# EXPANDER GRAPHS, PROPERTY $(\tau)$ AND APPROXIMATE GROUPS

EMMANUEL BREUILLARD

Contents



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# I. Lecture 1: Amenability and random walks

The final aim of these lectures will be to prove spectral gaps for finite groups and to turn certain Cayley graphs into expander graphs. However in order to do so it is useful to have some understanding of the analogous spectral notions of amenability and Kazhdan property $(T)$ which are important for infinite groups. In fact one important aspect of asymptotic group theory (the part of group theory concerned with studying the geometric and group theoretic properties of large finite groups) is the ability transfer results back and forth from the world of infinite groups to that of finite groups and vice-versa.

We begin by reviewing the definition of amenability for a (countable) group and several of its equivalent definitions.

(A). **Amenability, Folner criterion.** In this lecture $\Gamma$ will always denote a countable group.

**Definition I.1.** *We say that* $\Gamma$ *is* amenable *if there exists a sequence of finite subsets* $F_n \subset \Gamma$ *such that for every* $\gamma \in \Gamma$,

$$\frac{|\gamma F_n \Delta F_n|}{|F_n|} \to 0$$

*as* $n$ *tends to infinity.*

The $F_n$'s are called *Folner sets.* They do not need to generate $\Gamma$ (in fact $\Gamma$ is not assumed finitely generated). From this definition it follows easily however that $\Gamma$ is the union of all $F_n F_n^{-1}$ and the $|F_n|$ tends to infinity unless $\Gamma$ is finite.

The following properties can be easily deduced from this definition (exercise):

- $\Gamma$ is amenable if and only if every finitely generated subgroup of $\Gamma$ is amenable,
- $\mathbb{Z}^d$ is amenable,
- if $\Gamma$ has subexponential growth (i.e. $\limsup \frac{1}{n} \log |S^n| = 0$, for some (all) finite symmetric generating set $S$), then there is a sequence of word metric balls $S^{n_k}$ of radius $n_k$ tending to infinity which is a Folner sequence.

Amenability is preserved under group extensions (see Exercise I.11 below), and thus every solvable group is amenable.

(B). **Isoperimetric inequality, edge expansion.** If $\Gamma$ is finitely generated, say by a finite symmetric (i.e. $s \in S \Rightarrow s^{-1} \in S$) set $S$, then we can consider its Cayley graph $\mathcal{G}(G,S)$, which is the graph whose vertices are the elements of $\Gamma$ and edges are defined by putting an edge[1] between $x$ and $y$ if there is $s \in S \setminus \{1\}$ such that $x = ys$.

Given a finite set $F$ of elements in $\Gamma$, we let $\partial_S F$ be the set of group elements $x$ which are not in $F$ but belong $FS$. This corresponds to the boundary of $F$ in the Cayley graph (i.e. points outside $F$ but at distance at most 1 from $F$).

The following is straightforward (exercise):

**Proposition I.2.** *The group $\Gamma$ is non-amenable if and only if its Cayley graph satisfies a linear isoperimetric inequality. This means that there is $\varepsilon > 0$ such that for every finite subset $F$ of $\Gamma$*

$$|\partial_S F| \geqslant \varepsilon |F|.$$

*Exercise:* Show that amenability is preserved under quasi-isometry.
*Exercise:* Show that the non-abelian free groups $F_k$ are non-amenable.

It follows from the last exercise that if a countable group contains a free subgroup, it is non-amenable. The converse is not true. In fact there are finitely generated torsion groups (i.e. every element is of finite order) which are non-amenable. Adyan and Novikov showed that the Burnside groups $B(n,k) := \langle a_1, \dots, a_k | \gamma^n = 1 \ \forall \gamma \rangle$ are infinite, and in fact, as Adyan later proved [1], they are non-amenable for $n$ odd and large enough ($\geqslant 665$).

(C). **Invariant means.** Amenable groups were introduced by John von Neumann in 1929 ([91]). His definition was in terms of invariant means (amenable = admits a mean).

