2,i}\mathcal{I}_{\frac{\ell-1}{2}}(D_iP_rD_i\overline{P}\otimes P_2\otimes\cdots\check{P}_r\cdots\otimes P_\ell)\\ &+\sum_i\mathcal{I}_{\frac{\ell+1}{2}}(\partial_iD_i\bar{P}\otimes\cdots\otimes P_\ell).\end{aligned}$$

We can deduce from these identities a control on these enumerations : For all $g\geq 0$, there exists $\varepsilon>0$ such that for $\mathbf{t}\in B(0,\varepsilon)$, $\mathcal{I}^g$ is absolutely convergent and has a bounded support i.e. there exists $M>0$ such that for all polynomials $P_1,\cdots,P_\ell$,

$$|\mathcal{I}_g(P_1\otimes\cdots\otimes P_\ell)|\leqslant\|P_1\|_M\cdots\|P_\ell\|_M$$

. **Proof.**

It is sufficient to show that for all $g \geq 0$, there exists $A_g, B_g > 0$ such that for all h, for all monomials P_1,...,P_ℓ, and all integers k_i :

$$\frac{\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)}{\prod_i k_i!} \leq A_g^{\Sigma deg P_i} B_g^{\Sigma k_i}$$

This is easy by induction using the induction relation on the $\mathcal{M}_g^{\mathbf{k}}(P_1 \otimes \cdots \otimes P_\ell)$ and in the same spirit than the control for planar maps in [GMS06].

□

Finally we need to know the effect of derivation on these generating function. In fact, derivation adds some vertices to the enumeration. For all $\mathbf{j} = (j_1, \cdots, j_n)$, we have

$$\mathcal{D}_{\mathbf{j}}\mathcal{I}(P) = (-1)^{\mathbf{j}} \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}^{\mathbf{k}+\mathbf{j}}(P)$$

and

$$\mathcal{D}_{\mathbf{j}}\mathcal{I}_g(P_1 \otimes \cdots \otimes P_\ell) = (-1)^{\mathbf{j}} \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}+\mathbf{j}}(P_1 \otimes \cdots \otimes P_\ell).$$

Besides, these series are absolutely convergent and has a bounded support. **Proof.**

The proof is straightforward, $\mathcal{I}$,$\mathcal{I}_g$ are analytic in a neighborhood of the origin thus their derivatives are analytic and their series are given by differentiating term by term $\mathcal{I}$ and $\mathcal{I}_g$.

□

Thus derivatives fix some vertices in the enumeration a fact often used in combinatorics to find relation between generating functions of graphs.

4.5 High order observable

We have already seen that $\nu(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\ell}{2}}(P_1 \otimes \cdots \otimes P_\ell)$ satisfy the limit equation of ν^N. This is our candidate for the limit of the ν^N's. In fact this suggests a statement closely related to 4.1.1. For all ℓ, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all polynomials P_1,...,P_ℓ,

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\ell}{2}}(P_1 \otimes \cdots \otimes P_\ell) + \frac{1}{N^2}\mathcal{I}_{\frac{\ell}{2}+1}(P_1 \otimes \cdots \otimes P_\ell) + \cdots + \frac{1}{N^{2g}}\mathcal{I}_{\frac{\ell}{2}+g}(P_1 \otimes \cdots \otimes P_\ell) + o(\frac{1}{N^{2g}}).$$

To prove this we have to define all the correction to the convergence. We define $\nu_1^N = \nu^N$ and by induction on h, for all N, all polynomials $P_1, \ldots, P_\ell$,

$$\nu_{h+1}^N(P_1 \otimes \cdots \otimes P_\ell) = N^2(\nu_h^N - \mathcal{I}_{\frac{\ell}{2}+h-1})(P_1 \otimes \cdots \otimes P_\ell).$$

Those quantities satisfy also some induction relation similar to those of proposition 4.3.2 For all $h \geqslant 2$, ℓ in $\mathbb{N}$, for all polynomial $P_1,\ldots,P_\ell$, for all N, the "finite Schwinger-Dyson's equation of order h" ($\mathbf{SD}_{h,\ell}^N$) is satisfied by ν^N : if ℓ is even

$$\begin{aligned}\nu_h^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)\\ &+ \sum_{i,r} \nu_{h-1}^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (\mathbf{SD}_{h,\ell}^N)\\ &+ \sum_{i} \nu_{h-1}^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

and if ℓ is odd

$$\begin{aligned}\nu_h^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,r} \mu(D_i\overline{P}D_iP_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)\\ &+ \sum_{i,r} \nu_h^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \qquad (\mathbf{SD}_{h,\ell}^N)\\ &+ \sum_{i} \nu_h^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

Proof.

Remember that we have shown in Proposition 4.3.2, for ℓ even

$$\begin{aligned}\nu_1^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = \sum_{i,r} &\mu_{\mathbf{t}}(D_i\overline{P}D_iP_r)\nu_1^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)\\ &+ \frac{1}{N^2}\sum_{i,r} \nu_1^N(D_i\overline{P}D_iP_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)\\ &+ \frac{1}{N^2}\sum_{i} \nu_1^N(\partial_i D_i\overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned}$$

and according to (4.15)

$$\mathcal{I}_{\frac{\ell}{2}}(\Xi P \otimes \cdots \otimes P_\ell) = \sum_{m \geq 2, i} \mathcal{I}(D_iP_rD_i\bar{P})\mathcal{I}_{\frac{\ell-2}{2}}(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell).$$

Thus if we subtract these two equalities and multiply the result by N^2 we obtain ($\mathbf{SD}_{2,\ell}^N$)

(Observe that with our convention $\nu^N(\lambda) = \mathcal{I}_0(\lambda)$).

$$\begin{aligned}\nu_2^N(\Xi P \otimes P_2 \otimes \cdots \otimes P_\ell) = &\sum_{i,r} \mu(D_i \overline{P} D_i P_r) \nu_2^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,r} \nu_1^N(D_i \overline{P} D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_i \nu_1^N(\partial_i D_i \overline{P} \otimes P_2 \otimes \cdots \otimes P_\ell).\end{aligned}$$

Now suppose that for ℓ even, $h \geqslant 2$, for all polynomial $P_1, \ldots, P_\ell$, for all N, $(\mathbf{SD}_{h,\ell}^N)$ is satisfied. Then according to (4.14) :

$$\begin{aligned}\mathcal{I}_{\frac{\ell}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) = &\sum_i \mathcal{I}_{\frac{(\ell+1)+1}{2}+h-2}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mathcal{I}_{\frac{(\ell-1)+1}{2}+h-2}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{\frac{(\ell-2)}{2}+h-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).\end{aligned}$$

and this can be translated into

$$\begin{aligned}\mathcal{I}_{\frac{\hat{\ell}}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) = &\sum_i \mathcal{I}_{\widehat{\frac{\ell+1}{2}}+h-2}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mathcal{I}_{\widehat{\frac{\ell-1}{2}}+h-2}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{\widehat{\frac{\ell-2}{2}}+h-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).\end{aligned}$$

Subtracting this equality from $(\mathbf{SD}_{h,\ell}^N)$ we get $(\mathbf{SD}_{h+1,\ell}^N)$. This proves by induction $(\mathbf{SD}_{h,\ell}^N)$ for all h, and for all ℓ even.

We proceed in the same way for ℓ odd. Observe that the equation for ℓ odd and $h = 1$ is satisfied according to Proposition 4.3.2. Then observe that for ℓ odd, $(\mathbf{SD}_{h+1,\ell}^N)$ can be obtained by subtracting (4.14) with $g = \frac{\ell+1}{2} + h - 1$

$$\begin{aligned}\mathcal{I}_{\frac{\hat{\ell}}{2}+h-1}(\Xi P \otimes \cdots \otimes P_\ell) = &\sum_i \mathcal{I}_{\widehat{\frac{\ell+1}{2}}+h-2}(\partial_i D_i \bar{P} \otimes \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mathcal{I}_{\widehat{\frac{\ell-1}{2}}+h-2}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{r \geq 2, i} \mu(D_i P_r D_i \bar{P}) \mathcal{I}_{\widehat{\frac{\ell-2}{2}}+h-1}(D_i P_r D_i \bar{P} \otimes \cdots \check{P}_r \cdots \otimes P_\ell).\end{aligned}$$

from $(\mathbf{SD}_{h,\ell}^N)$.

□

4.6 Asymptotic of the matrix model

The issue with the previous relations is that they only give us the moments of products of polynomials such that the first polynomial is in the image of Ξ. Thus we need to invert Ξ. We define the operator norm with respect to $\|.\|_M$:

$$|||A|||_M = \sup_{\|P\|_M \leqslant 1} \|AP\|_M.$$

In [GMS07], we gave some estimates on the operator norm of Ξ.

1. The operator Ξ_0 is invertible on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$.
2. There exists $M_0 > 0$ such that for all $M > M_0$, the operators Ξ_2, Ξ_0 and Ξ_0^{-1} are continuous and their norm are uniformly bounded for $\mathbf{t}$ in B_η.
3. For all polynomials P, $\deg \Xi_0^{-1} P \leqslant \deg P$ and $\deg \Xi_1 P \leqslant \deg P + \deg V - 2$
4. For all $\varepsilon, M > 0$, there exists $\eta_\varepsilon > 0$ such for $|\mathbf{t}| < \eta_\varepsilon$, Ξ_1 is continuous as an operator on $\mathbb{C}\langle X_1, \cdots, X_m\rangle$ and $|||\Xi_1|||_M \leqslant \varepsilon$.

The last step to proves Theorem 4.1.1 is to control the ν_N^h. This done by induction using the recursive relation for those quantities. The only issue is to invert the operator Ξ. This was doable in the context of [GMS06] by completing the space of polynomials for an appropriate norm. This can not be done here and use some approximate inversion of Ξ with the degree of the inversion depending on the dimension N. This leads to a quite technical proof.

For all $\ell, h \in \mathbb{N}^*$, $\alpha > 0$, there exists $C, \eta, M_0 > 0$, such that for all $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ and all polynomials $P_1, \ldots, P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,

$$|\nu_h^N(P_1 \otimes \cdots \otimes P_\ell)| \leq C\|P_1\|_M \cdots \|P_\ell\|_M$$

Proof.

The case $h = 1$, ℓ even is a direct consequence of Lemma 4.3.1. We treat the other cases by induction using Proposition 4.5.2. As the equations are different according to the parity of ℓ, we have to be careful : we prove the result by an induction on h and for a fixed h we deal first with the case ℓ even and then with the case ℓ odd (Note that both time we will do an induction on ℓ). Now we choose $\ell, h \in \mathbb{N}^*$, $\alpha > 0$ and polynomials $P_1, \ldots, P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$.

Then, the idea to nearly invert Ξ on a polynomial P is to approximate P_1 by $\Xi Q_n = (\Xi_0 + \Xi_1) Q_n$ with

$$Q_n = \sum_{k=0}^{n-1} (-\Xi_0^{-1}\Xi_1)^k \Xi_0^{-1} P_1.$$

The remainder is

$$R_n = P_1 - \Xi Q_n = (-\Xi_1 \Xi_0^{-1})^n P_1.$$

As Ξ_1 is the multiplication by a derivative of V it should have a small norm and the remainder should be easily controlled.

We can make the decomposition :

$$\nu_h^N(P_1 \otimes \cdots \otimes P_\ell) = \nu_h^N(\Xi Q_n \otimes \cdots \otimes P_\ell) + \nu_h^N(R_n \otimes \cdots \otimes P_\ell) \tag{4.16}$$

Now we let n goes to infinity with N, for example $n = [\sqrt{N}]$. It is important that n goes to infinity no too slowly but we must have $n = O(\sqrt{N})$ in order to use all the induction hypothesis. An important fact is that the degrees of R_n and Q_n are $O(\sqrt{N})$ since $\Xi_0^{-1}\Xi_1$ change the degree by at most $D-2$.

We first control the term with R_n, by definition of the ν_N^h,

$$\nu_h^N(R_n \otimes \cdots \otimes P_\ell) = (N^{2(h-1)}\nu_1^N + N^{2(h-1)}\mathcal{I}_{\frac{\ell}{2}} + \cdots + \mathcal{I}_{\frac{\ell}{2}+h-1})(R_n \otimes \cdots \otimes P_\ell).$$

Each of the $\mathcal{I}_g$ are compactly supported according to Lemma 4.4.3 so that if η is sufficiently small, for $\mathbf{t}$ in $B_{\eta,c}$, $\mathcal{I}_{\frac{\ell}{2}}$, …,$\mathcal{I}_{\frac{\ell}{2}+h-1}$ are convergent and we can take M bigger than the radius of their support. Besides, Proposition 4.3.1 shows that for polynomials P of degree of order $N^{\frac{1}{2}}$,

$$|\nu_1^N(P_1 \otimes \cdots \otimes P_\ell)| \leqslant CN\|P_1\|_M \cdots \|P_\ell\|_M.$$

Thus according to Lemma 4.6.1, for η small, $\|R_n\|_M \leqslant |||\Xi_1\Xi_0^{-1}|||^n\|P_1\|_M$ decrease exponentially fast in n and this uniformly for $\mathbf{t} \in B_{\eta,c}$. Then since $n \sim \sqrt{N}$

$$|\mathcal{I}_g(R_n \otimes \cdots \otimes P_\ell)| \leqslant \|R_n\|_M\|P_2\|_M \ldots \|P_\ell\|_M \leqslant Ce^{-C'\sqrt{N}}\|P_1\|_M \ldots \|P_\ell\|_M.$$

Thus,

$$|\nu_h^N(R_n \otimes \cdots \otimes P_\ell)| \leqslant N^{2h}Ce^{-C'\sqrt{N}}\|P_1\|_M \ldots \|P_\ell\|_M \tag{4.17}$$

and $N^{2h}Ce^{-C'\sqrt{N}}$ is bounded.

Finally we have to deal with $\nu_h^N(\Xi Q_n \otimes \cdots \otimes P_\ell)$. We can use $(\mathbf{SD}_{h,\ell}^N)$:

$$\begin{aligned}\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell) &= \sum_{i,m} \mu_{\mathbf{t}}(D_i\bar{Q}_n D_i P_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i,m} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(D_i\bar{Q}_n D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell) \\ &+ \sum_{i} \nu_{h-\mathbb{1}_{\ell \text{ even}}}^N(\partial_i D_i\bar{Q}_n \otimes P_2 \otimes \cdots \otimes P_\ell)\end{aligned} \tag{4.18}$$

We now use the induction hypothesis. Indeed if h is even, on the right hand side either h decreases or h remains constant and ℓ decreases and remains even. If h is odd, either ℓ becomes even or ℓ decreases. Now, let C be an uniform bound on the norm of Ξ_0^{-1} (which exists according to Lemma 4.6.1) by definition of Q_n,

$$\|Q_n\|_M \leqslant \sum_{k=0}^{n-1} |||\Xi_0^{-1}\Xi_1|||_M^k|||\Xi_0^{-1}|||_M\|P\|_M \leqslant \frac{C}{1-C|||\Xi_1|||_M}\|P\|_M.$$

Thus, using Lemma 4.6.1, if η is sufficiently small, for $\mathbf{t}$ in $B_{\eta,c}$, $\|Q_n\|_M \leqslant 2\|P\|_M$. Note that

$$\deg Q_n \leqslant \deg P_1 + 2\sqrt{N}(D-1) \leqslant (\alpha + 2(D-1))\sqrt{N}.$$

We can now apply the induction hypothesis with $\alpha' = \alpha + 2(D-1)$, there exists M, C, η such that for $\mathbf{t}$ in $B_{\eta,c}$,

$$\begin{aligned}&|\mu_{\mathbf{t}}(D_i\bar{Q}_n D_i P_r)\nu_h^N(P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)| \\ &\leqslant C\|D_i\bar{Q}_n D_i P_r\|_M \|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|\end{aligned} \tag{4.19}$$

where we have assumed that M is bigger than the radius of the support of $\mu_{\mathbf{t}}$ which is bounded according to (4.4). Besides, by the induction hypothesis we can obtain with the same constant,

$$\begin{aligned}&|\nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(D_i\bar{Q}_n D_i P_r \otimes P_2 \otimes \cdots \check{P}_r \cdots \otimes P_\ell)| \\ &\leqslant C\|D_i\bar{Q}_n D_i P_r\|_M \|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|_M\end{aligned} \tag{4.20}$$

and

$$\begin{aligned}&|\nu^N_{h-\mathbb{1}_{\ell \text{ even}}}(\partial_i D_i\bar{Q}_n \otimes P_2 \otimes \cdots \otimes P_\ell)| \\ &\leqslant \|\partial_i D_i\bar{Q}_n\|_M \|P_2\|_M \cdots \|\check{P}_r\|_M \cdots \|P_\ell\|_M.\end{aligned} \tag{4.21}$$

Now remember, that if $M' > M$,

$$D_i : (\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_{M'}) \to (\mathbb{C}\langle X_1, \cdots, X_m\rangle, \|.\|_M)$$

is continuous, thus

$$\begin{aligned}\|D_i\bar{Q}_n D_i P_r\|_M &\leqslant \|D_i\bar{Q}_n\|_M \|D_i P_r\|_M \\ &\leqslant C\|Q_n\|_{M'}\|P_r\|_{M'} \leqslant C\|P_1\|_{M'}\|P_r\|_{M'}\end{aligned}$$

and

$$\|\partial_i D_i\bar{Q}_n\|_M \leqslant C\|P_1\|_{M'}.$$

If we use inequalities (4.19), (4.20) and (4.21) in the decomposition (4.18), we get

$$|\nu_h^N(\Xi Q_n \otimes P_2 \otimes \cdots \otimes P_\ell)| \leqslant C\|P_1\|_{M'}\|P_2\|_{M'} \cdots \|P_\ell\|_{M'}.$$

Finally, we conclude with (4.17) and (4.16)

□

Now, for all ℓ, h there exists $\eta > 0$ such that for $\mathbf{t} \in B_{\eta,c}$, for all polynomials $P_1, \cdots, P_\ell$,

$$\nu_g^N(P_1 \otimes \cdots \otimes P_\ell) = N^2(\nu_g^N - \mathcal{I}_{\frac{\hat{\ell}}{2}+g-1})(P_1 \otimes \cdots \otimes P_\ell)$$

is a bounded sequence for all $g \leqslant h+1$. Thus for all $g \leqslant h$, $\nu_g^N(P_1 \otimes \cdots \otimes P_\ell)$ goes to $\mathcal{I}_{\frac{\hat{\ell}}{2}+g-1}$ and

$$\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{I}_{\frac{\hat{\ell}}{2}} + \frac{1}{N^2}\mathcal{I}_{\frac{\hat{\ell}}{2}+1} + \cdots + + \frac{1}{N^{2h}}\mathcal{I}_{\frac{\hat{\ell}}{2}+h} + o(\frac{1}{N^{2h}}).$$