**Definition I.3.** *An* invariant mean *on a countable group $\Gamma$ is a finitely additive probability measure $m$ defined on the set of all subsets of $\Gamma$, which is invariant under the group action by left translations, i.e. $m(\gamma A) = m(A)$ for all $\gamma \in \Gamma$ and $A \subset \Gamma$.*

It is easily checked that an invariant mean is the same thing as a continuous linear functional $m : \ell^\infty(\Gamma) \to \mathbb{R}$ such that

- $f \geqslant 0 \Rightarrow m(f) \geqslant 0$,
- $\forall \gamma \in \Gamma$, $\gamma_* m = m$,
- $m(\mathbf{1}) = 1$.

where $\gamma_* m$ is the push forward of $m$ by the left translation by $\gamma$, and $\mathbf{1}$ is the constant function equal to 1 on $\Gamma$.

Note that if $\Gamma$ has an invariant mean, then it has a bi-invariant mean, that is a mean $m$ as above which is also invariant under right translations: just take the mean of $f(y^{-1}x)$ with respect to $y$, then with respect to $x$.

Folner [50] showed the following:

[1] We allow multiple edges between two distinct points, but no loop at a given vertex.

**Proposition I.4.** *(Folner criterion) A group* $\Gamma$ *is amenable (in the sense of Definition I.1 above) if and only if admits an invariant mean.*

The proof of the existence of the invariant mean from the Folner sequence follows by taking a weak-$\star$ limit in $\ell^\infty(\Gamma)$ of the "approximately invariant" probability measures $\frac{1}{|F_n|}\mathbf{1}_{F_n}$. For the converse, one needs to approximate $m$ in the weak topology by functions in $\ell^1$, then take appropriate level sets of these functions. For the details of this proof and that of the next proposition, we refer the reader to the appendix of the book by Bekka, de la Harpe and Valette [8]. We will also give an alternate argument for the converse in the exercises using Tarski's theorem on paradoxical decompositions.

There is also a related characterization of amenability in terms of actions on compact metric spaces:

**Proposition I.5.** *A group* $\Gamma$ *is amenable if and only if every action of* $\Gamma$ *by homeomorphisms on a compact (metric) space* $X$ *preserves a Borel probability measure.*

*Sketch of proof.* If one averages any probability measure on $X$ by Folner sets and takes a weak limit, one obtains an invariant probability measure. Conversely $\Gamma$ acts on the space of means on $\Gamma$. This is a convex compact space (for the weak-* topology) and if $\Gamma$ preserves a probability measure on it, it must fix its barycenter (which will be an invariant mean).

(D). **Random walks on groups, the spectral radius and Kesten's criterion.** In his 1959 Cornell thesis [76], Kesten studied random walks on Cayley graphs of finitely generated groups and he established yet another characterization of amenability relating it to the rate of decay of the probability of return to the identity, and to the spectrum of the Markov operator associated to the random walk.

Before we state Kesten's theorem, let us first give some background on random walks on groups. This will be useful later on in Lectures 3 and 4 when we discuss the Bourgain-Gamburd method.

Suppose $\Gamma$ is finitely generated and $\mu$ is a finitely supported symmetric (i.e. $\forall \gamma \in \Gamma, \mu(\gamma) = \mu(\gamma^{-1})$) probability measure on $\Gamma$ whose support generates $\Gamma$.

We can associate to $\mu$ an operator $P_\mu$ on $\ell^2(\Gamma)$, *the Markov operator*, by setting for $f \in \ell^2(\Gamma)$

$$P_\mu f(x) = \sum_{\gamma \in \Gamma} f(\gamma^{-1}x)\mu(\gamma).$$

Clearly $P_\mu$ is self-adjoint (because $\mu$ is assumed symmetric) and moreover

$$P_\mu \circ P_\nu = P_{\mu * \nu},$$

for any two probability measures $\mu$ and $\nu$ on $\Gamma$, where $\mu * \nu$ denotes the convolution of the two measures, that is the new probability measure defined by

$$\mu * \nu(x) := \sum_{\gamma \in \Gamma} \mu(x\gamma^{-1})\nu(\gamma).$$

The convolution is the image of the product $\mu \otimes \nu$ under the product map $\Gamma \times \Gamma \to \Gamma$, $(x, y) \mapsto xy$ and is the probability distribution of the product random variable $XY$, if $X$ is a random variable taking values in $\Gamma$ with distribution $\mu$ and $Y$ is a random variable with distribution $\nu$ independent of $X$.