Thus Proposition 4.5.1 is proved. The special case $\ell = 1$ is exactly Theorem 4.1.2 :

$$E[\hat{\mu}^N(P)] = \mathcal{I}(P) + \frac{1}{N^2}\nu_1^N(P)$$
$$= \mathcal{I}(P) + \frac{1}{N^2}\mathcal{I}_1(P) + \cdots + \frac{1}{N^{2g}}\mathcal{I}_g(P) + o(\frac{1}{N^{2g}}).$$

Thus we can prove Theorem 4.1.1. For all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,

$$F_{V_\mathbf{t}}^N = F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

and F^g is the generating function for maps of genus g associated with V :

$$F^g(\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}^\mathbf{k}$$

where if $\mathbf{k} = (k_1, \cdots, k_n)$, $\mathcal{C}^\mathbf{k}$ is the number of maps on a surface of genus g with k_i vertices of type q_i. **Proof.**

Note that the estimate we get in Lemma 4.7.2 are uniform in $\mathbf{t}$ provided we are in $B_{\eta,c}$. Now observe that if $V_\mathbf{t}$ is c-convex then for α in $[0,1]$, $V_{\alpha\mathbf{t}}$ is c-convex if $c \leqslant 1$ and 1-convex if $c > 1$. Thus if $\mathbf{t}$ is in $B_{\eta,c}$, for all $0 < \alpha < 1$, $\alpha\mathbf{t} \in B_{\eta,\min(c,1)}$. This allow us to use Proposition 4.7.3 with an uniformly bounded remainder.

$$F_{V_\mathbf{t}}^N = \int_0^1 -E_{\mu_{V_{\alpha\mathbf{t}}}^N}[\hat{\mu}^N(V_\mathbf{t})]d\alpha$$
$$= -\int_0^1 \mu_{\alpha\mathbf{t}}(V_\mathbf{t})d\alpha - \frac{1}{N^2}\int_0^1 \nu_{\alpha\mathbf{t}}^N(\hat{\mu}^N(V_\mathbf{t}))d\alpha$$
$$= F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}})$$

with

$$F^0 = -\int_0^1 \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}\mathcal{M}^\mathbf{k}(V_\mathbf{t})d\alpha$$
$$= \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}^\mathbf{k}$$

since it can be easily checked that

$$\sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-ut_i)^{k_i}}{k_i!}\mathcal{C}^\mathbf{k}$$

has the same derivative in u than $-\int_0^u \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}\mathcal{M}^{\mathbf{k}}(V_{\mathbf{t}})d\alpha$. With the same technique,

$$\begin{aligned}
F^g &= -\int_0^1 \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_i \frac{(-\alpha t_i)^{k_i}}{k_i!}\mathcal{M}_g^{\mathbf{k}}(V_{\mathbf{t}})d\alpha \\
&= \sum_{k_1,\cdots,k_n\in\mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}_g^{\mathbf{k}}
\end{aligned}$$

This proves the Theorem :

$$F_{V_{\mathbf{t}}}^N = F^0 + \cdots + \frac{1}{N^2 g}F^g + o(\frac{1}{N^{2g}}).$$

□

4.7 Higher derivatives.

In this section we will show that one can differentiate these expansions term by terms. Indeed, the family of the ν_h^N's is sufficiently rich to express any of its own derivatives. Thus, we will be able to find a recursive decomposition of this derivatives.

For all $1 \leqslant j \leqslant n$, for all polynomials $P_1, \cdots, P_\ell$, if ℓ is even,

$$\begin{aligned}
\frac{\partial}{\partial t_j}\nu_1^N(P_1\otimes\cdots\otimes P_\ell) &= -\nu_1^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\
&\quad -\sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_1^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\
&\quad +\nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu_1^N(q_j)
\end{aligned}$$

and if ℓ is odd,

$$\begin{aligned}
\frac{\partial}{\partial t_j}\nu_1^N(P_1\otimes\cdots\otimes P_\ell) &= -\nu_2^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\
&\quad -\sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_2^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\
&\quad +\nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu_1^N(q_j).
\end{aligned}$$

Proof.

We simply need to differentiate

$$\frac{1}{Z_V^N}\int\prod_r(\frac{1}{N}\mathrm{Tr}-\mu)(P_r)e^{-N\mathrm{Tr}V}d\mu^N.$$

In that expression we can either differentiate the potential, $\frac{1}{Z_V^N}$,or one of the $\mu(P_r)$, this leads to

$$\begin{aligned}\frac{\partial}{\partial t_j}E[\prod_r(\frac{1}{N}\mathrm{Tr}-\mu)(P_r)] &= -N^2E[\prod_r(\frac{1}{N}\mathrm{Tr}-\mu)(P_r)(\frac{1}{N}\mathrm{Tr}-\mu)(q_j)] \\ &+ E[\prod_r(\frac{1}{N}\mathrm{Tr}-\mu)(P_r)]N^2E[(\frac{1}{N}\mathrm{Tr}-\mu)(q_j)] \\ &- \sum_r \frac{\partial}{\partial t_j}\mu(P_r)E[\prod_{r'\neq r}(\frac{1}{N}\mathrm{Tr}-\mu)(P_{r'})]\end{aligned}$$

Where one can notice that we have added in the two first terms of the right hand side the quantity $\mu(q_j)$ but these two modifications cancel each other.

Now multiply by the normalisation N^ℓ to get the equation in the case ℓ even. In the case ℓ odd, if we multiply by $N^{\ell+1}$ we get

$$\begin{aligned}\frac{\partial}{\partial t_j}\nu_1^N(P_1\otimes\cdots\otimes P_\ell) &= -N^2\nu_1^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\ &+ \nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu^N(q_j) \\ &- N^2\sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_1^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\ &= -\nu_2^N(P_1\otimes\cdots\otimes P_\ell\otimes q_j) \\ &+ \nu_1^N(P_1\otimes\cdots\otimes P_\ell)\nu^N(q_j) \\ &- \sum_{r=1}^{\ell}\frac{\partial}{\partial t_j}\mu(P_r)\nu_2^N(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell) \\ &+ N^2 r_N\end{aligned}$$

with by definition of ν_2^N,

$$r_N = -\mathcal{I}_{\frac{l+1}{2}}(P_1\otimes\cdots\otimes P_\ell\otimes q_j) - \sum_r \frac{\partial}{\partial t_j}\mu(P_r)\mathcal{I}_{\frac{l-1}{2}}(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell).$$

Let's have a closer look at this expression, $\mathcal{I}_{\frac{l+1}{2}}(P_1\otimes\cdots\otimes P_\ell\otimes q_j)$ counts maps with $\frac{l+1}{2}$ handles and such none of the $l+1$ components P_1,…, P_ℓ, q_j is planar. Since each handle can break the planarity of at most two components, the only way to obtain such a configuration is to form $\frac{l+1}{2}$ couples among these l components and in each of these couples to put a handle between the two components. For example, one can decompose these maps according to the other vertex in the couple of q_j :

$$\mathcal{I}_{\frac{l+1}{2}}(P_1\otimes\cdots\otimes P_\ell\otimes q_j) = \sum_r \mathcal{I}_1(P_r\otimes q_j)\mathcal{I}_{\frac{l-1}{2}}(P_1\otimes\cdots\check{P}_r\cdots\otimes P_\ell).$$

Now observe that $\mathcal{I}_1(P_r\otimes q_j)$ counts maps of genus 1 such that the component of P_r is linked to the component of q_j, thus it is equivalent to the counting of planar maps with two prescribed vertices, one of type P_r and one of type q_j. According to Lemma 4.4.4, that exactly what count $-\frac{\partial}{\partial t_j}\mu(P_r)$. Thus $r_N = 0$ and the proposition is proved.

□

With this decomposition we can now show that the derivatives of the ν_h^N's are of order 1.

For all $\mathbf{j} = (j_1, \cdots, j_n)$, for all $\ell, h \in \mathbb{N}^*$, $\alpha > 0$, there exists constants $C, \eta, M_0 > 0$, such that for all $\mathbf{t} \in B_{\eta,c}$, $M > M_0$ and all polynomials $P_1,\ldots,P_\ell$ of degree less than $\alpha N^{\frac{1}{2}}$,

$$|\mathcal{D}_{\mathbf{j}}\nu_h^N(P_1 \otimes \cdots \otimes P_\ell)| \leq C\|P_1\|_M \cdots \|P_\ell\|_M$$

Proof.

Proposition 4.7.1 allow to relate any derivative of an observable in terms of derivatives of lesser degree of those observable. Note also that since it is an exact relation this is still true for the correction of these observable and thus this allows us to deduce the result by induction on the number of derivatives. The start of this induction, that is the case with $|\mathbf{j}| = 0$ is exactly Lemma 4.6.2.

□

From there we deduce For all ℓ, for all $g \in \mathbb{N}$, for all $\mathbf{j} = (j_1, \cdots, j_n)$ there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$, for all polynomials $P_1,\ldots,P_\ell$,

$$\mathcal{D}_{\mathbf{j}}\nu^N(P_1 \otimes \cdots \otimes P_\ell) = \mathcal{D}_{\mathbf{j}}\mathcal{I}_{\frac{\ell}{2}}(P_1 \otimes \cdots \otimes P_\ell) + \frac{1}{N^2}\mathcal{D}_{\mathbf{j}}\mathcal{I}_{\frac{\ell}{2}+1}(P_1 \otimes \cdots \otimes P_\ell) + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}}\mathcal{I}_{\frac{\ell}{2}+g}(P_1 \otimes \cdots \otimes P_\ell) + o(\frac{1}{N^{2g}}).$$

Finally we prove Theorem 4.1.3, For all $\mathbf{j} = (j_1, \cdots, j_n)$, for all $g \in \mathbb{N}$, there exists $\eta > 0$ such that for all $\mathbf{t}$ in $B_{\eta,c}$,

$$\mathcal{D}_{\mathbf{j}}F_{V_{\mathbf{t}}}^N = \mathcal{D}_{\mathbf{j}}F^0(\mathbf{t}) + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}}F^g(\mathbf{t}) + o(\frac{1}{N^{2g}}).$$

Besides, $\mathcal{D}_{\mathbf{j}}F^g$ is the generating function for rooted maps of genus g associated with V :

$$\mathcal{D}_{\mathbf{j}}F^g(\mathbf{t}) = \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!}\mathcal{C}_g^{\mathbf{k}+\mathbf{j}}(q_{i_1}, \cdots, q_{i_p})$$

where $\mathcal{C}_g^{k_1,\cdots,k_n}$ is the number of maps on a surface of genus g with k_i vertices of type q_i.

Proof.

The case $\mathbf{j} = 0$ is just Theorem 4.1.1. Thus we can assume $\mathbf{j} \neq 0$, for example $j_1 \neq 0$. Observe that for all i,

$$\frac{\partial}{\partial t_i}F_{V_{\mathbf{t}}}^N = -E[\hat{\mu}^N(q_i)] = -\mathcal{I}(q_i) - \frac{1}{N^2}\nu^N(q_i).$$

we can use the Proposition 4.7.3 : there exists $\eta > 0$ such that for $\mathbf{t} \in B_{\eta,c}$,

$$\begin{aligned}
\mathcal{D}_{\mathbf{j}} F^N_{V_{\mathbf{t}}} &= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}(\mathcal{I}(q_1) + \frac{1}{N^2}\nu^N(q_1)) \\
&= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) - \frac{1}{N^2}\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\nu^N(q_1) \\
&= -\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) - \frac{1}{N^2}\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_1(q_1) - \cdots - \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_g(q_1) \\
&+ o(\frac{1}{N^{2g}})
\end{aligned}$$

Observe now that according to Lemma 4.4.4

$$\begin{aligned}
\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}_g(q_1) &= \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{M}_g^{\mathbf{k}+\mathbf{j}-\mathbb{1}_{i=1}}(q_1) \\
&= - \sum_{k_1,\cdots,k_n \in \mathbb{N}} \prod_i \frac{(-t_i)^{k_i}}{k_i!} \mathcal{C}_g^{\mathbf{k}+\mathbf{j}} = -\mathcal{D}_{\mathbf{j}} F_g
\end{aligned}$$

and by the same method, $\mathcal{D}_{\mathbf{j}-\mathbb{1}_{i=1}}\mathcal{I}(q_1) = -\mathcal{D}_{\mathbf{j}} F_0$. Thus we get,

$$\mathcal{D}_{\mathbf{j}} F^N_{V_{\mathbf{t}}} = \mathcal{D}_{\mathbf{j}} F_0 + \cdots + \frac{1}{N^{2g}}\mathcal{D}_{\mathbf{j}} F_g + o(\frac{1}{N^{2g}}).$$

□

Chapitre 5

Asymptotics for unitary matrix models

Ce chapitre est l'article [CGMS06] écrit avec Benoît Collins et Alice Guionnet. Cet article est soumis.

Abstract

In this paper we solve the following two conjectures : firstly, we prove that in small parameter regions, arbitrary unitary matrix integrals converge in the large N limit and match their formal expansion, We give a combinatorial interpretation of our matrix integral convergence results and investigate examples related to free probability and the HCIZ integral. Secondly, the convergence result leads us to the proof of a conjecture of Voiculescu about smoothness of microstates.

Introduction

Matrix integrals provide models for physical systems (2D quantum gravitation, gauge theory, renormalization, etc...), and generating series for a wide family of combinatorial objects (see e.g [tH74, Zvo97]).

Gaussian integrals are the most studied. It was shown by Brezin, Itzykson, Parisi and Zuber [BIPZ78] that perturbations of Gaussian integrals expand formally as a generating function of maps, sorted by their genus when the dimension N of the matrices is regarded as a parameter. Such 'topological' expansions were shown also to hold in the large N limit, and then to match with the formal expansion on a mathematical level of rigor by two authors [GMS06, GMS07, MS06a] and previously in the one matrix case in [ASM01, ACKM93] and [EM03]. The relation of Gaussian matrices with the enumeration of maps is an easy consequence of Wick calculus -or equivalently Feynman diagrams- see [Zvo97] for a good introduction. However, Gaussian matrices are very special in the sense that their spectrum has a prescribed large N limit. According to 't Hooft [tH74], such topological expansion should hold in the more general context of models invariant under unitary conjugation. This leads us to concentrate in this article on the matrix integrals of type

$$I_N(V, A_i^N) := \int_{\mathcal{U}_N(\mathbb{C})^m} e^{N \mathrm{Tr}(V(U_i, U_i^*, A_i^N, 1 \le i \le m))} dU_1 \cdots dU_m \tag{5.1}$$

where A_i^N are $N \times N$ deterministic uniformly bounded matrices, dU denotes the Haar measure on the unitary group $\mathcal{U}_N(\mathbb{C})$ and V is polynomial in the non-commutative variables $(U_i, U_i^*, A_i^N, 1 \le i \le m)$.

We shall study in this article the first order asymptotics of matrix integrals given by (5.1) when the joint distribution of the $(A_i^N, 1 \le i \le m)$ converges ; namely for all polynomial function P in m non-commutative indeterminates

$$\lim_{N \to \infty} \frac{1}{N} \mathrm{Tr}(P(A_i^N, 1 \le i \le m)) = \tau(P) \tag{5.2}$$

for some linear functional τ on the set of polynomials. Without loss of generality, we shall assume that A_i^N are Hermitian matrices.

It is convenient to assume that the polynomial V is such that $\mathrm{Tr}(V(U_i, U_i^*, A_i^N, 1 \le i \le m))$ is real for all $U_i \in \mathcal{U}_N(\mathbb{C})$, all Hermitian matrices A_i^N, for all $i \in \{1, \cdots, m\}$ and $N \in \mathbb{N}$.

Under those very general assumptions, the only result proved so far is the formal convergence of these matrix integrals. Namely, it was proved in [Col03] by one author that for each k, the limit

$$\frac{\partial^k}{\partial z^k} N^{-2} \log \int_{\mathcal{U}_N(\mathbb{C})} e^{zNTr(V(U_i, U_i^*, A_i^N, 1 \le i \le m))} dU_1 \cdots dU_m|_{z=0}$$

converges towards an integer $f_k(V, \tau)$ depending only on the limiting distribution of the A_i^N's and V.

The questions of whether the limit of the matrix integrals actually exists (at least for small parameter z), of the convergence of the power series of term $f_k(V, \tau)$, of the equality between these two quantities were all open and we solve them affirmatively in the four first sections, as recapped in the following theorem : Under the above hypotheses and if we further assume that the spectral radius of the matrices $(A_i^N, 1 \le i \le m, N \in \mathbb{N})$ is uniformly bounded (by say M), there exists $\varepsilon = \varepsilon(M, V) > 0$ so that for $z \in [-\varepsilon, \varepsilon]$,

$$\lim_{N \to \infty} \frac{1}{N^2} \log \int_{\mathcal{U}_N(\mathbb{C})} e^{zN \mathrm{Tr}(V(U_i, U_i^*, A_i^N, 1 \le i \le m))} dU_1 \cdots dU_m =: F_{V,\tau}(z).$$

Moreover, $F_{V,\tau}(z)$ is an analytic function of $z \in \mathbb{C} \cap B(0, \varepsilon)$ and for all $k \in \mathbb{N}$,

$$\frac{\partial^k}{\partial z^k} F_{V,\tau}(z)|_{z=0} = f_k(V, \tau).$$

This also implies that the series $F_{V,\tau}(z)$ is analytic with a positive radius of convergence, a result which had not been proved by the techniques of [Col03] based on Weingarten function.

Our approach is based on non-commutative differential calculus and perturbation analysis as developed in the context of Gaussian matrices in [GMS06, GMS07, MS06a]. Another possibility to prove the equality between real and formal limits would have been to be able to show convergence of the integrals for complex parameters z. We have not yet been able to follow this line successfully, and this remains an open question.

An important example of unitary matrix integral is the so-called spherical integral, studied by Harisch-Chandra and by Itzykson and Zuber,

$$HCIZ(A,B) := \int_{U\in\mathbb{U}_n} e^{N\mathrm{Tr}(U^*AUB)}dU.$$

This integral is of fundamental importance in analytic Lie theory and was computed for the first time by Harisch-Chandra in [HC57]. In the last two decades it has also become an issue to study its large dimension asymptotics.

Theorem 5.0.5 holds true for the HCIZ integral as well. It thus relates the results of [Col03] which computed the formal limit of the HCIZ integral and those of [GZ02] where the limit of $HCIZ(A,B)$ was obtained (regardless of any small parameters assumptions) by using large deviations techniques. In fact, it implies that the free energy found in [GZ02] is analytic in the vicinity of the origin. Let

$$I(\mu) = \frac{1}{2}\mu(x^2) + \frac{1}{2}\int\int \log|x-y|d\mu(x)d\mu(y).$$

If μ_A (resp. μ_B) denote the limiting spectral measure of A (resp. B), assume that $I(\mu_A)$ and $I(\mu_B)$ are finite. Then, the limit of $N^{-2}\log HCIZ(A,B)$ is given, according to [GZ02], by

$$I(\mu_A,\mu_B) = -I(\mu_A) - I(\mu_B) - \frac{1}{2}\inf_{\rho,m}\left\{\int_0^1\int\left(\frac{m_t(x)^2}{\rho_t(x)} + \frac{\pi^2}{3}\rho_t(x)^3\right)dxdt\right\} \tag{5.3}$$

where the inf is taken over m,ρ so that $\mu_t(dx) = \rho_t(x)dx \in \mathcal{P}(\mathbb{R})$ is a continuous process, $\mu_0 = \mu_A$, $\mu_1 = \mu_B$ and

$$\partial_t\rho_t(x) + \partial_x m_t(x) = 0.$$

The inf over (ρ_t, m_t) is taken (see [Gui04]) at the solution of an Euler equation for isentropic flow with negative pressure $-\frac{\pi^2}{3}\rho^3$.

When μ_A and μ_B have a small compact support of width ℓ, our result shows also that $I(\mu_A,\mu_B)$ expends analytically in ℓ, a result which is not obvious from formula (5.3). Moreover, the coefficients of this expansion count certain planar graphs (see section 5.5), as summarized in the following theorem. Denote $\sqrt{\beta}\sharp\mu$ the probability measure

$$\sqrt{\beta}\sharp\mu(f) = \int f(\sqrt{\beta}x)d\mu(x).$$

Assume that μ_A and μ_B are two compactly supported probability measures. Then, there exists $\beta_0 > 0$ such that for all $\beta \in [-\beta_0, \beta_0]$,

$$I(\sqrt{\beta}\sharp\mu_A, \sqrt{\beta}\sharp\mu_B) = \sum_{n\geq 0}\beta^n\mathbb{M}_n(\mu_A,\mu_B)$$

where

$$\mathbb{M}_n(\mu_A, \mu_B) = \sum_{m \text{ admissible maps of } \Sigma_n} M_m(\mu_A, \mu_B).$$

Σ_n is the set of planar maps drawn above n stars of type U^*AUB by gluing pairwise oriented arrows and possibly rings and $M_m(\mu_A, \mu_B)$ is the weight of the map.

We refer the reader to section 5.5 for the definitions of stars, admissible maps and weights. Our definition of planar maps is more complicated than in the usual Wick calculus because the sums are signed and we have a notion of admissibility. However it was an open question in mathematical physics to have a graphical model for unitary integrals (see [ZJZ03]). Moreover, this graphical interpretation gives a new understanding of cumulants formulae (see section 5.6.2).

The convergence of other integrals was still unknown and it is one of the points of this paper to show their convergence. We use it to study Voiculescu's microstates entropy evaluated at a set of laws which are small perturbations of the law of free variables, and prove regularity of microstates Under suitable assumptions described in Theorem 5.8.1,

$$\chi(\mu) = \liminf_{\substack{\varepsilon \downarrow 0 \\ k \uparrow \infty}} \liminf_{N \to \infty} \frac{1}{N^2} \log \mu_N^{\otimes m} \left(\Gamma_R(\mu, \varepsilon, k)\right)$$

and a formula for $\chi(\mu)$ can be given.

This result generalizes section 4 in [GMS06].

The paper is organized as follows : after setting our working framework (section 5.1), we study the action of perturbations upon the integral $I_N(V, A_i^N)$ and deduce some properties of the related Gibbs measure ; namely that the so-called empirical distribution of the matrices under this Gibbs measure satisfies asymptotically an equation called the Schwinger-Dyson equation (section 5.2). Then, we study this equation and obtain uniqueness for parameters of the potential V small enough (section 5.3) and analyticity (section 5.4).

Then, we describe a combinatorial solution of Schwinger-Dyson equation (section 5.5) and deduce applications of these results to free probability (section 5.6) and to the convergence of matrix integrals $I_N(V, A_i^N)$ (section 5.7). Finally, we point out some consequence of our result for free entropy (section 5.8).

5.1 Notations

Let $\mathcal{U}_N(\mathbb{C})$ be the set of unitary matrices, $\mathcal{M}_N(\mathbb{C})$ the set of $N \times N$ matrices with complex entries, $\mathcal{H}_N(\mathbb{C})$ the subset of hermitian matrices of $\mathcal{M}_N(\mathbb{C})$ and $\mathcal{A}_N(\mathbb{C})$ the subset of antihermitian matrices of $\mathcal{M}_N(\mathbb{C})$. We let m be a fixed integer number throughout this article. We denote by $(A_i^N)_{1 \le i \le m}$ a m-tuple of $N \times N$ Hermitian matrices. We shall assume that the sequence $(A_i^N)_{1 \le i \le m}$ is uniformly bounded for the operator norm, and without loss of generality that they are bounded by one,

$$\sup_{N,i} \|A_i^N\|_\infty = \sup_{N,i} \lim_{p \to \infty} \left(\mathrm{Tr}((A_i^N)^{2p})\right)^{\frac{1}{2p}} \le 1.$$

5.1.1 Free $*$-algebra

Let $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$ be the set of polynomial functions in the non-commutative indeterminates $(U_i, U_i^{-1}, A_i)_{1\le i\le m}$ with the relation

$$U_i U_i^* = U_i^* U_i = 1.$$

Note that in general we may want to consider models with a number of "deterministic" indeterminates A_i different from the number of "random unitary" indeterminates U_i but this general case can be obtained from the previous one by looking only at a sub-algebra and our convention shortens a little the notations. The algebra $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$ is equipped with the involution $*$ so that $A_i^* = A_i$ and $U_i^* = U_i^{-1}$ and for any $X_1, \cdots, X_n \in (U_i, U_i^{-1}, A_i)_{1\le i\le m}$, any $z \in \mathbb{C}$,

$$(zX_1X_2\cdots X_{n-1}X_n)^* = \bar{z}X_n^*X_{n-1}^*\cdots X_2^*X_1^*.$$

Note that for any $U_i \in \mathcal{U}_N(\mathbb{C})$, $A_i \in \mathcal{H}_N(\mathbb{C})$, and $P \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$,

$$\big(P(U_i, U_i^{-1}, A_i, 1 \le i \le m)\big)^* = P^*(U_i, U_i^{-1}, A_i, 1 \le i \le m)$$

where in the left hand side $*$ denotes the standard involution on $\mathcal{M}_N(\mathbb{C})$. We denote by $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle_{sa}$ the set of self-adjoint polynomials; $P = P^*$, and in the same spirit $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle_a$ the set of anti-self-adjoint polynomials; $P^* = -P$. In the sequel, except when something different is explicitly assumed, we shall make the hypothesis that the potential V belongs to $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle_{sa}$, which insures that for all $U_i \in \mathcal{U}_N(\mathbb{C})$ and $A_i^N \in \mathcal{H}_N(\mathbb{C})$ $\mathrm{Tr}\left(V((U_i, U_i^{-1}, A_i^N)_{1\le i\le m})\right)$ is real-valued for all $U_i \in \mathcal{U}_N(\mathbb{C})$ and $A_i^N \in \mathcal{H}_N(\mathbb{C})$.

5.1.2 Non-commutative derivatives

On $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, we define the non-commutative derivatives ∂_i, $1 \le i \le m$, given by

$$\partial_i A_j = 0, \quad \partial_i U_j = 1_{i=j} U_j \otimes 1 \quad \partial_i U_j^{-1} = -1_{i=j} 1 \otimes U_j^{-1}, \quad \forall j,$$

and satisfying the Leibnitz rule : for $P, Q \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$,

$$\partial_i(PQ) = \partial_i P \times (1 \otimes Q) + (P \otimes 1) \times \partial_i Q. \tag{5.4}$$

Here, $\times$ denotes the product $P_1 \otimes Q_1 \times P_2 \otimes Q_2 = P_1P_2 \otimes Q_1Q_2$. We also let D_i be the corresponding *cyclic* derivatives such that if $m(A \otimes B) = BA$, $D_i = m \circ \partial_i$.

If q is a monomial in $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, we more specifically have

$$\partial_i q = \sum_{q=q_1U_iq_2} q_1U_i \otimes q_2 - \sum_{q=q_1U_i^{-1}q_2} q_1 \otimes U_i^{-1}q_2 \tag{5.5}$$

$$D_i q = \sum_{q=q_1U_iq_2} q_2q_1U_i - \sum_{q=q_1U_i^{-1}q_2} U_i^{-1}q_2q_1. \tag{5.6}$$

5.1.3 Bounded tracial states

Let $\mathcal{T}$ be the set of tracial states on the algebra generated by $(U_i, U_i^{-1}, A_i)_{1\le i\le m}$, i.e. the set of linear forms on $\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle$ such that

$$\mu(PP^*)\ge 0, \quad \mu(PQ)=\mu(QP), \quad \mu(1)=1.$$

Throughout this article, we restrict ourselves to tracial states $\mu\in\mathcal{T}$ such that

$$\mu((A_i(A_i)^*)^n)\le 1 \quad \forall n\in\mathbb{N}, \forall i\in\{1,\cdots,m\}.$$

We denote $\mathcal{M}$ this subset of $\mathcal{T}$.

Note that for any monomial $q\in\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle$, the Cauchy-Schwarz inequality implies that for any $\mu\in\mathcal{M}$,

$$\mu(qq^*)\le 1. \tag{5.7}$$

We endow $\mathcal{M}$ with its weak topology : μ_n converges to μ if and only if for all P,

$$\lim_{n\to\infty}\mu_n(P)=\mu(P).$$

By equation (5.7) and since the above topology is the product topology, $\mathcal{M}$ is a compact metric space.

We denote $\hat{\mu}^N$ the empirical distribution of matrices $A_i^N\in\mathcal{H}_N(\mathbb{C})$ and $U_i\in\mathcal{U}_N(\mathbb{C})$ which is given for all $P\in\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle$ by

$$\hat{\mu}^N(P)=\frac{1}{N}\mathrm{Tr}\left(P(U_i,U_i^{-1},A_i^N,1\le i\le m)\right).$$

This object will be of crucial interest for us.

The notation $\mathcal{M}|_{(A_i)_{1\le i\le m}}$ stands for the set of tracial states of $\mathcal{M}$ restricted to the algebra generated by the $(A_i)_{1\le i\le m}$. In particular, the limiting distribution τ given by (5.2) belongs to $\mathcal{M}|_{(A_i)_{1\le i\le m}}$.

5.1.4 Tracial power states

Let $V\in\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle_{sa}$ and μ_V^N be the distribution on $\mathcal{U}_N(\mathbb{C})$ given by

$$\mu_V^N(dU_1,\cdots,dU_m)=I_N(V,A_i^N)^{-1}\exp(N\mathrm{Tr}(V))dU_1\cdots dU_m.$$

We define, for all $P\in\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle$,

$$\bar{\mu}_V^N(P):=E_{\mu_V^N}[\hat{\mu}^N(P)]:=\frac{\int\frac{1}{N}\mathrm{Tr}Pe^{N\mathrm{Tr}V}dU_1\ldots dU_n}{\int e^{N\mathrm{Tr}V}dU_1\ldots dU_n}.$$

In the following, an n-tuple of monomials $(q_i)_{1\le i\le n}$ in $\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle$ will be fixed and we shall take $V=V_{\mathbf{t}}=\sum_{i=1}^n t_iq_i$. Then, $\bar{\mu}_{V_{\mathbf{t}}}^N(P)$ can be seen as a power series in the t_i's ;

$$\bar{\mu}_{V_{\mathbf{t}}}^N(P):=\sum_{\mathbf{k}\in\mathbb{N}^n}\frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!}\frac{\partial^{|\mathbf{k}|}}{\prod_i\partial t_i^{k_i}}\bigg|_{t_i=0}\frac{E[\hat{\mu}^N(P)e^{N^2\hat{\mu}^N(V_{\mathbf{t}})}]}{E[e^{N^2\hat{\mu}^N(V_{\mathbf{t}})}]}. \tag{5.8}$$

We will call μ a 'tracial power state' of $\mathcal{M}$ if and only if it is a map

$$\mu : \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle \to \mathbb{C}[[\mathbf{t}]]$$

with for all a, b, $\mu(ab) = \mu(ba)$. Here $\mathbb{C}[[\mathbf{t}]]$ is the algebra of power series in the variables $t_1, \cdots, t_n$. In particular, we may view $\mu^N_{V_\mathbf{t}}$ as a tracial power state of $\mathcal{M}$.

5.1.5 Cumulants.

The classical cumulants $\{C_k\}_{k\geq 0}$ are defined via their formal generating function :

$$\log E(e^{tX}) = \sum_{k\geq 0} t^k C_k(X, \ldots, X)/k!$$

This equality holds also in a complex neighborhood of 0 for t if X is bounded. We also define the cumulants $C_\mathbf{k}$ for $\mathbf{k}$ in $\mathbb{N}^n$:

$$\log E(e^{t_1X_1+\cdots+t_nX_n}) = \sum_{\mathbf{k}\in\mathbb{N}^n} \mathbf{t}^\mathbf{k} C_\mathbf{k}(X_1, \ldots, X_n)/\mathbf{k}!$$

where $\mathbf{k} = (k_1, \cdots, k_n)$, $\mathbf{k}! = \prod_i k_i!$, $|\mathbf{k}| = \sum_i k_i$ and $\mathbf{t}^\mathbf{k} = \prod_i t_i^{k_i}$. Note that :

$$C_\mathbf{k}(X_1, \ldots, X_k) = C_{|\mathbf{k}|}(X_1, \cdots, X_1, \cdots, X_n, \cdots, X_n)$$

where in the previous list the variable X_i appears k_i times.

Let us recall some properties of these cumulants. The following two statements hold true :

1.
$$\frac{E(Ye^{t_1X_1+\ldots+t_nX_n})}{E(e^{t_1X_1+\ldots+t_nX_n})} = \sum_{\mathbf{k}\in N^n} \mathbf{t}^\mathbf{k} C_{1,\mathbf{k}}(Y, X_1, \ldots, X_n)/\mathbf{k}!$$

2.
$$\frac{E(YZe^{t_1X_1+\ldots+t_nX_n})}{E(e^{t_1X_1+\ldots+t_nX_n})} - \frac{E(Ye^{t_1X_1+\ldots+t_nX_n})}{E(e^{t_1X_1+\ldots+t_nX_n})}\frac{E(Ze^{t_1X_1+\ldots+t_nX_n})}{E(e^{t_1X_1+\ldots+t_nX_n})}$$
$$= \sum_{k\geq 0} \mathbf{t}^\mathbf{k} C_{1,1,\mathbf{k}}(Y, Z, X_1, \ldots, X_n)/\mathbf{k}!$$

Proof.

Item (1) is obtained by replacing $t_1X_1 + \ldots + t_nX_n$ by $yY + t_1X_1 + \ldots + t_nX_n$ and differentiating the generating function of the cumulants in y at $y = 0$.

Item (2) is obtained by replacing tX by $yY + zZ + tX$ and differentiating the equality defining the cumulants in y and z at $y, z = 0$.

□

5.2 Matrix models

We first investigate the asymptotic behavior of the random state $\hat{\mu}^N$ and then we study its moments.

5.2.1 Behavior of $\hat{\mu}^N$

In this section, we investigate the behavior of $\hat{\mu}^N$ under μ_V^N when N goes to infinity. Note that $\hat{\mu}^N$ belongs to $\mathcal{M}$.

The main result of this section is the following Assume that V is self-adjoint. For all polynomial
$P \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$,

$$\lim_{N\to\infty} \left\{\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P) + \hat{\mu}^N(D_i V P)\right\} = 0 \quad \mu_V^N \; a.s.$$

In particular, any limit point $\mu \in \mathcal{M}$ of $\hat{\mu}^N$ under μ_V^N satisfies the Schwinger-Dyson equation

$$\mu \otimes \mu(\partial_i P) + \mu(D_i V P) = 0 \tag{5.9}$$

for all $P \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$ and $\mu|_{(A_i)_{1\le i\le m}} = \tau$. The idea of the proof, rather common in quantum field theory and successfully used in [GMS06, GMS07, MS06a], is to obtain equations on $\hat{\mu}^N$ by performing an infinitesimal change of variables in $I_N(V, A_i^N)$. More precisely we make the change of variables $\mathbf{U} = (U_1, \cdots, U_m) \to \Psi(\mathbf{U}) = (\Psi_1(\mathbf{U}), \cdots, \Psi_m(\mathbf{U}))$ with

$$\Psi_j : \mathbf{U} \to U_j e^{\frac{\lambda}{N} P_j(\mathbf{U})}$$

where the P_j are antisymmetric polynomials (i.e. $P^* = -P$). This change of variables becomes very close to the identity as N goes to infinity, reason why it is called "infinitesimal".

The function Ψ is a local diffeomorphism and its Jacobian has the following expansion when N goes to infinity

$$J_\Psi = e^{\frac{\lambda}{N}\sum_i \mathrm{Tr}\otimes\mathrm{Tr}(\partial_i P_i) + O(1)}$$

Proof.

Let us first recall the following two elementary results of differential geometry :

1. The map $\exp : \mathcal{M}_N(\mathbb{C}) \longrightarrow \mathcal{M}_N(\mathbb{C})$ is differentiable and :

$$\mathrm{Diff}_M \exp .H := \lim_{\varepsilon\to 0} \varepsilon^{-1}(e^{M+\varepsilon H} - e^M) = \left(\sum_{k=0}^{+\infty} \frac{(\mathrm{Ad}_M)^k}{(k+1)!} H\right) e^M$$

   where Ad_M is the operator defined by $\mathrm{Ad}_M H = MH - HM$.