The probability measure $\mu$ induces a *random walk* on $\Gamma$, i.e. a stochastic process $(S_n)_{n \geqslant 1}$ defined as

$$S_n = X_1 \cdot \ldots \cdot X_n,$$

where the $X_i$'s are independent random variables with the same probability distribution $\mu$ on $\Gamma$. The process $(S_n)_{n \geqslant 1}$ is a Markov chain and $p_{x \to y} := \mu(x^{-1}y)$ are the transition probabilities. This means that the probability that $S_{n+1} = y$ given that $S_n = x$ is $p_{x \to y}$, independently of $n \geqslant 1$.

When $\mu$ is the probability measure

$$\mu = \mu_S := \frac{1}{|S|} \sum_{s \in S} \delta_s,$$

where $\delta_s$ is the Dirac mass at $s \in S$ and $S$ is a finite symmetric generating set for $\Gamma$, we say that $\mu$ and its associated process $(S_n)_{n \geqslant 1}$ is the *simple random walk* on $(\Gamma, S)$. It corresponds to the nearest neighbor random walk on the Cayley graph $\mathcal{G}(\Gamma, S)$, where we jump at each stage from one vertex to a neighboring vertex with equal probability.

Kesten was the first to understand that studying the probability that the random walk returns to the identity at time $n$ could be useful to classify infinite groups[2]. This quantity is

$$Proba(S_n = 1) = \mu^n(1),$$

where we have denoted the $n$-th convolution product of $\mu$ with itself by $\mu^n := \mu * \ldots * \mu$.

We will denote the identity element in $\Gamma$ sometimes by 1 sometimes by $e$.

**Proposition I.6.** *Here are some basic properties of $\mu^n$.*

- $\mu^{2n}(1)$ *is non-increasing,*
- $\mu^{2n}(x) \leqslant \mu^{2n}(1)$ *for all $x \in \Gamma$.*

[2]His secret goal was to use his criterion to establish that the Burnside groups are infinite by showing that they are non-amenable, see the comments at end of [76].

Note that $\mu^{2n+1}(1)$ can be zero sometimes (e.g. the simple random walk on the free group), but $\mu^{2n}(1)$ is always positive.

The Markov operator $P_\mu$ is clearly a contraction in $\ell^2$ (and in fact in all $\ell^p$, $p \geqslant 1$), namely $||P_\mu|| \leqslant 1$. A basic tool in the theory of random walks on groups is the *spectral theorem* for self-adjoint operators applied to $P_\mu$. This will yield Kesten's theorem and more.

*Proof of Proposition I.6.* Let $\delta_x$ be the Dirac mass at $x$. Observe that $P_{\mu^n} = P_\mu^n$ and that $\mu^n(x) = P_\mu^n \delta_e(x) = \langle P_\mu^n \delta_e, \delta_x \rangle$ (in $\ell^2(\Gamma)$ scalar product). Denoting $P_\mu$ by $P$ for simplicity it follows that

$$\mu^{2(n+1)}(1) = \langle P^n \delta_e, P^{n+2}\delta_e \rangle \leqslant ||P^n \delta_e|| \cdot ||P^2 P^n \delta_e|| \leqslant ||P^n \delta_e||^2 = \mu^{2n}(1).$$

and that

$$\mu^{2n}(x) = \langle P^{2n}\delta_e, \delta_x \rangle \leqslant ||P^n \delta_e|| \cdot ||P^n \delta_x|| = \mu^{2n}(1),$$

where the last equality follows from the fact that $P^n \delta_x(y) = P^n \delta_e(yx^{-1})$. □

**Proposition-Definition I.7.** *(Spectral radius of the random walk) The* spectral radius *$\rho(\mu)$ of the Markov operator $P_\mu$ acting on $\ell^2(\Gamma)$ is called the spectral radius of the random walk.*

Note that since $P_\mu$ is self-adjoint, its spectral radius coincides with its operator norm $||P_\mu||$, and with $\max\{|t|, t \in spec(P_\mu)\}$.

Let us apply the spectral theorem for self-adjoint operators to $P_\mu$. This gives a resolution of identity $E(dt)$ (measure taking values into self-adjoint projections), and a probability measure $\eta(dt) := \langle E(dt)\delta_e, \delta_e \rangle$ on the interval $[-1, 1]$ such that for all $n \geqslant 1$,

$$\langle P^n \delta_e, \delta_e \rangle = \int_{[-1,1]} t^n \eta(dt). \tag{1}$$

**Definition I.8.** *(Spectral measure) The* spectral measure *of the random walk is the measure $\eta$ associated to the Markov operator $P_\mu$ by the spectral theorem as above.*

*Exercise:* Show that $\langle E(dt)\delta_x, \delta_x \rangle = \eta(dt)$ for all $x \in \Gamma$ and that the other spectral measures $\langle E(dt)f, g \rangle$, with $f, g \in \ell^2(\Gamma)$ are all absolutely continuous w.r.t $\eta$.