2. If $P \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$ is considered as a function of the U_i's, then it is differentiable and its differential with respect to the i-th variable in the direction A, for A in $\mathcal{A}_N(\mathbb{C})$ is :

$$\mathrm{Diff}_i P.A := \lim_{\varepsilon\to 0} \varepsilon^{-1}(P(U_1, \cdots, U_{i-1}, U_i e^{\varepsilon A}, U_{i+1}, \cdots) - P(\mathbf{U})) = \partial_i P \sharp A.$$

As a consequence, if we fix A in $\mathcal{A}_N(\mathbb{C})$, $1 \leqslant i \leqslant m$, one has

$$\begin{aligned}\mathrm{Diff}_i\Psi_j(\mathbf{U}).A &= 1_{i=j}U_jA + U_jDiff_{\frac{\lambda}{N}P_j(U)}\exp.(\frac{\lambda}{N}\partial_iP_j\sharp A)\\ &= 1_{i=j}U_jA + \frac{\lambda}{N}\sum_{k=0}^{+\infty}U_j\frac{(\mathrm{Ad}_{\frac{\lambda}{N}P_j(U)})^k}{(k+1)!}(\partial_iP_j\sharp A)e^{\frac{\lambda}{N}P_j(U)}\\ &= 1_{i=j}U_jA + U_j\frac{\lambda}{N}\Phi_{ij}A.\end{aligned}$$

with Φ_{ij} the linear map from $\mathcal{A}_N(\mathbb{C})$ into $\mathcal{M}_N(\mathbb{C})$ given by :

$$\Phi_{ij}A := \sum_{k=0}^{+\infty}\frac{(\mathrm{Ad}_{\frac{\lambda}{N}P_j(\mathbf{U})})^k}{(k+1)!}(\partial_iP_j\sharp A)e^{\frac{\lambda}{N}P_j(U)}.$$

We can factorize the term U_j to obtain :

$$\mathrm{Diff}\Psi(\mathbf{U}) = U \circ (\mathrm{Id}_{\mathcal{A}_N(\mathbb{C})^m} + \frac{\lambda}{N}\Phi) \tag{5.10}$$

with $U(M_1,\cdots,M_m) = (U_1M_1,\cdots,U_mM_m)$ and Φ is the linear operator from $\mathcal{A}_N(\mathbb{C})^m$ to $\mathcal{M}_N(\mathbb{C})^m$ whose blocks are the Φ_{ij}.

Since the operator norms of the A_i's and the U_i's are uniformly bounded in N, the operator norm of $\mathrm{Ad}_{\frac{\lambda}{N}P_j(\mathbf{U})}$ as an operator on $(\mathcal{M}_N(\mathbb{C}), \|.\|_\infty)$ is also bounded. Thus, Φ_{ij} is a uniformly bounded operator from $\mathcal{A}_N(\mathbb{C})$ to $\mathcal{M}_N(\mathbb{C})$. Since Φ comes with a factor $\frac{\lambda}{N}$, we can deduce that for N large enough such that, the norm of $\frac{\lambda}{N}\Phi$ is less than $1/2$. For those N, Ψ is a local diffeomorphism with positive eigenvalues.

We can now compute the factor coming from the Jacobian in the integral :

$$J_\Psi := |\det \mathrm{Diff}\Psi(\mathbf{U})| = |\det U||\det(I + \frac{\lambda}{N}\Phi)|.$$

It can be easily checked that $|\det U| = 1$.

Besides, the positivity of the eigenvalues of Φ allows us to replace the determinant by the exponential of a trace :

$$J_\Psi = \exp(\mathrm{Tr}\log(I + \frac{\lambda}{N}\Phi)) = \exp\left(-\sum_{p\geq 1}\frac{(-\lambda)^p}{pN^p}\mathrm{Tr}(\Phi^p)\right).$$

Note that since Φ is a bounded operator on $\mathcal{A}_N(\mathbb{C})$ which is a space of dimension N^2, the p-th term in the previous sum is at most of order N^{2-p}. We only look at the terms up to the order $O(N)$. A quick computation shows that if

$$\varphi : \begin{array}{ccc}\mathcal{A}_N(\mathbb{C}) & \to & \mathcal{A}_N(\mathbb{C})\\ X & \to & \sum_l A_lXB_l\end{array}$$

is considered as a real endomorphism, $\mathrm{Tr}\varphi = \sum_l \mathrm{Tr}A_l \mathrm{Tr}B_l$ (this can be checked by decomposing φ on the canonical base of $\mathcal{A}_N(\mathbb{C})$). This is sufficient to obtain the first term of the Jacobian :

$$\frac{\lambda}{N}\mathrm{Tr}(\Phi) = \frac{\lambda}{N}\sum_i \mathrm{Tr}(\Phi_{ii}) = \sum_i \frac{\lambda}{N}\mathrm{Tr}\otimes\mathrm{Tr}(\partial_i P_i) + O(1).$$

□

Before making the change of variable we show that Ψ is a bijection. For N large enough, Ψ is a diffeomorphism of $\mathcal{U}_N(\mathbb{C})^m$. **Proof.**

The only non-trivial property is the injectivity of Ψ. If $\Psi(U) = \Psi(V)$ then for all $j \in \{1, \cdots, m\}$,

$$U_j^* V_j - I = e^{\frac{\lambda}{N}P_j(U)} e^{-\frac{\lambda}{N}P_j(V)} - I.$$

Thus, we obtain if N is sufficiently large so that $\frac{\lambda}{N}P_j(U)$ is in a domain where the function exp is 2-Lipschitz, then if $\|.\|_\infty$ is the operator norm,

$$\begin{aligned}\|U_j - V_j\|_\infty = \|U_j V_j^* - 1\|_\infty &= \|e^{\frac{\lambda}{N}P_j(U)} e^{\frac{-\lambda}{N}P_j(V)} - 1\|_\infty \\ &= \|e^{\frac{\lambda}{N}P_j(U)} - e^{\frac{\lambda}{N}P_j(V)}\|_\infty \leq \frac{2|\lambda|}{N}\|P_j(U) - P_j(V)\|\end{aligned}$$

and the results follows since $(P_j, 1 \leq j \leq m)$ are uniformly lipschitz on $\mathcal{U}_N(\mathbb{C})^m$ so that $\sum_{j=1}^m \|U_j - V_j\|_\infty$ vanishes for sufficiently large N.

□

Proof.
[Proof of Theorem 5.2.1.]
Let us define

$$Y^N(P) = \sum_i \frac{1}{N}\mathrm{Tr}(D_i V P_i) + \left(\frac{1}{N}\mathrm{Tr}\otimes\frac{1}{N}\mathrm{Tr}\right)(\partial_i P_i)$$

We expand $\mathrm{Tr}V(\Psi(\mathbf{U}))$ as

$$\mathrm{Tr}(V(\Psi(\mathbf{U}))) - \mathrm{Tr}(V) = \frac{\lambda}{N}\sum_i \mathrm{Tr}(D_i V P_i) + O(N^{-1})$$

and perform the change of variables $\mathbf{U} \to \Psi(\mathbf{U})$ in $I_N(V, A_i^N)$;

$$\begin{aligned}I_N(V, A_i^N) &:= \int e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m \\ &= \int e^{N\left(\mathrm{Tr}(V(\Psi(\mathbf{U})) - \mathrm{Tr}(V)\right)} J_\Psi(\mathbf{U}) e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m \\ &= \int e^{NY^N(P)+0(1)} e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m\end{aligned}$$

where $O(1)$ is of order one independently of N and uniformly on the unitary matrices $U_1, \ldots,$ U_m. Thus we have proved that

$$\int e^{NY^N(P)} d\mu_V^N(\mathbf{U}) = O(1).$$

Borel-Cantelli's lemma thus insures that

$$\limsup_{N\to\infty} Y^N(P) \leq 0 \quad a.s.$$

and the converse inequality holds by changing P into $-P$ since Y^N is linear. This proves the first statement of Theorem 5.2.1. The last result is simply based on the compactness of $\mathcal{M}$ and the fact that any limit point must then satisfy the same asymptotic equations than $\hat{\mu}^N$.

□

Another consequence of this convergence is the existence of solutions to (5.9) for any self-adjoint potential V (since any limit point of $\hat{\mu}^N$ in the compact metric space $\mathcal{M}$ will satisfy it) a fact already proved in [Bia03]. Moreover, since these solutions are limit points of $\hat{\mu}^N$, they belong to $\mathcal{M}$ and in particular $|\mu(q)| \leq 1$ for any monomial q.

5.2.2 Moments of $\hat{\mu}^N$

In the rest of the paper, we denote by E the expectation with respect to the Haar measure on the unitary group. The goal of this section is to show (see Proposition 5.2.5) that cumulants also satisfy a formal version of Schwinger-Dyson equation. We start with the following lemma :

One has, for all i all N, all monomials $q_1, \cdots, q_n$ and all $\mathbf{k} = (k_1, \cdots, k_n)$ in $\mathbb{N}^n$,

$$\begin{aligned} &N^2 E\left(\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P) \cdot \left(\hat{\mu}^N(q_1)\right)^{k_1} \cdots \left(\hat{\mu}^N(q_n)\right)^{k_n}\right) \\ &+ \sum_j k_j E\left(\left(\hat{\mu}^N(q_1)\right)^{k_1} \cdots \left(\hat{\mu}^N(q_j)\right)^{k_j-1} \cdots \left(\hat{\mu}^N(q_n)\right)^{k_n} \hat{\mu}^N(D_i q_j \cdot P)\right) = 0 \end{aligned}$$

Proof.

Following Lemma 5.2.2, we write down the change of variable

$$\Psi_i : \mathbf{U} \to (U_1, \cdots, U_{i-1}, U_i e^{\lambda P_i(\mathbf{U})}, U_{i+1}, \cdots, U_m)$$

in the integral $\int((\hat{\mu}^N q_1)^{k_1} \cdots (\hat{\mu}^N q_n)^{k_n}) dU_1 \cdots dU_m$, where the integration is taken over the unitary Haar measure. Its Jacobian satisfies

$$J_\Psi = 1 + \lambda \mathrm{Tr} \otimes \mathrm{Tr}(\partial_i P) + o(\lambda).$$

and we have the expansion

$$\mathrm{Tr}(q_j(U')) = \mathrm{Tr}(q_j(U)) + \lambda \mathrm{Tr}(D_i q_j \cdot P) + o(\lambda).$$

The first order of a Taylor expansion around $\lambda = 0$ proves the claim.

□

As a formal series equality, one has, for all i, for all $\mathbf{t}$,

$$E[\hat{\mu}^N \otimes \hat{\mu}^N(\partial_i P)e^{N^2\hat{\mu}^N(V_{\mathbf{t}})}] + E[\hat{\mu}^N(D_i V_{\mathbf{t}} \cdot P)e^{N^2\hat{\mu}^N(V_{\mathbf{t}})}] = 0.$$

Proof.
Multiplying the equality of Lemma 5.2.4 by $\mathbf{t}^{\mathbf{k}} N^{2|\mathbf{k}|-2}/\mathbf{k}!$ and summing over $\mathbf{k}$ in $\mathbb{N}^n$ gives the desired identity.

□

Finally we would like to study the large N limit μ^f of these formal states (the index f stands for "formal"). Let $V_{\mathbf{t}}$ be the polynomial $\sum_{j=1}^n t_j q_j$. For all P, the sequence $\bar{\mu}^N_{V_{\mathbf{t}}}(P)$ converges as a formal series (i.e. coefficientwise) when N goes to infinity to some $\mu^f(P)$. Besides, μ^f satisfies the family of equations, for all i, for all P,

$$\mu^f \otimes \mu^f(\partial_i P) + \mu^f(D_i V_{\mathbf{t}} \cdot P) = 0.$$

Proof.
First, we prove the existence of a limit. By the first item of Proposition 5.1.1, we can express $\bar{\mu}^N_{V_{\mathbf{t}}}(P)$ as a sum over cumulants,

$$\bar{\mu}^N_{V_{\mathbf{t}}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \mathbf{t}^{\mathbf{k}} C_{1,\mathbf{k}}(\frac{1}{N}\mathrm{Tr}P, N\mathrm{Tr}q_1, \cdots, N\mathrm{Tr}q_n)/\mathbf{k}!.$$

The limit in N, of the $C_{1,\mathbf{k}}(\frac{1}{N}\mathrm{Tr}P, N\mathrm{Tr}q_1, \cdots, N\mathrm{Tr}q_n)$ was proved to exists in [Col03] so that μ^f is well defined.

Item (2) from Proposition 5.1.1 implies

$$\frac{E(\frac{1}{N}\mathrm{Tr}P_1\frac{1}{N}\mathrm{Tr}P_2 e^{N\mathrm{Tr}V})}{E(e^{N\mathrm{Tr}V})} - \frac{E(\frac{1}{N}\mathrm{Tr}P_1 e^{N\mathrm{Tr}V})}{E(e^{N\mathrm{Tr}V})}\frac{E(\frac{1}{N}\mathrm{Tr}P_2 e^{N\mathrm{Tr}V})}{E(e^{N\mathrm{Tr}V})}$$
$$= \sum_{k\geq 0} \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!} C_{1,1,\mathbf{k}}(\frac{1}{N}\mathrm{Tr}P_1, \frac{1}{N}\mathrm{Tr}P_2, N\mathrm{Tr}q_1, \cdots, N\mathrm{Tr}q_n).$$

Now, it follows from [Col03] that elements on the right hand side have decay N^{-2} so that the coefficientwise limit is zero. This can be interpreted as a formal convergence of measure result for the states $\hat{\mu}^N$.

The proof of the Theorem follows from this observation and from Proposition 5.2.5.

□

5.3 Study of Schwinger-Dyson equation

We have shown that the limit points of the matrix model satisfy (5.9). The aim of this section is to study this equation and show that it has a unique solution.

Let $\tau \in \mathcal{M}|_{(A_i)_{1\le i\le m}}$. A tracial state $\mu \in \mathcal{M}$ is said to satisfy Schwinger-Dyson equation **SD[V,τ]** if and only if for all $P \in \mathbb{C}\langle (A_i)_{1\le i\le m}\rangle$,

$$\mu(P) = \tau(P)$$

and for all $P \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, all $i \in \{1, \cdots, m\}$,

$$\mu \otimes \mu(\partial_i P) + \mu(D_i V\, P) = 0.$$

Let $V \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$. One can decompose V in a sum

$$V = \sum_{i=1}^{n} t_i q_i(U_j, U_j^{-1}, A_j, 1 \le j \le m)$$

with monomial functions q_i and complex numbers t_i. We let D be the maximal degree of the monomials q_i.

Here we prove that τ is uniquely defined provided that V is small enough.

Let D an integer and τ a tracial state in $\mathcal{M}|_{(A_i)_{1\le i\le m}}$ be given. There exists $\varepsilon = \varepsilon(D, m) > 0$ such that if $|t_i| \le \varepsilon$, there exists at most one solution μ to **SD[V,τ]**. From this and Theorem 5.2.1 we deduce the following Assume that V is self-adjoint. Let D an integer and τ a tracial state in $\mathcal{M}|_{(A_i)_{1\le i\le m}}$ be given. There exists $\varepsilon = \varepsilon(D, m) > 0$ such that if $|t_i| \le \varepsilon$, $\hat{\mu}^N$ converges almost surely to the unique solution μ of the Schwinger-Dyson equation. Moreover, $\bar{\mu}_V^N = \mu_V^N(\hat{\mu}^N)$ converges as well to this solution as N goes to infinity.

This result is obvious since Theorems 5.2.1 and 5.3.2 show that $\hat{\mu}^N$ has a unique limit point, and thus converges almost surely. The convergence of $\bar{\mu}_V^N$ is then a direct consequence of bounded convergence theorem since $\hat{\mu}^N \in \mathcal{M}$.

Actually Theorem 5.2.1 and Corollary 5.3.3 do not use the assumption that the matrices A_i^N are deterministic, but only that they are bounded and have almost surely a converging joint distribution. Therefore these two results still hold almost surely in that framework. This observation implies that our result can also encompass the case of the truncated GUE or other classical bounded matrix models.

Proof.

[Proof of Theorem 5.3.2.] Let μ be a solution to **SD[V,τ]**. Note that if we take q a monomial in $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, either q does not depend on $U_j, U_j^{-1}, 1 \le j \le m$ and then $\mu(q) = \tau(q)$ is uniquely defined, or q can be written as $q = q_1 U_i^n q_2$ for some $i \in \{1, \cdots, m\}$ and $n \in \{-1, +1\}$. Then, by the traciality assumption, $\mu(q) = \mu(q_2 q_1 U_i^n) = \mu(U_i^n q')$ with $q' = q_2 q_1$. Remark that we can assume without loss of generality that the last letter of q' is not U_i^{-n}. We next use **SD[V,τ]** to compute $\mu(U_i^n q)$ for some monomial q. We assume first that $n = -1$. Then, by (5.4),

$$\partial_i \left(U_i^{-1} q\right) = -1 \otimes (U_i^{-1} q) + U_i^{-1} \otimes 1 \times \partial_i q.$$

Taking the expectation, we thus find by (5.5) that

$$\begin{aligned}\mu(U_i^{-1}q) &= \mu\otimes\mu(U_i^{-1}\otimes 1\partial_i q)+\mu(D_iVq)\\ &= \sum_{q=q_1U_iq_2}\mu(U_i^{-1}q_1U_i)\mu(q_2)-\sum_{q=q_1U_i^{-1}q_2}\mu(U_i^{-1}q_1)\mu(U_iq_2)\\ &\quad+\sum_j t_{ij}\mu(q_{ij}q) \qquad (5.11)\end{aligned}$$

where $D_iV=\sum_j t_{ij}q_{ij}$. Note that the sum runs at most on Dn terms and that all the t_{ij} are bounded by $\max|t_i|$. A similar formula is found when $n=+1$ by differentiating qU_i.

We next show that (5.11) characterizes uniquely $\mu\in\mathcal{M}$ when the t_{ij} are small enough. It will be crucial here that $\mu(q)$ is bounded independently of the t_i's (here by the constant 1).