We can now state:

**Theorem I.9.** *(Kesten) Let $\Gamma$ be a finitely generated group and $\mu$ a symmetric probability measure with finite support generating $\Gamma$.*

- *$\forall n \geqslant 1$, $\mu^{2n}(1) \leqslant \rho(\mu)^{2n}$ and $\lim_{n \to +\infty} (\mu^{2n}(1))^{\frac{1}{2n}} = \rho(\mu)$,*
- *(Kesten's criterion) $\rho(\mu) = 1$ if and only if $\Gamma$ is amenable.*

*Proof of the first item.* The existence of the limit and the upper bound follows from the subadditive lemma (i.e. if a sequence $a_n \in \mathbb{R}$ satisfies $a_{n+m} \leqslant a_n + a_m$, for all $n, m \in \mathbb{N}$, then $\frac{a_n}{n}$ converges to $\inf_{n \geqslant 1} \frac{a_n}{n}$). Indeed $\mu^{2(n+m)}(1) \geqslant \mu^{2n}(1)\mu^{2m}(1)$ (the chance to come back at 1 at time $2n+2m$ is at least the chance to come back at time $2n$ and to come back again at time $2n+2m$). Take logs.

In order to identify the limit as the spectral radius, we apply the spectral theorem (see equation (1) above) to $P_\mu$, so that $\mu^{2n}(1)^{\frac{1}{2n}} = \langle P^{2n}\delta_e, \delta_e\rangle^{\frac{1}{2n}}$ takes the form

$$\Big(\int_{[-1,1]} t^{2n}\eta(dt)\Big)^{\frac{1}{2n}}.$$

However when $n \to +\infty$, this tends to $\max\{|t|, t \in spec(P_\mu)\} = \rho(\mu)$. □

Below we sketch a proof of Kesten's criterion via an analytic characterization of amenability in terms of Sobolev inequalities. The following proposition subsumes Kesten's criterion.

**Proposition I.10.** *Let $\Gamma$ be a group generated by a finite symmetric set $S$ and let $\mu$ a symmetric probability measure whose support generates $\Gamma$. The following are equivalent:*

1. *$\Gamma$ is non-amenable,*
2. *there is $C = C(S) > 0$ such that $||f||_2 \leqslant C||\nabla f||_2$ for every $f \in \ell^2(\Gamma)$,*
3. *there is $\varepsilon = \varepsilon(S) > 0$ such that $\max_{s\in S} ||s \cdot f - f||_2 \geqslant \varepsilon||f||_2$ for all $f \in \ell^2(\Gamma)$,*
4. *$\rho(\mu) < 1$.*

Here $\nabla f$ is the function on the set of edges of the Cayley graph of $\Gamma$ associated with $S$ given by

$$\nabla f(e) = |f(e^+) - f(e^-)|,$$

where $e^+$ and $e^-$ are the end-points of the edge $e$.

*Proof.* Note that condition (3) does not depend on the generating set (only the constant $\varepsilon$ may change). For the equivalence between (3) and (4) observe further that a finite collection of unit vectors in a Hilbert space average to a vector of norm strictly less than 1 if and only if the angle between at least two of them is bounded away from zero (and the bounds depend only on the number of vectors).

The equivalence between (2) and (3) is clear because $||\nabla f||_2$ is comparable (up to multiplicative constants depending on the size of $S$ only) to $\max_{s\in S} ||s \cdot f - f||_2$.

Condition (2) easily implies (1), because the linear isoperimetric inequality $|\partial_S F| \geqslant \varepsilon|F|$ is immediately derived from (2) by taking $f = \mathbf{1}_F$ the indicator function of $F$.