Now, let $\mu,\mu'\in\mathcal{M}$ be two solutions to **SD[V,τ]** and set

$$\Delta(\ell)=\sup_{\deg(q)\le\ell}|\mu(q)-\mu'(q)|$$

where the supremum holds over monomials of $\mathbb{C}\langle U_1,\cdots,U_m,A_1,\cdots,A_m\rangle$ with total degree in the U_j and U_j^{-1} less than l. Namely, if the monomial (or word) q contains U_j a_j^+ times and U_j^{-1} a_j^- times, we assume $\sum_{j=1}^m(a_j^++a_j^-)\le\ell$. Note that

$$\Delta(\ell) = \max_{\substack{1\le i\le m\\ a\in\{+1,-1\}}}\sup_{\deg q\le\ell-1}|\mu(U_i^aq)-\mu'(U_i^aq)| \qquad (5.12)$$

and that by (5.11), we find that, for q with degree less than $\ell-1$,

$$\begin{aligned}|\mu(U_i^*q)-\mu'(U_i^*q)| &\le \sum_{q=q_1U_iq_2}|(\mu-\mu')(q_1)|+\sum_{q=q_1U_iq_2}|(\mu-\mu')(q_2)|\\ &+\sum_{q=q_1U_i^{-1}q_2}|(\mu-\mu')(U_i^{-1}q_1)|+\sum_{q=q_1U_i^{-1}q_2}|(\mu-\mu')(U_iq_2)|\\ &+\sum_j t_{ij}|(\mu-\mu')(q_{ij}q)|\end{aligned}$$

and thus

$$\Delta(\ell)\le 2\sum_{p=1}^{\ell-2}\Delta(p)+2\sum_{p=1}^{\ell-1}\Delta(p)+nD\varepsilon\Delta(\ell+D-1)$$

where we used that $\deg(q_1)\in\{0,\cdots,\ell-2\}$, $\deg(q_2)\in\{0,\cdots,\ell-2\}$ (but $\Delta(0)=0$) and $\deg(q_{ij})\le D$ and assumed $|t_i|\le\varepsilon$. Hence, we have proved that

$$\Delta(\ell)\le 4\sum_{p=1}^{\ell-1}\Delta(p)+nD\varepsilon\Delta(\ell+D).$$

Multiplying these inequalities by γ^ℓ we get, since $H(\gamma) := \sum_{\ell \geq 1} \gamma^\ell \Delta(\ell) < \infty$ for $\gamma < 1$,

$$H(\gamma) \leq \frac{\gamma}{1-\gamma} H(\gamma) + \frac{nD\varepsilon}{\gamma^D} H(\gamma)$$

resulting with $H(\gamma) = 0$ for γ so that $1 > \frac{\gamma}{1-\gamma} + \frac{nD\varepsilon}{\gamma^D}$. Such a $\gamma > 0$ exists when ε is small enough. This proves the uniqueness.

□

As a corollary, we characterize asymptotic freeness by a Schwinger-Dyson equation, a result which was already obtained in [Voi99], Proposition 5.17.

A tracial state μ satisfies **SD[0,τ]** if and only if , under μ, the algebra generated by $\{A_i, 1 \leq i \leq m\}$ and $\{U_i, U_i^{-1}, 1 \leq i \leq m\}$ are free and the U_i's are two by two free and satisfy

$$\mu(U_i^a) = 0 \quad \forall a \in \mathbb{Z} \backslash \{0\}.$$

Proof.

By the previous theorem, it is enough to verify that the law μ of $(A_i, U_i, U_i^{-1})_{1 \leq i \leq m}$ satisfies **SD[0,τ]**. So take $P = U_{i_1}^{a_1} B_1 \cdots U_{i_p}^{a_p} B_p$ with some B_k's in the algebra generated by $(A_i, 1 \leq i \leq m)$. We wish to show that for all $i \in \{1, \cdots, m\}$,

$$\mu \otimes \mu(\partial_i P) = 0.$$

Note that by linearity, it is enough to prove this equality when $\mu(B_j) = 0$ for all j. Now, by definition, we have

$$\partial_i P = \sum_{k: i_k = i, a_k > 0} \sum_{l=1}^{a_k} U_{i_1}^{a_1} B_1 \cdots B_{k-1} U_i^l \otimes U_i^{a_k - l} B_k \cdots U_{i_p}^{a_p} B_p$$

$$- \sum_{k: i_k = i, a_k < 0} \sum_{l=0}^{a_k - 1} U_{i_1}^{a_1} B_1 \cdots B_{k-1} U_i^{-l} \otimes U_i^{a_k + l} B_k \cdots U_{i_p}^{a_p} B_p.$$

Taking the expectation on both sides, since $\mu(U_j^i) = 0$ and $\mu(B_j) = 0$ for all $i \neq 0$ and j, we see that freeness implies that the right hand side is null (recall here that in the definition of freeness, two consecutive elements have to be in free algebras but the first and the last element can be in the same algebra). Thus, $\mu \otimes \mu(\partial_i P) = 0$ which proves the claim.

□

5.4 Formal solution and analyticity

We have shown in Theorem 5.2.6 that the limit points of the formal model also satisfy an equation similar to Schwinger-Dyson's equation. The only difference is that one of these equations is on the space of tracial states while the other one is on the space of tracial power states. In order to prove that the formal model matches the matrix model we need to study this formal equation and show that the series have a positive radius of convergence, hence providing a solution to **SD[V,τ]** as defined in Definition 5.3.1.

Let $V_{\mathbf{t}} = \sum_i t_i q_i$ be a polynomial and τ a tracial power state in $\mathcal{M}|_{(A_i)_{1\leq i\leq m}}$. A tracial power state $\mu \in \mathcal{M}$ is said to satisfy Schwinger-Dyson equation $\mathbf{SD}^f[V_{\mathbf{t}},\tau]$ if and only if for all $P \in \mathbb{C}\langle(A_i)_{1\leq i\leq m}\rangle$,

$$\mu(P) = \tau(P)$$

and for all $P \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, all $i \in \{1, \cdots, m\}$,

$$\mu \otimes \mu(\partial_i P) + \mu(D_i V_{\mathbf{t}}\, P) = 0.$$

(Here, both terms of the above equality are elements of $\mathbb{C}[[\mathbf{t}]]$ and the equality is formal.)

We already know, due to Theorem 5.2.6, that there exists a solution to this equation. We now prove that this solution is unique.

There exists a unique tracial power state μ which satisfies Schwinger-Dyson equation $\mathbf{SD}^f[V_{\mathbf{t}},\tau]$.

Proof.

Let $\mu_{\mathbf{t}}$ be a tracial power state solution of $\mathbf{SD}^f[V_{\mathbf{t}},\tau]$. There exists a family $\mu^{\mathbf{k}}, \mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n$ in $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle^*$ such that for all P,

$$\mu_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{t_i^{k_i}}{k_i!} \mu^{\mathbf{k}}(P).$$

We will now show that the $\mu_{\mathbf{k}}$ are uniquely inductively defined by the relation given by $\mathbf{SD}^f[V_{\mathbf{t}},\tau]$. Let us define 1_j the vector in $\mathbb{N}^n$ which vanishes on every coordinate except the j-th which is 1. We get the following equalities, for all $\mathbf{k}$,

1. If P is in $\mathbb{C}\langle(A_i)_{1\leq i\leq m}\rangle$, $\mu^{\mathbf{k}}(P) = \tau(P)1_{\mathbf{k}=0}$,
2. If $P = RU_iS$ with S in $\mathbb{C}\langle(A_i)_{1\leq i\leq m}\rangle$, $\mu^{\mathbf{k}}(P) = \mu^{\mathbf{k}}(SRU_i)$,
3. If $P = RU_i^*S$ with R in $\mathbb{C}\langle(A_i)_{1\leq i\leq m}\rangle$ and S does not contain any U_j (but may contain the U_j^*), $\mu^{\mathbf{k}}(P) = \mu^{\mathbf{k}}(U_i^*SR)$,
4. If q does not contain any U_j,

$$\mu^{\mathbf{k}}(U_i^*q) = -\sum_{q=q_1U_i^*q_2} \binom{\mathbf{k}}{\mathbf{k}'} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \mu^{\mathbf{k}'}(U_i^*q_1)\mu^{\mathbf{k}''}(U_i^*q_2) + \sum_j k_j \mu^{\mathbf{k}-1_j}(D_i q_j q).$$

5. And for all q,

$$\mu^{\mathbf{k}}(qU_i) = -\sum_{q=q_1U_iq_2} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mu^{\mathbf{k}'}(q_1U_i)\mu^{\mathbf{k}''}(q_2U_i) + \sum_{q=q_1U_i^*q_2} \sum_{\mathbf{k}'+\mathbf{k}''=\mathbf{k}} \binom{\mathbf{k}}{\mathbf{k}'} \mu^{\mathbf{k}'}(q_1)\mu^{\mathbf{k}''}(q_2) + \sum_j k_j \mu^{\mathbf{k}-1_j}(D_i q_j q).$$

One can see that this allows to compute uniquely any $\mu^{\mathbf{k}}(P)$. The first relation takes care of the non random case, the relations 2 and 3 use the traciality to place a variable U in a convenient place. Finally relations 4 and 5 allow to compute $\mu^{\mathbf{k}}(P)$ as a function which depends on the $\mu^{\mathbf{k}'}(Q)$ with $\deg Q < \deg P$ and $\mathbf{k}' \leqslant \mathbf{k}$ (first terms) or on the $\mu^{\mathbf{k}'}(Q)$ with $\mathbf{k}' < \mathbf{k}$ (last term). This is a well founded induction. Thus the $\mu^{\mathbf{k}}$ are uniquely defined.

□

We next show that this solution is not only formal but gives a family of solution $\mu_{\mathbf{t}}$ of the non-formal equation $\mathbf{SD}[\mathbf{V_t}, \tau]$ which depend analytically on the parameters $(t_i)_{1\leq i\leq n}$.

There exists $\varepsilon > 0$ such that for $\mathbf{t} \in \mathbb{C}^n, \max_{1\leq i\leq n} |t_i| \leq \varepsilon$, the formal solution $\mu_{\mathbf{t}}$ of $\mathbf{SD^f}[\mathbf{V_t}, \tau]$ is indeed a convergent series. For all polynomials P, $\mathbf{t} \longrightarrow \mu_{\mathbf{t}}(P)$ is analytic.

In other words, there exists a family $(\mu^{\mathbf{k}}, \mathbf{k} = (k_1, \cdots, k_n) \in \mathbb{N}^n)$ of elements in $\mathcal{M}$ such that

$$\mu_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \prod_{i=1}^{n} \frac{t_i^{k_i}}{k_i!} \mu^{\mathbf{k}}(P)$$

converges absolutely for $\max_{1\leq i\leq n} |t_i| \leq \varepsilon$. An immediate consequence of this result is to deduce that the formal solution is a real solution of $\mathbf{SD}[\mathbf{V_t}, \tau]$ in a small parameters region, and therefore by Theorem 5.3.2, equals the real solution. This will be a key to prove Theorem 5.0.5 (see section 5.7). The formal solution of Schwinger-Dyson equation converges for small $\mathbf{t}$. In addition it matches the real solution which thus depends analytically in the parameters $\mathbf{t}$ of the potential.

Let us now prove Theorem 5.4.3. **Proof.**

According to the proof of Theorem 5.4.2 the $\mu^{\mathbf{k}}$ are uniquely defined by the family of relations (1)-(5). We only need to control the growth of the coefficient $\mu^{\mathbf{k}}$ to show that this give for small parameters a convergent expansion.

To find a bound we use the Catalan's numbers

$$C_0 = 1, C_{k+1} = \sum_{0\leqslant p\leqslant k} C_p C_{k-p}$$

and the fact that they do not explode too fast; $C_{k+1} \leqslant 4C_k$. We denote $C_{\mathbf{k}} := \prod_i C_{k_i}$ and $D_k := A^{k-1}C_{k-1}$ for $k \geq 1$, $d_0 := 0$. The two key properties of this sequence is first that it is sub-geometric ($D_{k+1} \leqslant 4AD_k$) and secondly it satisfies $D_k = A\sum_{0<p<k} D_p D_{k-p}$. Now our induction hypothesis is that there exists $A, B > 0$ such that for all $\mathbf{k}$, for all monomial P of degree p,

$$\frac{|\mu^{\mathbf{k}}(P)|}{\mathbf{k}!} \leqslant C_{\mathbf{k}} B^{\mathbf{k}} D_p. \tag{5.13}$$

We prove this bound by induction. We will only show how it works for a polynomial of the form qU_i since it is the most complicated case.

$$\frac{|\mu^{\mathbf{k}}(qU_i)|}{\mathbf{k}!} \leqslant \sum_{\substack{q=q_1U_iq_2 \\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{|\mu^{\mathbf{k}'}(q_1U_i)|}{\mathbf{k}'!}\frac{|\mu^{\mathbf{k}''}(q_2U_i)|}{\mathbf{k}''!}$$
$$+ \sum_{\substack{q=q_1U_i^*q_2 \\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{|\mu^{\mathbf{k}'}(q_1)|}{\mathbf{k}'!}\frac{|\mu^{\mathbf{k}''}(q_2)|}{\mathbf{k}''!} + \sum_{k_j\neq 0} \frac{|\mu^{\mathbf{k}-\mathbb{1}_j}(D_iq_jq)|}{(\mathbf{k}-\mathbb{1}_j)!}$$

Now we use the induction hypothesis. Note that the number of terms in the last sum is less than nD with D the degree of V. If q is of degree $p-1$,

$$\frac{|\mu^{\mathbf{k}}(qU_i)|}{\mathbf{k}!C_{\mathbf{k}}B^{\mathbf{k}}D_p} \leqslant 2\sum_{\substack{0<q<p \\ \mathbf{k}'+\mathbf{k}''=\mathbf{k}}} \frac{C_{\mathbf{k}'}B^{\mathbf{k}'}D_qC_{\mathbf{k}''}B^{\mathbf{k}''}D_{p-q}}{C_{\mathbf{k}}B^{\mathbf{k}}D_p}$$
$$+ r\frac{C_{\mathbf{k}-\mathbb{1}_j}B^{\mathbf{k}-1}D_{p+D}}{C_{\mathbf{k}}B^{\mathbf{k}}D_p}$$
$$\leqslant 2\prod_i \frac{C_{k_i+1}}{C_{k_i}}\frac{1}{A} + r\frac{(4A)^D}{B}.$$

The point is that we can choose $A, B > 0$ such that this last quantity is lesser than 1. For example take $A > 4^{n+1}$ and then $B > 2r(4A)^D$.

Thus, for $\|t\| := \max_i |t_i| < 1/4B$, for all P in $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$,

$$\nu_{\mathbf{t}}(P) = \sum_{\mathbf{k}} \prod_i \frac{t_i^{k_i}}{k_i!}\mu^{\mathbf{k}}(P)$$

is an absolutely convergent series.

□

5.5 Combinatorics.

The purpose of this section is to provide a graphical approach to the solution of the Schwinger-Dyson equation, and therefore to the computation of unitary matrix integrals and free entropy (see sections 5.6, 5.7 and 5.8). Actually, the proof of Theorem 5.4.2 gives a recursive way of computing formal solutions to the Schwinger-Dyson equation, and therefore numerical solutions with arbitrary precision.

Before giving a detailed description of our combinatorial model, we start with an overview. We need the notions of a **star**, which is a pictorial encoding of a monomial in the algebra $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, of **root star**, which is a distinguished star, and of a **map**, which is a specific planar decoration over a set of stars and one root star. Thus there will be a forgetful application of maps onto sets of stars (which we will call multistars).

The goal of this section is to show that the limits of integrals on the space of unitary matrices are generating function of the number of some maps as described above. However we are not interested in all maps, but rather on some that arise from an admissible construction, which leads us to the third concept of **admissible maps**. Last, we need the notion of **weight** of a map, and our result will be in terms of sum over admissible maps of weights.

Let us point out that for the sake of clarity, although our natural playground is the algebra $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$ and our definitions work in full generality, we restrict ourselves in the examples to the case of one single unitary matrix U and two variables $A_1 =: A$ and $A_2 =: B$. We first start with the definition of a star and root star, in the spirit of [GMS06, GMS07].

First we define the base elements that we will use to construct our maps.

1. A **star** is a circle endowed with the clockwise orientation, decorated with elements such as colored incoming or outgoing arrows, and colored diamonds. One of the element is marked.

2. To each letter X_i in the alphabet $(A_i, U_i, U_i^*)_{1\leq i\leq m}$, we associate (bijectively) an **element** as follows; a diamond of color i if $X_i = A_i$ and a ring of color i if $X_i = U_i$ or U_i^*; in the case of U_i (resp. U_i^*) we attach before the ring an outgoing arrow of color i (resp. we attach after the ring an incoming arrow of color i) outside of the circle.

3. To a monomial $q \in \mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, we associate in a canonical way a **star of type** q by drawing on the oriented circle the elements associated to the successive letters of q, while the element corresponding to the first letter of q is marked.

4. A **root star of type** q is obtained by drawing on the oriented circle the elements associated to the successive letters of q in the counter clockwise order, the arrows being drawn inside the circle. Although the maps are on the sphere, in the graphical representation of this section we will draw them on the plane and, to highlight the role of the root star we will draw it in this section such that it contains all the other stars. It can be viewed as the star centered in infinity or as the outer face of the dual map. Besides, on a root star we can distinguish a root element. If q contains no U_i nor U_i^*, there are no root element. If q contains a U_i, the ring associated to the last $(U_i, 1 \leq i \leq m)$ is the root element. If q contains no U_i but some U_i^*, the ring associated to the first $(U_i^*, 1 \leq i \leq m)$ is called the root element.

5. A **multistar** is a set of k stars inside a root star drawn on the same plane with a coherent orientation.

The figure 5.1 shows a concrete example of a multistar. In the middle of the picture there is a star of type U^*AUB and, surrounding it, a root star of type $U^*A^5UB^2U^*A^3UB$.

We are now ready to introduce the main objects in our combinatorial model, namely, maps :

A map is a decoration of a multistar into a connected graph embedded in the plane by drawing two species of edges between rings :

1. A first category of edges, called “dotted edges", can be drawn between rings either attached to two outgoing arrows of the same color or to incoming arrows of the same

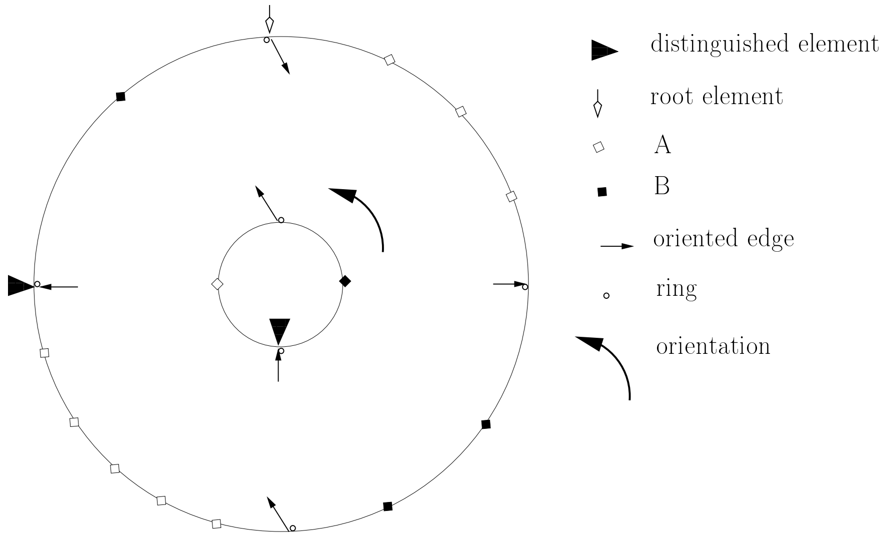

Fig. 5.1 – Star of type U^*AUB and root star of type $U^*A^5UB^2U^*A^3UB$.

color. Edges can not have diamonds as endpoints. Rings can have any number of dotted edges going out of them, possibly none.