The only less obvious implication is (1) ⇒ (2) as we need to go from sets to arbitrary functions. The idea to do this is to express $f$ as a sum of indicator functions of sublevel sets. Namely, for $t \geqslant 0$, let $A_t = \{\gamma \in \Gamma; f(\gamma) > t\}$. Then for $x \in \Gamma$

$$f(x) = \int_0^{+\infty} \mathbf{1}_{t<f(x)} dt = \int_0^{+\infty} \mathbf{1}_{A_t}(x) dt$$

and for an edge $e$ of $\mathcal{G}(\Gamma, S)$

$$|f(e^+) - f(e^-)| = \int_0^{+\infty} \mathbf{1}_{f(e^-)<t<f(e^+)} dt = \int_0^{+\infty} \mathbf{1}_{\partial A_t}(e) dt$$

where $\partial A$ (for any subset $A \subset \Gamma$) is the set of edges connecting a point in $A$ to a point outside $A$.

Summing over all vertices $x$ and all edges $e$ we obtain the *co-area formulae*:

$$\int_0^{+\infty} |A_t| dt = ||f||_1$$

and

$$\int_0^{+\infty} |\partial A_t| dt = ||\nabla f||_1$$

If $\Gamma$ is non-amenable, it satisfies the linear isoperimetric inequality, namely there is $\varepsilon > 0$ such that $|\partial A| \geqslant \varepsilon |A|$ for every finite subset $A$ of $\Gamma$. Applying this to $A_t$ and using the co-area formulae, we conclude:

$$||\nabla f||_1 \geqslant \varepsilon ||f||_1$$

for every $\ell^1$ function on $\Gamma$. To get the $\ell^2$ estimate, simply note that $||f||_2^2 = ||f^2||_1$ and

$$||\nabla f^2||_1 \leqslant 2||\nabla f||_2 ||f||_2$$

for every $f \in \ell^2$ as one can see by applying the Cauchy-Schwartz inequality (combined with $|x^2 - y^2| = |x - y||x + y|$ and $|x + y|^2 \leqslant 2(x^2 + y^2)$). □

*Remark.* Kesten's theorem and the above results extend without much difficulty to arbitrary connected graphs of bounded degree showing that positive edge expansion is equivalent to exponential decay of transition probabilities for the simple random walk and equivalent to the spectrum of the Markov operator being contained in $(-1, 1)$. See for example [38]. Analogues exist for Riemannian manifolds (Cheeger-Buser inequality, Brooks' theorem) and finite graphs as we will see below.

**Exercise I.11.** *If* $1 \to \Gamma_1 \to \Gamma_2 \to \Gamma_3 \to 1$ *is an exact sequence, then* $\Gamma_2$ *is amenable if and only if* $\Gamma_1$ *and* $\Gamma_3$ *are amenable [hint: one can use property* (3) *as a working definition for amenability]. Conclude that solvable groups are amenable.*

(E). **Further facts and questions about growth of groups and random walks.** Varopoulos uncovered a relation between the growth of a finitely generated group (how fast $|S^n|$ grows for a given generating set $S$) and the rate of decay of the probability of return to the identity of its symmetric random walks. For example he showed:

**Theorem I.12** (Varopoulos, see [99])**.** *Let* $V_S(n) = |S^n|$ *be the growth function of a group generated by a finite symmetric set* $S$. *Let* $\mu$ *be the symmetric probability measure on* $\Gamma$ *with finite support generating* $\Gamma$.

- *If* $V_S(n) \gg n^d$*, then* $\mu^{2n}(e) \ll \frac{1}{n^{d/2}}$*,*
- *If* $V_S(n) \gg \exp(n^\alpha)$*, then* $\mu^{2n}(e) \ll \exp(n^{-\frac{\alpha}{\alpha+2}})$*.*

[here we say $f \ll g$ if there is $c_1, c_2 > 0$ s.t. $\forall n, f(n) \leqslant c_1 g(c_2 n)$].

The case of polynomial growth is pretty well-understood. By Gromov's theorem [58], groups of polynomial growth are virtually nilpotent. In this case Varopoulos showed that $\frac{c_1}{n^{d/2}} \leqslant \mu^{2n}(e) \leqslant \frac{c_2}{n^{d/2}}$ for some $c_1, c_2 > 0$ and in fact much more is true (namely $n^{d/2}\mu^{2n}(e)$ converges to a non-zero constant and there are gaussian estimates for $\mu^{2n}(x)$ depending on $d(e,x)$, see the work of Alexopoulos [2] and Hebisch-Saloff-Coste [65]).

Note that this implies in particular that every group of exponential growth must have a decay of the probability of return at least in $\exp(n^{-\frac{1}{3}})$. This rate is achieved by polycyclic groups (that is solvable discrete subgroups of $\mathrm{GL}_d(\mathbb{C})$) as was shown by Alexopoulos [3].