2. A second category of edges, called "colored oriented edge" arises from the connection of an arrow going out of a star (associated with a variable U_i) into an incoming arrow (associated to a variable U_i^*) of the same color. These colored oriented edges is a pairing between the set of U_i's and the set of U_i^*'s : exactly one incoming arrow is glued to each outgoing arrow.

In addition, all the above edges do not cross, all arrows are paired but rings can be attached to any number of dotted edges (including to none).

In the rest of this section we keep considering pictures drawn on spheres. They therefore give rise to graphs with vertices, edges and faces - together with additional decoration. For our forthcoming definitions, we need to clarify the notion of 'face' : we consider that faces of a graph are the connected components of the complementary of the graph on the sphere. However, we take the convention that the original stars are 'fattened vertices'. Therefore the interior of stars will not be considered as faces (neither is the exterior of the root star).

If the graph is connected, each 'face' component is isomorphic to a disc ; thus this is an actual face. This is due to the fact that our map is embedded into a sphere. This condition would not be granted in the case of an embedding into a higher genus oriented 2D compact manifold. In this case it would have to stand in the definition of a map of 'higher genus' : this will be of use for future work but for the sake of simplicity we do not emphasize this notion in this paper.

Next, we define the weight of a map. The boundary of a face is homeomorphic to a disc, it is given an orientation (the orientation of the sphere) and is decorated with diamonds (note that all arrows have been paired) ; it thus has the structure of a star except for the distinguished element.

As we said before, not all maps will contribute and we need to define now the notion of admissible maps. Admissibility can be checked by an inductive procedure **IP**, which ressembles Tutte's surgery [Tut63] and which amounts to check one after the other whether edges of the map are admissible. Once an edge has been checked, it is frozen and we continue by checking the other edges.

Inductive Procedure IP :

a- If the root star has no root element, then it can not be connected to any other star and hence the graph can not be a map unless there is no other star in which case the map is just the trivial graph with no edges.

b- The root star has a root element which is associated to a U_i (resp. a U_i^*), for some $i \in \{1, \cdots, m\}$.

1-Then, we first check the admissibility of the dotted edges. We first consider the dotted edge which is the farthest from the arrow and declare it admissible if its other vertex is a ring of an outgoing (resp. ingoing) arrow and that *there is no other dotted edge* attached to this ring which is farther (amongst the unfrozen dotted edges) from its arrow. Once this condition is verified, we freeze this dotted edge and the root element remains the root element. We check all dotted edges of the element root inductively. Once the dotted edges are frozen, the map may have been cut into disjoint subgraphs whose boundary is homeomorphic to a disc. In each of these subgraphs, we declare the first element after the dotted edge as marked. The boundary of each subgraph is then a star which will be the root star of the resulting submap.

2- When all dotted edges are frozen, we check that the arrow of the root is paired with an arrow of the opposite direction (note that if the element root comes from a U_i^*, it can only be paired with an element of another star since by definition there is no more outgoing arrows on the root star). The oriented edge is seen as a fat edge. In particular, if the oriented edge link the root star with another star, we see this other star as part of the root star for the next step, i.e we identify the root star of type QU_iP glued to the star of type RU_i^*S by the marked U_i's with the root star of type $PQSR$ with, by convention, the distinguished element chosen to be the closest element after the glued U_i^*. If the oriented edge link two rings of the root star, two disjoint subgraphs are formed and we proceed as in -1-.

c-We continue the inductive procedure on the submaps.

Now we can define weighted sum of admissible maps.

Assume we are given the tracial state τ of (5.2).

- First we define the weight of the faces of a map. The faces of the map are the connected components of the map (with boundaries made of dotted edges, oriented edges, piece of circles and decorated with diamonds). The interior of the circle of a star is not considered to be a face of the map (neither is the exterior ot the root star). The faces can also be seen as the submaps defined in the inductive procedure, in the last step where their boundary is only decorated with diamonds. In particular, each of their boundary have the structure of a star and therefore are associated with a monomial in the A_i's. The weight of a face is the trace $\tau(q)$ of the monomial q associated with its boundary.
- The weight of the map m, denoted by $M_m(\tau)$, is the product of the weights of the faces times a sign given by -1 to the power the number of dotted edges.
- We define the weighted sum of admissible maps constructed above the stars $r_1, \cdots, r_n$

and the root star P :

$$\mathcal{M}_{r_1,\cdots,r_n}(P) = \sum M_m(\tau)$$

where the sum runs over all admissible maps m constructed above $r_1, \cdots, r_n$ with root star P. Assuming that $V_{\mathbf{t}} = t_1 q_1 + \ldots + t_n q_n$ where q_i are monomials, we define the formal series :

$$\mathcal{M}_{\mathbf{t}}(P) = \sum_{\mathbf{k}\in\mathbb{N}^n} \frac{\mathbf{t}^{\mathbf{k}}}{\mathbf{k}!} \mathcal{M}_{\mathbf{k}}(P)$$

with $\mathcal{M}_{k_1,\cdots,k_n}(P) = \mathcal{M}_{q_1,\cdots,q_1,\cdots,q_n,\cdots,q_n}(P)$ where the monomial q_j appears in k_j successive position and $\mathbf{t}^{\mathbf{k}} = \prod t_i^{k_i}$, $\mathbf{k}! = \prod k_i!$.

Remark that we do not count all the maps which contain the stars $r_1,\ldots,r_n$ but only those that are constructed using our inductive rules ; they for instance forbid to glue the two same rings more than twice.

However, a given map is counted at most once since there is only one way to decompose c it using the procedure **IP**. Indeed, it is easy to check that at each step we have only one possibility for the next step since the dotted edges have to be drawn one after the other following the orientation and no new dotted edge can be drawn after the arrow of the root has been glued.

Example

Let us show some example. We start from one root star and a star on the sphere (see figure 5.1). We want to construct maps above these stars with our rules, starting with the root element shown by the arrow outside the root star. Figures 5.2, 5.3 and 5.5 are examples of such maps. Note that the weights of the maps of figures 5.2 and 5.3 are the same, the only difference is the way the three rings are glued. There is a third way to glue those three rings shown in figure 5.4 which is a map but can not be obtained by our rule of construction (and thus is not admissible).

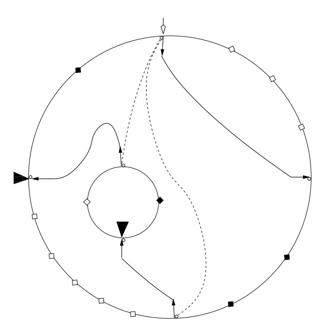

FIG. 5.2 – A possible map. Its weight is $\tau^{\otimes 5}(A^6 \otimes B \otimes B^2 \otimes A^3 \otimes B)$

We now come to the main theorem of this section, namely the graphical expansion result for $\mathcal{M}_{\mathbf{t}}$:

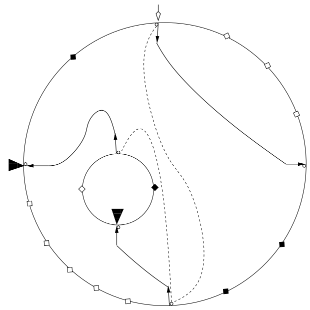

FIG. 5.3 – Another one. Its weight is $\tau^{\otimes 5}(A^6 \otimes B \otimes B^2 \otimes A^3 \otimes B)$

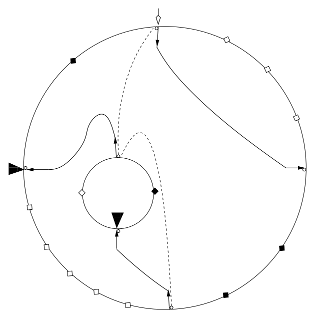

FIG. 5.4 – A counterexample : **IP** is violated because of the order of the dotted edges at the root element

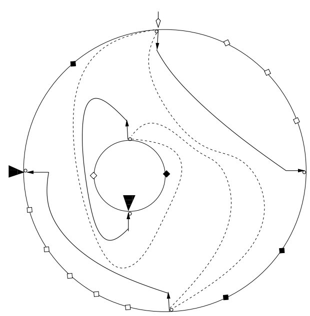

FIG. 5.5 – Another map. Its weight is $\tau^{\otimes 6}(A^5 \otimes A \otimes A^3 \otimes B \otimes B^2 \otimes B)$

Let $V = \sum t_i q_i$. Let $\mu_{\mathbf{t}}$ be a solution of $\mathbf{SD}[V_{\mathbf{t}}, \tau]$ and $\mathcal{M}_{\mathbf{t}}$ be the formal series

$$\mathcal{M}_{\mathbf{t}}(P) = \sum_{\mathbf{k} \in \mathbb{N}^n} \prod_{i=1}^{n} \frac{t_i^{k_i}}{k_i!} \mathcal{M}_{\mathbf{k}}(P)$$

where $\mathcal{M}^{\mathbf{k}}(P)$ is the weighted sum of planar maps with one root star of type P and k_i stars of type q_i. Then, $\mathcal{M}_{\mathbf{t}}$ is absolutely convergent in a neighborhood of the origin and for all polynomial P,

$$\mathcal{M}_{\mathbf{t}}(P) = \mu_{\mathbf{t}}(P).$$

Proof.

For the sake of clarity we first prove the case $V = 0$.

We proceed by induction on the total degree in U_i, $1 \leq i \leq m$, in q.

Suppose that there is no variable U_i in q. Then either there is no variable U_i^* and both sides of the equality are equal to $\tau(q)$, or there is a U_i^* and both sides vanish : the l.h.s by freeness property and the r.h.s because one can not glue the arrow coming out from this U_i^* anywhere.

We assume our identification proved when the degree of q in the U_i's is less than k. We next take q with degree in the U_i's equal to $k+1$. Thus we can assume that there is a U_i in q,and we consider the last one in q so that $q = pU_ib$ with b a polynomial in the U_j^* and the A_j's, $1 \leq j \leq m$. By definition, $\mathcal{M}(pU_ib) = \mathcal{M}(bpU_i)$ since it depends only on the position of the last U_i. Thus, we may assume that q is of the form PU_i with P of degree k. We apply Schwinger-Dyson equation to this quantity :

$$\mu(PU_i) = -\sum_{P=RU_iS} \mu(RU_i) \otimes \mu(SU_i) + \sum_{P=RU_i^*S} \mu(R) \otimes \mu(S) \tag{5.14}$$

Now, we can apply our induction hypothesis since all polynomial appearing above have degree strictly smaller than $k+1$.

We need to show that this is exactly the induction relation for maps. To construct a map above a star of type PU_i. We first look at the root element U_i and we have to decide what to do first with the edge linked to the hyperedge and then with the arrow. There are two possibilities :

1. The first possibility is that there is no dotted edge going outside of the ring of the root. In such a case, we can glue the arrow to any other arrow of opposite direction and of the same color (corresponding to a variable U_i^*). This implies that P decomposes into RU_i^*S and we construct an oriented edge between U_i and U_i^*. Thus we separate the map in two parts and we have to construct a map above the R part and another one above the S part (this is the case 2 of **IP**). This is exactly the possibilities counted by the second term in the right hand side of (5.14).
2. The second possibility is that we glue the root ring to another ring. Thus P must decompose into RU_iS and the creation of the dotted edge amounts to decompose the map into RU_i and SU_i. In this procedure, we have fixed one dotted edge and thus multiplied the contribution of the resulting map by -1. (this is the case 1 of **IP**). We again separate the map in two parts by planarity. Besides, we separate once again the map in two and choose a new root ring inside the new face. The second sum computes the operation of gluing rings by dotted edges.

Putting these two possibilities together we see that the state μ and the enumeration of maps $\mathcal{M}_0$ satisfy the same induction so that they are equal ; $\mathcal{M}_0(pU_ib) = \mu(pU_ib)$ for any b monomial which does no contain any of the $(U_i, 1 \leq i \leq m)$. Note here that no dotted edges between rings of incoming arrows can be drawn since if there are no outgoing arrows in a map, but some U_i^*, there is no contribution. By traciality of μ, we deduce as well that $\mathcal{M}_0$ is tracial.

Now we turn to the general V case.

We first check the induction relation when the root star P contains a U_i for some $i \in \{1, \cdots, m\}$. Let us denote for n-tuples $\mathbf{k}$ and ℓ, $\binom{\mathbf{k}}{\ell} = \prod_i \binom{k_i}{l_i}$. We check the formal equality by considering the induction relation, now given by :

$$\begin{aligned}\mu^{\mathbf{k}+\mathbf{1_j}}(PU_i) = &-\sum_{\ell \leq \mathbf{k}+\mathbf{1_j}} \sum_{P=RU_iS} \binom{\mathbf{k}+1_j}{\ell} \mu^{\ell}(RU_i) \otimes \mu^{\mathbf{k}+\mathbf{1_j}-\ell}(SU_i) \\ &+ \sum_{\ell \leq \mathbf{k}+\mathbf{1_j}} \sum_{P=RU_i^*S} \binom{\mathbf{k}+1_j}{\ell} \mu^{\ell}(R) \otimes \mu^{\mathbf{k}+\mathbf{1_j}-\ell}(S) \\ &- \sum_{q_j=RU_iS} k_j \mu^{\mathbf{k}}(PU_iSRU_i) + \sum_{q_j=RU_i^*S} k_j \mu^{\mathbf{k}}(PSR)\end{aligned} \tag{5.15}$$

We need to show that the enumeration of maps satisfies the same relation. We start by putting stars of type $(q_j, 1 \leq j \leq n)$ inside a root star of type PU_i and we wonder what happens to the root U_i. We apply one step of IP. Two things can happen. Either we link U_i to another part of P and in that case we have already shown that the possibilities are enumerated by the first two terms of the induction relation. Here, note that the product of $\binom{k_i}{\ell_i}$ corresponds to the possible distribution of stars in each part of the map, since all the stars are labeled.

Thus we need to show that the two other terms take into account the case where U_i is linked to another star q_j. According to our construction rules we have two possibilities :

1. Starting from U_i we glue the arrow to an arrow of the same color entering a star of type q. This rule forbids any other gluing from U_i, this is counted by

$$\sum_{q_j=RU_i^*S} k_j \mu(PSR).$$

   The coefficient k_j counts the number of choices for the star of type q_j since they are all labelled.

2. The other possibility is to glue the ring to a ring of the same color. This leads to

$$-\sum_{q_j=RU_iS} k_j \mu(PU_iSRU_i)$$

   possibilities.

In the case where P does not contain any U_i, $1 \leq i \leq m$ but still some U_i^*, the root of the root star can only be glued by a dotted edge to any other U_i^*, or by a directed edge to a U_i of a star.The resulting induction relation is exactly given by the formula obtained by conjugation of (5.15), hence again $M_{\mathbf{k}}(P) = \mu^{\mathbf{k}}(P)$.

Thus the proof is complete.

□

This theorem gives a combinatorial interpretation in term of maps to the unitary integrals. The fact that we do not take the sum on all maps but only on admissible ones makes this interpretation less transparent than the one for the gaussian case found in [BIPZ78]. However, now that we know that the series can be identified to the matrix integral, we obtain some combinatorial identities which show that **IP** is less rigid than it looks like.

Let $V = \sum t_i q_i$.

1. For all P, Q,
$$\mathcal{M}_{\mathbf{t}}(PQ) = \mathcal{M}_{\mathbf{t}}(QP).$$
2. For all monomials $r_1, \ldots, r_n, r_{n+1}$, and all permutation σ of $n+1$ elements,
$$\mathcal{M}_{r_1,\cdots,r_n}(r_{n+1}) = \mathcal{M}_{r_{\sigma(1)},\cdots,r_{\sigma(n)}}(r_{\sigma(n+1)}).$$
3. Assume that we define another procedure to define the root element of the root star (for example we pick the root element to be the second ring avaible if possible, or we pick a ring at random, or any other choice which may change during **IP** for the root stars that are created during the procedure when new faces are added). This will change the notion of admissible maps and we can define a new weighted sum $\mathcal{M}'_{r_1,\cdots,r_n}(P)$ and a new series $\mathcal{M}'_{\mathbf{t}}(P)$ where the sum occurs on these new maps. For all $r_1, \ldots, r_n, P$,
$$\mathcal{M}_{r_1,\cdots,r_n}(P) = \mathcal{M}'_{r_1,\cdots,r_n}(P)$$
$$\mathcal{M}_{\mathbf{t}}(P) = \mathcal{M}'_{\mathbf{t}}(P).$$

Note that due to the definition of admissible maps via the procedure **IP**, those properties are far from being obvious from a purely combinatorial point of view. Still they will appear as easy consequence of the identification with the matrix model.

Obviously different roots lead to a different procedure **IP**, and thus potentially to different maps. It is actually possible to see through examples that this phenomenon actually happens.

However, it follows from the second point of the corollary that the choice of the root does not affect the weighted sum. The first and third points show that the choice of the root element and of the root star does not affect the final series. We were not able to give a more direct combinatorial proof of that result.

To be more specific on the impact of the choice of the roots on the maps, let us call clusters the equivalence class of rings for the equivalence relation generated by $a \sim b$ if the ring a is glued to the ring b by a dotted edge. Changing the choices of the roots will lead to different admissible maps since it will allow different positions for the dotted edges. For example, they were three choices for the starting root in figure 5.1. For each of these choices, two of the three maps represented in figures 5.2, 5.3 and 5.4 would have been reachable by the inductive construction **IP** but not the third one. The one who is not constructible depends on the choice of the first root. It seems that if the maps are different, nevertheless the clusters are the same and in that simple case, knowing this cluster is sufficient to define the faces created by the dotted edges and thus the weight of the maps.

One may wonder why we choose this representation instead of representating the clusters by an additional structure and putting edges between the rings and this structure. In that case we would miss the possibility of figure 5.5. In the case of a one map with only one vertex this can not occur and we will use that fact in section 5.6.

Proof.

Changing the root element of a star is the same thing than making a circular permutation of the variable of the associated monomial. The Theorem shows that weighted sums are equal to the limit of the empirical measure of the matrix model which are tracial. The first and third items are a direct consequence of this identification.