The theorem is a special case of a more general result proved by Varopoulos which says that if $u(t)$ is the solution to the ODE

$$u' + \frac{u}{\psi(u)^2} = 0,$$

where $\psi(u) := \inf\{n, V_S(n) > 1/u\}$ for $u \in (0,1)$, then

$$\mu^{2n}(e) \ll u(n)$$

Varopoulos's proof is a refinement of Kesten's argument used in the proof of Proposition I.10 above, in which the Sobolev inequality is weakened so as to make the constant depend on the size of the support of $f$ (Nash inequality). We refer the reader to the survey [99] and the book [119] for the details of this argument.

The possible growth behaviors of finitely generated groups are still quite mysterious. For example I think it is an open question to determine whether every real number $\geqslant 0$ can arise as the exponential growth rate $\lim \frac{1}{n}\log|S^n|$ of a finitely generated group. A consequence of the uniform Tits alternative (see Theorem II.15 below) is that the exponential growth rate of non-virtually solvable linear groups (linear = subgroup of $\mathrm{GL}_d$ over some field) is bounded away from 0 by a positive constant depending only on $d$ and not on the field.

The situation for groups of intermediate growth is also very interesting. Grigorchuk ([54, 55]) proved in the early 1980's that there exist finitely generated groups whose growth function is not exponential, yet not polynomial either. We refer the reader to the nice recent exposition [56] for the description of Grigorchuk's examples.

Very recently Bartholdi and Erschler [6] showed that for every $\alpha \in [0.77, 1]$ there exists a finitely generated group $\Gamma = \langle S \rangle$ and constants $c_1, c_2 > 0$ such that for all large $n \geqslant 1$,

$$\exp(c_1 n^\alpha) \leqslant V_S(n) \leqslant \exp(c_2 n^\alpha)$$

Their construction builds on Grigorchuk's own constructions using certain permutational wreath products.

Grigorchuk conjectures [57] the following *Gap Conjecture*: if given any finitely generated group either $\Gamma$ is virtually nilpotent ($\Leftrightarrow$ of polynomial growth by Gromov's theorem [58]) or there is $c > 0$ such that $V_S(n) \geqslant \exp(c\sqrt{n})$ for all large $n$. It is known when $\Gamma$ is residually nilpotent (i.e. $\forall \gamma \in \Gamma \setminus \{1\}$ there is a nilpotent quotient in which the image of $\gamma$ is non-trivial), and recently Grigorchuk reduced the conjecture to two cases: residually finite groups and simple groups.

It is also still an open question to determine the possible rates of decay of $\mu^{2n}(e)$ for an arbitrary group. For example can all rates of the form $\exp(-n^\alpha)$ for $\alpha \in [\frac{1}{5}, 1]$ be achieved ? (Pittet and Saloff-Coste showed that the values $\alpha = \frac{d}{d+2}$, $d \in \mathbb{N}$, are achieved by wreath products $\mathbb{Z}^d \wr F$ for a finite group $F$). See [98] and [48, Theorem 2] for more on this topic.

(F). **Exercise: Paradoxical decompositions, Ponzi schemes and Tarski numbers.** We conclude this lecture by proving, in the form of a exercise, yet another characterization of amenability, which is due to Tarski [114] following an argument from [38].

**Theorem I.13** (Tarksi). *A group is non-amenable if and only if it is paradoxical.*

Let us define "paradoxical". Let $\Gamma$ be a group acting on a set $X$. This $\Gamma$-action is said to be *$N$-paradoxical* if one can partition $X$ into $n + m \leqslant N$ disjoint pieces

$$X = A_1 \cup \ldots \cup A_n \cup B_1 \cup \ldots \cup B_m$$

in such a way that there are elements $a_1, \ldots, a_n \in \Gamma$ and $b_1, \ldots, b_m \in \Gamma$ such that we get new partitions of $X$ into disjoint pieces

$$X = \bigcup_{i=1}^{n} a_i A_i \text{ and } \bigcup_{j=1}^{m} b_j B_j$$

We say that $\Gamma$ is paradoxical if it is $N$-paradoxical for some finite $N \in \mathbb{N}$ for the action of $\Gamma$ on itself by left translations.

**1)** Prove that the non-abelian free group $F_2$ and in fact any group $\Gamma$ containing the free group $F_2$ is 4-paradoxical.