For the second item, observe that permuting the first n monomials doesn't change the sum by its definition. Thus we only need to show that

$$\mathcal{M}_{r_1,\cdots,r_n}(P) = \mathcal{M}_{P,r_2,\cdots,r_n}(r_1).$$

Let us define $V = \sum_i u_i r_i + tP$. We will again use the identification with the matrix model but now we will use the formal version. The coefficient $\mathcal{M}_{r_1,\cdots,r_n}(P)$ appears as the coefficient of the limit tracial power state μ^f by Corollary 5.3.3 and Theorem 5.5.4. More precisely,

$$\mathcal{M}_{r_1,\cdots,r_n}(P) = (-1)^n \lim_N \left.\frac{\partial^n}{\partial u_1 \cdots \partial u_n}\right|_{u_i=0} \mu^f(P).$$

We now use the fact that μ^f is the limit coefficientwise of the formal model defined in (5.8). Thus,

$$\begin{aligned}\mathcal{M}_{r_1,\cdots,r_n}(P) &= \lim_N \left.\frac{\partial^n}{\prod_i \partial u_i}\right|_{u_i=0} \frac{E[\hat{\mu}^N(P)e^{N^2\hat{\mu}^N(V_{t=0})}]}{E[e^{N^2\hat{\mu}^N(V_{t=0})}]} \\ &= \lim_N \left.\frac{\partial^{n+1}}{\partial t \prod_i \partial u_i}\right|_{u_i=0,t=0} \frac{1}{N^2} \ln E[e^{N^2\hat{\mu}^N(V)}].\end{aligned}$$

We conclude by noticing that this last expression is symmetric in the monomials $r_1,\ldots,r_n,P$.

□

5.6 Application to free probability

In this section we look at applications of the combinatorial results of section 5.5 to free probability.

Let us assume the U_i's are chosen independently according to the Haar measure. If we define $X_i = U_i^* A_i U_i$ then the X_i's are asymptotically free (according to a theorem of Voiculescu [Voi91]) and with fixed distribution μ uniquely defined by the distribution of the A_i's. We are interested in using our setup to compute limit of moments of these variables or in other word to compute the moments of free variables :

$$\mu(X_{i_1}...X_{i_k}).$$

According to our interpretation this can be computed by looking at the maps above the star of type $X_{i_1}...X_{i_k}$ without any other stars, in other words we have to focus on computations of $\mathcal{M}(q)$ which turns out to be equal to $\mu(q)$ where μ is the free state product (see Corollary 5.3.4)

We are interested in using this method to compute some non-commutative moments of free variables, in relation with Speicher's non-crossing cumulants theory, cf [Spe94].

5.6.1 One star maps

For these purposes we need to find a simplified interpretation of $\mathcal{M}(q)$ in the single star map.

For this one vertex-case, the combinatorial interpretation can be slightly modified. First, we do not need to consider dotted edges between incoming arrows since if there is a U_i^* there must be a U_i which can be chosen as the root element or we can not build any map. But the main difference is that now each time we glue two rings, the edge newly created separate these two rings in two different faces so that they can no longer be glued together. Thus, we can forget about the restriction of the construction rules and present a simpler description in that case. Instead of gluing the ring two by two we will now glue them together. We define a new structure which we will call a node and now rings can only be glued to node and a node can be glued to any number of rings. A one vertex map is a map with one vertex where the arrows has been glued two by two while respecting the orientation and rings may be glued to exactly one node, each node is glued to an arbitrary number of rings but at least one. Figure 5.6 shows the new representation of a one vertex map. The trick to go from the previous interpretation to this one is to glue together to a node all the rings that are in the same class of the equivalence relation generated by being glued. In order to compute the weight of such a map, observe that several maps give the same one-star map, but the weight is easy to compute since as we will see we only need to add a factor C_{d-1} for each node of degree d.

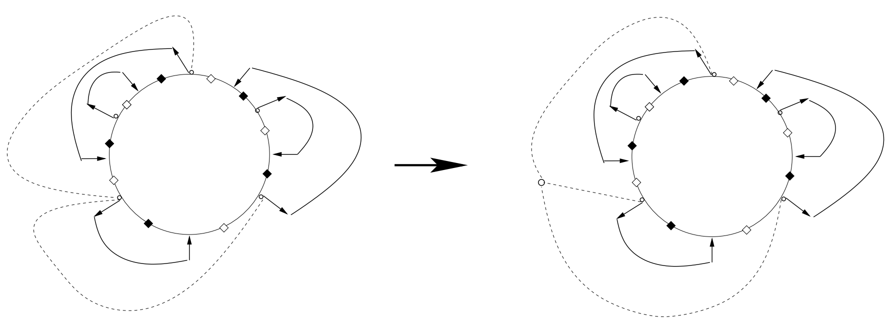

FIG. 5.6 – Reduction of a map on one star to a one-star map.

A one-star map is a connected graph embedded on a sphere above one star and with some edges such that

1. Edges are drawn only between rings and must not intersect.

2. Arrows must be glued two by two while respecting the orientation and the color : an arrow going out of a star (associated with a variable U_i) is always glued to exactly one other arrow going into a star (associated to a variable U_i^*) of the same color. This pair of arrows creates an oriented edge.

3. Any number of rings may be glued together on a node.

The weight of a one-star map is the product of the weight of its faces which is defined as before as trace of products of A_i's times the product of the weight of the nodes. The weight of a node of degree d is $(-1)^{d-1}C_{d-1}$. We define $\widetilde{\mathbb{M}}_0(q)$ the weighted sum of one-star map above a star of type q. Note that we no longer need to take care of roots and of maps that can be built with some set of rules.

For all monomial q,

$$\mu(q) = \widetilde{\mathbb{M}}_0(q).$$

Proof.

We only need to show that $\mathcal{M}(q) = \widetilde{\mathbb{M}}_0(q)$. For this we need to compute the number of maps above one star that are reduced to a given one-star map. The reduction goes as follows : two rings are glued to the same node if they are linked by a sequence of dotted-edges. We have to count how many configurations of dotted edges lead to a node of degree d. When one of the ring glued to this node becomes the root in the recursive construction, it has to be glued to one of the other ring glued to the node. Thus it separates the set of ring in two subsets, so according to our inductive procedure of section 5.5, we have to continue to glue this ring to other ones while we continue the construction in the face newly created. This yields a structure of tree on this set of rings. We have as many choices as they are trees with $d-1$ edges (to glue the d ring we need exactly $d-1$ edges). This explains the factor C_{d-1}. The factor $(-1)^{d-1}$ simply comes from the factor -1 which comes with each edge.

□

5.6.2 Maps and cumulants

Let $A_1, \ldots, A_n$ be self-adjoint variables and U a unitary matrix, free from the A_i's. Then choosing k indices $i_1, \ldots, i_k$ in $\{1, n\}$ one has

$$\mu(A_{i_1} \ldots A_{i_k}) = \mu(U^*UA_{i_1} \ldots U^*UA_{i_k})$$

Let us apply Schwinger-Dyson equation with respect to U to the above equality, and let us rearrange the sum according to the non-crossing partition of A_i's generated by the oriented edges. Obviously one obtains a formula of type

$$\mu(A_{i_1} \ldots A_{i_k}) = \sum_{\pi \in NC(k)} \tilde{K}_\pi(A_{i_1}, \ldots, A_{i_k}) \tag{5.16}$$

where $NC(k)$ is the non-crossing partitions and $\tilde{K}_\pi$ is a k-linear form multiplicative along the blocks of π in the sense of Speicher : if $\pi = \{V_1, \dots, V_n\}$ with the block $V_i = \{a_1^i, \dots, a_{r_i}^i\}$

$$\tilde{K}_\pi(X_1 \dots X_k) = \prod_i \tilde{K}_{(r_i)}(X_{a_1^i}, \dots, X_{a_{r_i}^i})$$

where (r_i) represents the partition on r_i elements with only one block.

The fact that such a formula holds true for any choice of non-commutative laws for A_i's proves via the moment-cumulant formula that $\tilde{K}_\pi$ has to be Speicher's non-crossing cumulants K_π. But it is also given as a sum on maps by our graphical model.

Let us recap this in the following proposition :

The n-th non-crossing cumulant of the variables A_1, …, A_p is the weight of all one-star maps over the star build by putting in the clockwise order a ring, a diamond of color i_1, a ring, a diamond of color i_2,…, a ring, a diamond of color i_p. Note that we have defined this map above a star which is not of type q for any monomial q. This would be a problem for admissible maps since **IP** requires the presence of oriented edges. But the definition of one-star map is fine in this context.

Actually, Proposition 5.6.2 gives us a new proof of the following Corollary, due to Speicher and known as non-crossing Moebius formula The following inversion formula holds true :

$$K_n(A_1, \dots, A_n) = \sum_{\pi \in NC(n)} \mu_\pi(A_1, \dots, A_n)(-1)^{n-|blocks(\pi^c)|} \prod_{B \text{ block of} \pi^c} C_{|B|-1},$$

where π^c is the Kreweras complement (see [NS06]) and C_q the catalan number. **Proof.**

This is a direct consequence of the previous proposition. Remember that $K_n(A_1, \dots, A_n)$ is a weighted sum over maps with dotted edges since the star contains some rings and no arrows. These dotted edges form a non-crossing partition of $[|1, \dots, n|]$ by saying that two rings are in the same component if their are linked to a same node. The weight associated to this map is a product whose factors are : $(-1)^{d-1}C_{d-1}$ for each node of degree d and the weight of each face. The faces are by definition the component of the Kreweras complement of π'. Thus we obtain :

$$K_n(A_1, \dots, A_n) = \sum_{\pi' \in NC(n)} \mu_{(\pi')^c}(A_1, \dots, A_n) \prod_{B \text{ block of} \pi'} (-1)^{|B|-1} C_{|B|-1}.$$

The formula follows after taking $\pi' = \pi^c$.

□

As a further remark, one can also read graphically the main properties of cumulants, for example, $K_n(X_1, \dots, X_n) = 0$ as soon as there are occurence of free elements. More precisely, assume that we can partition the X_i's in two families the A_j's and the B_k's with the algebra generated by the A_j's free from the algebra generated by the B_k's. Then if all the X_i's do not take value in the same algebra, $K_n(X_1, \dots, X_n) = 0$. Indeed, one can replace all the family of A_j's by the one of V^*A_jV with V unitary and free from the other variables. Now when looking at the combinatorial interpretation of $\mu(X_1, \dots, X_n)$ we can see that the oriented edges coming from V separate the components containing the A_j's from the others. By following those edges we see that the faces they are defining contain only variable from one of the two algebras. Thus, in the decomposition (5.16), the terms corresponding to partitions with one component containing both some A_i's and some B_j's does not occur. By uniqueness of the decomposition into cumulants we deduce that those elements vanish i.e. $K_n(X_1, \dots, X_n) = 0$.

These remarks are not new but this shows that our graphical model fully encompasses the theory of non-crossing cumulants and that the Schwinger-Dyson equation can also be read in terms of cumulants.

It is interesting to mention here that papers [MŚS07] and [MJE07] have developed a calculus on annuli which seems to be related to our graphical model. However these approaches only deal with the asymptotics of second order cumulants whereas our approach via formal calculus, see section 5.4, allows us to deal with arbitrary order cumulants.

The actual relation can be found in [CMŚS07], where convolution on partitioned permutations is introduced and showed to be the relevant algebraic tool to handle higher order freeness, namely, the asymptotic behaviour of cumulants of unitarily invariant random matrices.

But the results in our paper give an explicit algorithmic description of the Moebius inversion formula and therefore of higher order cumulants. As in the one star case, cumulants are also obtained by inserting an outer U^*U between each variable of each star and by looking at generating function where U is linked to its neighboring U^*.

It is interesting to see that a direct (yet difficult to describe) graphical reading of the Schwinger-Dyson equation, which is our main tool of investigation of unitarily invariant matrix models, yields non-crossing and could yield higher order moments related series and operations similar to convolution, although these latter results rely on more representation theoretic grounds (Weingarten function theory as developed in [CŚ06]).

It is not obvious to us how the Schwinger-Dyson equation can be read off from the results of [CMŚS07] (without writing a change of variable invariance formula), and it would be interesting to attempt to figure out the meaning of Schwinger-Dyson equation at the representation theoretic level.

5.7 Application to the asymptotics of $I_N(V, A_i^N)$

Let $(q_1, \cdots, q_n)$ be fixed monomials in $\mathbb{C}\langle U_1, \cdots, U_m, A_1, \cdots, A_m\rangle$, let $V = \sum t_i q_i$ and $I_N(V, A_i)$ be given by (5.1). Then There exists $\eta = \eta(q_1, \cdots, q_n)$ so that for any $\mathbf{t} \in \mathbb{C}^n \cap B(0, \eta)$ we have

$$\lim_{N\to\infty} \frac{1}{N^2} \log I_N(V_{\mathbf{t}}, A_i^N) = \sum_{\mathbf{k}\in\mathbb{N}^n\setminus(0,..,0)} \prod_{1\le i\le n} \frac{t_i^{k_i}}{k_i!} I_{\mathbf{k}}(q_1, \cdots, q_n, \tau).$$

Moreover, for any j such that $k_j \neq 0$

$$I_{\mathbf{k}}(q_1, \cdots, q_n, \tau) = \sum_{\mathbf{k}} \frac{(-\mathbf{t})^{\mathbf{k}}}{\mathbf{k}!} \mathcal{M}_{\mathbf{k}}$$

where $\mathcal{M}_{\mathbf{k}}$ is the weighted sum of maps constructed above k_i stars of degree i for all i, after choosing one of them as a root star (this is well defined according to corollary 5.5.5). **Proof.**

Let

$$F_{\mathbf{t}}^N = \frac{1}{N^2} \log I_N(V_{\mathbf{t}}, A_i^N).$$

Then, if $\alpha \in \mathbb{R}$,

$$\partial_\alpha F_{\alpha\mathbf{t}}^N = \int \hat{\mu}^N(V_{\mathbf{t}}) d\mu_{V_{\mathbf{t}}}^N.$$

Assume that $\mathbf{t}$ is small enough so that Corollary 5.3.3 holds and remark that $V_{\alpha\mathbf{t}}$ is self-adjoint and such that $|\alpha t_i| \leq \varepsilon$ for all i and all $0 \leq \alpha \leq 1$. Thus, for $\alpha \in [0,1]$,

$$\lim_{N\to\infty} \partial_\alpha F_{\alpha\mathbf{t}}^N = \mu_{\alpha\mathbf{t}}(V_{\mathbf{t}})$$

with $\mu_{\alpha\mathbf{t}}$ the solution to $\mathbf{SD}[\alpha V_{\mathbf{t}}, \tau]$. By dominated convergence theorem, we deduce that

$$\lim_{N\to\infty} F_{\alpha\mathbf{t}}^N = \int_0^1 \mu_{\alpha\mathbf{t}}(V_{\mathbf{t}}) d\alpha$$

□

Here also, we obtain the following important corollary :

The following holds true :

$$\lim_{N\to\infty} \frac{\partial^k}{\partial z^k} \frac{1}{N^2} \log \int_{\mathcal{U}_N(\mathbb{C})} e^{zNTrV((U_i, U_i^*, A_i^N)_{1\leq i\leq m})} d\mathbf{U}|_{z=0} = \frac{\partial^k}{\partial z^k} F_{V,\tau}(z)|_{z=0}$$

In particular, with this result we can give an expansion of the Harisch-Chandra-Itzykson-Zuber integral as a generating function of the number of some maps. Let us recall the exact expression of this integral :

$$F_N^{A,B}(z) := \frac{1}{N^2} \log HCIZ(zA, B) = \frac{1}{N^2} \log \int_{\mathcal{U}_N(\mathbb{C})} e^{zNTr(U^*AUB)} dU.$$

The maps appearing in the expansion contain only stars of type U^*AUB (see the star in the middle of figure 5.1). Besides we can build these maps without considering the rings attached to variable U^* since we will always be able to choose the root element to be a U (a U^* always comes with a U for this potential).

Since the number of diagrams is growing quickly we compute only the first term of the expansion. Note that when gluing the arrow of the root of the root star, we must always glue it to another incoming arrow of another star and hence we shall never face the case of a root star with no U_i's. Again, we therefore do not see dotted edges between incoming arrows.

Besides, we consider only the case where the distribution is centered, that is when $\tau(A) = \tau(B) = 0$. The other cases can be deduced easily from this one since we have the relation

$$F_N^{a+A,b+B}(z) = F_N^{A,B}(z) + \frac{z}{N}(b\text{Tr}A + a\text{Tr}B) + zab.$$

In terms of diagrams, this means that we only need to consider diagrams such that no face contains only one diamond.

According to the previous theorem, $\lim_{N\to\infty} F_N^{A,B}(z)$ has, for small parameters z, an expansion $\sum_n F_n z^n$. We now use this graphical representation to compute the first terms of this integral.

Since the distributions are centered, the first term F_1 is zero.

The second term F_2 consists of maps constructed with two stars of type U^*AUB. There is only one way to add edges between these two stars to construct a connected map without faces which contains only one diamond, this is represented by figure 5.7. We obtain a map with two faces. One has two diamonds associated to A and the other one two diamonds associated to B. Thus the weight of this map is $\tau(A^2)\tau(B^2)$. Since there is no gluing between the rings they are no other signs. They are only one way to distribute the labels on this picture (that is the second distribution leads to the same map) thus to obtain F_2 we only need to divide by 2!,

$$F^2 = \frac{1}{2}\tau(A^2)\tau(B^2).$$

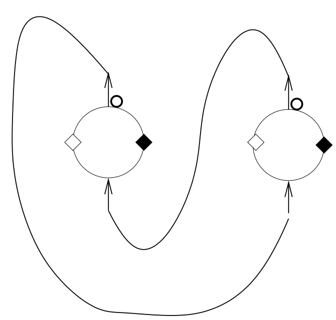

FIG. 5.7 – Second term in the expansion of the IZ integral.

We can continue this for the next terms in the expansion, the third term (see figure 5.8) is in the same spirit and leads to

$$F^3 = \frac{1}{3}\tau(A^3)\tau(B^3).$$

The fourth term is the first one where gluings between the rings appear. Thus weigths with negative coefficients can occur. The sign of a map is easy to compute, it is -1 to the power the number of dotted lines in the map. Equivalently since in the case of Itzykson-Zuber integral the number of oriented edges is equal to the number of stars, this number is also equal to the number of faces of the map and thus to the number of factor in the product of moments of the weight. In figure 5.9, we have drawn all unlabelled planar maps one can construct with 4 stars. To compute the exact coefficient of each map one has to multiply it by the number of way to distribute the labels and divide by 4!.

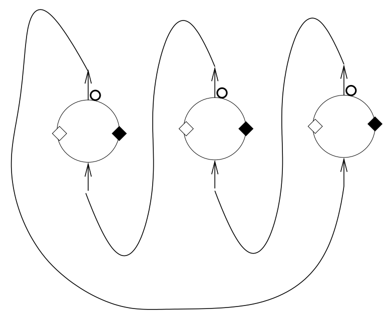

FIG. 5.8 – Third term in the expansion of the IZ integral.

This leads to,

$$F^4 = \frac{1}{4}\tau(A^4)\tau(B^4) - \frac{1}{2}\tau(A^2)^2\tau(B^4) - \frac{1}{2}\tau(A^4)\tau(B^2)^2$$

$$+\frac{1}{2}\tau(A^2)^2\tau(B^2)^2 + \frac{1}{4}\tau(A^2)^2\tau(B^2)^2.$$

Here the weight are given in the same order than the maps in the figure. Note a new and interesting feature that appears in the third map : two rings are linked by more than one dotted edge.

The other terms can be computed in the same way, for example figure 5.10 represents the fifth term and gives

$$F^5 = \frac{1}{5}\tau(A^5)\tau(B^5) - \tau(A^2)\tau(A^3)\tau(B^5) - \tau(A^5)\tau(B^2)\tau(B^3)$$

$$+4\tau(A^2)\tau(A^3)\tau(B^2)\tau(B^3).$$

Thus the first terms agree with the expansion given in [ZJZ03] on page 23, besides this allows us to answer a question raised in this paper. Indeed, the authors ask if there is an explanation to the fact that the coefficient of F_n all seem to be integer multiple of $\frac{1}{n}$. This is easy to prove with this graphical interpretation. To compute the contribution of a given unlabelled map we must distribute the labels $\{1, .., n\}$ on its stars, count the number of different map that we obtain and divide by $n!$. But after choosing the star which received the label 1 we have $(n-1)!$ ways to distribute the remaining labels and they all lead to different maps (note that on the other hand, due to possible symmetry in the unlabelled map, different choices for the star with the label 1 may lead to the same maps). Thus the coefficient in front of this map is a multiple of $\frac{(n-1)!}{n!} = 1/n$. More precisely it is $1/n$ times the number of choices of the star wich carry the label 1 that will lead to different maps, in particular it is always less than 1.

To finish, we wish to point out that we can recover results in [Col03] and [GM05a] about scalings of IZ integral. In these two papers, one considers the scaling where A has small rank, which amounts to considering only terms $\tau(A^k) \times P(B)$. Here the transformation depicted in

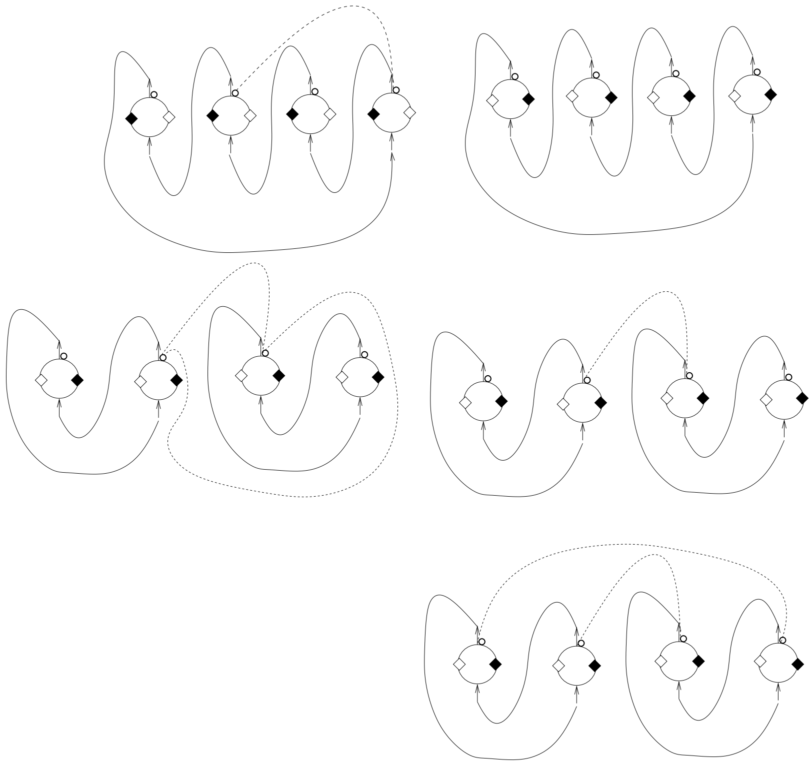

FIG. 5.9 – Fourth term in the expansion of the IZ integral.

section 5.6 applies and one sees that $P(B)$ has to be $k^{-1}K_k(B)$. In particular this means in the case that A is a rank 1 projection, that $N^{-1}\log IZ$ tends to the primitive of Voiculescu's R-transform.

5.8 Application to Voiculescu free entropy

Voiculescu's microstates free entropy is given as the asymptotic the volume of matrices whose empirical distribution approximates sufficiently well a given tracial state. Up to a Gaussian factor, it is given by

$$\chi(\mu) = \limsup_{\substack{\varepsilon\downarrow 0 \\ k\uparrow\infty, R\uparrow\infty}} \limsup_{N\to\infty} \frac{1}{N^2}\log \mu_N^{\otimes m}\left(\Gamma_R(\mu,\varepsilon,k)\right)$$

with μ_N the Gaussian measure on $\mathcal{H}_N(\mathbb{C})$ and $\Gamma_R(\mu,\varepsilon,k)$ the microstates

$$\Gamma_R(\mu,\varepsilon,k) = \{X_1,\cdots,X_m \in \mathcal{H}_N(\mathbb{C}) : |\frac{1}{N}\mathrm{Tr}(X_{i_1}\cdots X_{i_p}) - \mu(X_{i_1}\cdots X_{i_p})| < \varepsilon$$

$$p \le k, i_\ell \in \{1,\cdots,m\}, \|X_i\|_\infty \le R\}.$$

When $m = 1$, it is well known [Voi00] that $\mu \in \mathcal{P}(\mathbb{R})$ and

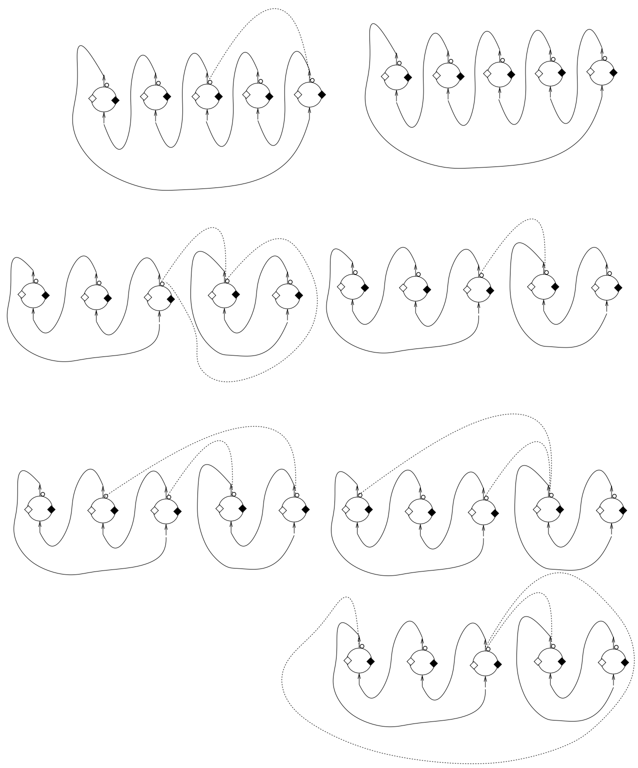

FIG. 5.10 – Fifth term in the expansion of the IZ integral.

$$\chi(\mu) = I(\mu) = \int\int \log|x-y|d\mu(x)d\mu(y) - \frac{1}{2}\int x^2 d\mu(x) + \text{const}.$$

Moreover, one can replace the lim sup by a lim inf in the definition of χ. Such answers (convergence and formula for χ) are still open in general when $m \geq 2$ (see [BCG03] for bounds). However, if μ is the law of m free variables with respective laws μ_i, then these questions are settled and

$$\chi(\mu) = \sum_{i=1}^{m} I(\mu_i).$$

We here want to emphasize that our result provides a small step towards dependent variables by showing convergence and giving a formula for the type of laws μ solutions of Schwinger-Dyson's equations $\mathbf{SD}[\mathbf{V}, \tau]$. Indeed, we shall prove that Let μ be the law of m self-adjoint variables X_i with fixed marginal distribution $(\mu_1, \cdots, \mu_m)$. Assume that X_i can be decomposed as $X_i = U_i D_i U_i^*$ with U_i unitary matrices such that the joint law of $(D_i, U_i, U_i^*)_{1\leq i\leq m}$ satisfy $\mathbf{SD}[\mathbf{V}, \tau]$ with τ the law of m free variables with marginal distribution $\mu_1, \cdots, \mu_m$ and some potential $V = \sum_{i=1}^n t_i q_i$. Assume that the t_i's are small enough so that Corollary 5.3.3 holds. Assume also that the hypotheses of Theorem 5.7.1 hold. Then,

$$\chi(\mu) = \liminf_{\substack{\varepsilon\downarrow 0\\ k\uparrow\infty}} \liminf_{N\to\infty} \frac{1}{N^2} \log \mu_N^{\otimes m}\left(\Gamma_R(\mu, \varepsilon, k)\right)$$

and a formula of $\chi(\mu)$ can be given in terms of the $\mu^{\mathbf{k}}$'s of Theorem 5.4.3.

Proof.

Indeed, let us consider $V = V(U_i A_i U_i^*, 1 \le i \le m)$ with V a self-adjoint polynomial and μ the unique solution of $\mathbf{SD}[\mathbf{V}, \tau]$ with τ the law of the $A_i, 1 \le i \le m$ which is now chosen to be the law of m free variables with marginals distribution μ_i, $1 \le i \le m$. Under the law $\mu_N^{\otimes m}$, we can diagonalize the matrices $X_i = U_i D_i U_i^*$ with U_i following the Haar measure on $\mathcal{U}_N(\mathbb{C})$, and id d is the Dudley metric, we find that for N sufficiently large

$$\begin{aligned}
\Lambda_N &:= \mu_N^{\otimes m}\left(\Gamma_R(\mu, \varepsilon, k)\right) \\
&= \mu_N^{\otimes m}\left(d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon; \hat{\mu}^N_{U_i D_i U_i^*, 1 \le i \le m} \in \Gamma_R(\mu, \varepsilon, k)\right) \\
&= \int_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon \\ \|D_i\|_\infty \le R}} \left(\int_{\hat{\mu}^N_{(U_i D_i U_i^*)_{1 \le i \le m}} \in \Gamma_R(\mu, \varepsilon, k)} dU_1 \cdots dU_m \right) \prod_{1 \le i \le m} d\sigma_N(\lambda_i)
\end{aligned}$$

where we denoted $\Delta(\lambda_j) = \prod_{k \ne j} |\lambda_k - \lambda_j|$ and $d\sigma_N$ the probability measure

$$d\sigma_N(\lambda) := Z_N^{-1} \prod_{k \ne j} |\lambda_k - \lambda_j|^2 e^{-\frac{N}{2}\sum (\lambda^j)^2} \prod_{1 \le j \le N} d\lambda^j.$$

In these notations, $D_i = \mathrm{diag}(\lambda_1^i, \cdots, \lambda_N^i)$ and $\lambda = (\lambda_1, \cdots, \lambda_N)$. Hereafter, $\hat{\mu}^N_{\{E_i\}_{1 \le i \le n}}$ denotes the empirical ditribution of $\{E_i\}_{1 \le i \le n}$; $\hat{\mu}^N_{\{E_i\}_{1 \le i \le n}}(P) = N^{-1}\mathrm{Tr}(P(E_i, 1 \le i \le n))$. As a consequence, applying the large deviations result of [BAG97] to the diagonal matrices D_i, we find that there exists $o(1)$ going to zero with ε such that

$$\begin{aligned}
\Lambda_N &\le e^{N^2 \sum_{i=1}^m I(\mu_i) + N^2 o(1)} \sup_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon \\ \|D_i\|_\infty \le R}} \int_{\hat{\mu}^N_{\{U_i D_i U_i^*\}_{1 \le i \le m}} \in \Gamma_R(\mu, \varepsilon, k)} dU_1 \cdots dU_m \\
&:= e^{N^2 \sum_{i=1}^m I(\mu_i) + N^2 o(1)} \Lambda_N^1
\end{aligned}$$

with for k greater than the degree of V,

$$\begin{aligned}
\Lambda_N^1 &= \sup_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon \\ \|D_i\|_\infty \le R}} \int_{\hat{\mu}^N_{\{U_i D_i U_i^*\}_{1 \le i \le m}} \in \Gamma_R(\mu, \varepsilon, k)} e^{N\mathrm{Tr}(V) - N\mathrm{Tr}(V)} dU_1 \cdots dU_m \\
&= e^{-N^2 \mu(V) + N^2 \varepsilon} \sup_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon \\ \|D_i\|_\infty \le R}} \int_{\hat{\mu}^N_{\{U_i D_i U_i^*\}_{1 \le i \le m}} \in \Gamma_R(\mu, \varepsilon, k)} e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m \\
&\le e^{-N^2 \mu(V) + N^2 \varepsilon} \sup_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon \\ \|D_i\|_\infty \le R}} \int e^{N\mathrm{Tr}(V)} dU_1 \cdots dU_m \\
&= e^{-N^2 \mu(V) + N^2 \varepsilon} \sup_{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon, \|D_i\|_\infty \le R} I_N(V, D_i)
\end{aligned}$$

Now, for fixed R, any D_i, D_i' in $d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon, \|D_i\|_\infty \le R$

$$\left| \frac{1}{N^2} \log I_N(V, D_i) - \frac{1}{N^2} \log I_N(V, D_i') \right| \le \eta(\varepsilon, R),$$

with $\eta(\varepsilon, R)$ going to zero as ε goes to zero for any fixed R. Hence,

$$\limsup_{N\to\infty} \frac{1}{N^2} \log I_N(V, D_i) \le F(V, \mu_i) + \eta(\varepsilon, R)$$

with $F(V, \mu_i)$ the limit of $N^{-2} \log I_N(V, A_i)$ given in Theorem 5.7.1 when the distribution of the A_i converges to free variables with marginal distribution μ_i. We thus have proved, letting ε going to zero and then R, k to infinity, that

$$\chi(\mu) \le \sum_{i=1}^m I(\mu_{A_i}) - \mu(V) + F(V, \mu_i).$$

Conversely, we have

$$\Lambda_N \quad \ge \quad e^{N^2 \sum_{i=1}^m I(\mu_i) + N^2 o(\varepsilon)} \Lambda_N^2$$

with

$$\begin{aligned}
\Lambda_N^2 &:= \inf_{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon, \|D_i\|_\infty \le R} \int_{\hat{\mu}^N_{(U_i D_i U_i^*)_{1\le i\le m}} \in \Gamma_R(\mu, \varepsilon, k)} dU_1 \cdots dU_m \\
&= e^{-N^2 \mu(V) + N^2 o(\varepsilon)} \inf_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \varepsilon \\ \|D_i\|_\infty \le R}} \int_{\hat{\mu}^N_{\{U_i D_i U_i^*\}_{1\le i\le m}} \in \Gamma_R(\mu, \varepsilon, k)} e^{N \mathrm{Tr}(V)} dU_1 \cdots dU_m \\
&\ge e^{-N^2 \mu(V) + N^2 o(\varepsilon)} \inf_{\substack{d(\hat{\mu}^N_{D_i}, \mu_i) < \delta \\ \|D_i\|_\infty \le R}} \int_{\hat{\mu}^N_{\{U_i D_i U_i^*\}_{1\le i\le m}} \in \Gamma_R(\mu, \varepsilon, k)} e^{N \mathrm{Tr}(V)} dU_1 \cdots dU_m
\end{aligned}$$

for any $\delta < \varepsilon$. Now, choosing δ and using the continuity of $\hat{\mu}^N_{\{U_i D_i U_i^*\}_{1\le i\le m}}$ in the distribution of the uniformly bounded variables D_i, we find by Corollary 5.3.3 and our hypothesis that

$$\liminf_{N\to\infty} \frac{\int_{\hat{\mu}^N_{U_i D_i U_i^*, 1\le i\le m} \in \Gamma_R(\mu, \varepsilon, k)} e^{N \mathrm{Tr}(V)} dU_1 \cdots dU_m}{\int e^{N \mathrm{Tr}(V)} dU_1 \cdots dU_m} = 1$$

which insures that

$$\chi(\mu) \ge \sum_{i=1}^m I(\mu_i) - \mu(V) + F(V, \mu_i).$$

Thus we have proved that

$$\chi(\mu) = \sum_{i=1}^m I(\mu_i) - \mu(V) + F(V, \mu_i).$$

Note that $\mu(V)$ and $F(V, \mu_i)$ can be written in terms of the $\mu^{\mathbf{k}}$ of Theorem 5.4.3 by Theorem 5.7.1.

□

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Résumé :

Cette thèse étudie les liens entre modèles matriciels et modèles combinatoires. Il est connu depuis plus d'une vingtaine d'années, notamment grâce à des travaux en physique théorique, que formellement, le développement en la dimension de l'énergie libre de modèles proches du Modèle Unitaire Gaussien pouvait s'identifier en tant que série formelle à une énumération selon leur genre de graphes plongés dans des surfaces.

Nous prouvons que les perturbations du Modèle Unitaire Gaussien par un potentiel convexe à petits paramètres convergent vers la série génératrice de graphes dans la sphère. L'objet central qui nous permet de faire ce lien est l'équation de Schwinger-Dyson. Cette équation qui apparaît de manière naturelle dans les modèles matriciels peut être vue comme un type d'équation utilisé par Tutte pour énumérer des objets combinatoires. Nous obtenons ainsi l'extrême régularité des limites de modèles de matrices et une structure supplémentaire aux énumérations combinatoires puisque celles-ci viennent de limite de probabilités sur les matrices.

Ces résultats sont ensuite raffinés. Tout d'abord à l'ordre suivant : la correction à la convergence de l'énergie tend vers une série énumérant des graphes plongés dans le tore. Cette preuve passe notamment par un théorème central limite sur la distribution des matrices.

Enfin, nous prouvons la convergence de tous les termes du développement de l'énergie libre vers des séries génératrices de graphes plongés sur des surfaces de genre de plus en plus grand.

Le second aspect de cette thèse est l'étude de modèle de matrices unitaires aléatoires obtenu en prenant de petites perturbations de la loi de Haar. Un des objectifs est la compréhension de l'intégrale de Itzykson-Zuber pour de petits paramètres. Nous prouvons la convergence au premier ordre d'intégrales unitaires et nous donnons une interprétation combinatoire de la série limite ainsi obtenue.