**2)** Suppose that $\Gamma$ is a 4-paradoxical group and $\Gamma = A_1 \cup A_2 \cup B_1 \cup B_2$ is a paradoxical decomposition as defined above. Show that $\Gamma$ *plays ping-pong* on itself, where the ping-pong players are $a := a_1^{-1} a_2$ and $b := b_1^{-1} b_2$. Deduce that $\Gamma$ contains a non-abelian free subgroup $F_2$.

**3)** Define the *Tarski number* $\mathcal{T}(\Gamma)$ of a group $\Gamma$ to be the smallest integer $N$ if it exists such that $\Gamma$ is $N$-paradoxical. By the above $\mathcal{T}(\Gamma) = 4$ if and only if $\Gamma$ contains $F_2$. Show that if $\Gamma$ is amenable, then $\mathcal{T}(\Gamma) = +\infty$.

**4)** Suppose that $\Gamma$ is finitely generated with symmetric generating set $S$ and is endowed with the corresponding word metric $d$ (i.e. $d(x,y) := \inf\{n \in \mathbb{N}, x^{-1}y \in S^n\}$). Given $k \in \mathbb{N}$, let $\mathcal{G}_k$ be the bi-partite graph obtained by taking two copies $\Gamma_1$ and $\Gamma_2$ of $\Gamma$ as the left and right vertices respectively and by placing an edge between $\gamma \in \Gamma_1$ and $\gamma' \in \Gamma_2$ if and only if $d(\gamma, \gamma') \leqslant k$ in the word metric of $\Gamma$. Verify that if there is some finite $k \in \mathbb{N}$ such that $\mathcal{G}_k$ admits a $(2,1)$ perfect matching[3] if and only if there exists a surjective 2-to-1 mapping $\phi : \Gamma \to \Gamma$ with the property that $\sup_{\gamma \in \Gamma} d(\gamma, \phi(\gamma)) < +\infty$ (we call such $\phi$ a "Ponzi scheme").

**5)** Show that the property of 4) takes place if and only if $\Gamma$ is paradoxical.

**6)** Prove the following version of *Hall's marriage lemma* for infinite bi-partite graphs. Let $k$ be a positive integer (we will need the result for $k = 2$ only). Suppose $\mathcal{B}$ is a bi-partite graph whose set of left vertices is countable infinite as is the set of right vertices. Suppose that for every finite subset of left vertices $L$, the number of right vertices connected to some vertex in $L$ has size at least $k|L|$, while for every finite subset $R$ of right vertices, the number of left vertices connected to some vertex in $R$ has size at least $|R|$. Show that $\mathcal{B}$ admits a $(k,1)$ perfect matching. [Hint: first treat the case $k = 1$, then reduce to this case.]

**7)** Using 6) that if $\Gamma$ is a non-amenable finite generated group, then there is $k \geqslant 1$ such that $\mathcal{G}_k$ has a $(2,1)$ perfect matching.

**8)** Conclude the proof of Tarski's theorem for arbitrary (not necessarily finitely generated) groups.

*Remark.* There are finitely generated groups with finite Tarski number $> 4$. For example the large Burnside groups with odd exponent. See [38] for some more examples.

[3] By definition this is a subset of edges of $\mathcal{G}_k$ such that the induced bi-partite graph has the property that every vertex on the left hand side is connected to exactly two vertices on the right hand side, while every vertex on the right hand side is connected to exactly one vertex on the left hand side.

## II. Lecture 2: The Tits alternative and Kazhdan's property $(T)$

(A). **The Tits alternative.** *Linear groups* over a field $K$, namely subgroups of $\mathrm{GL}_d(K)$, form a very interesting large class of groups. While there are few general tools to study arbitrary finitely generated groups (often one has to resort to combinatorics and analysis as we did in Lecture 1 for example), the situation is very different for linear groups as a wide range of techniques (including algebraic number theory and algebraic geometry) becomes available. Hence proving that a group of geometric origin is linear can have a big pay off (e.g. braid groups, or the more recent example of small simplification groups, cf. Sageev's lectures in this volume).

Jacques Tits determined in 1972 which linear groups are amenable as a consequence of his famous alternative:

**Theorem II.1.** *(Tits alternative* [115]*) Let $\Gamma$ be a finitely generated linear group (over some field $K$). Then*

- *either $\Gamma$ is virtually solvable (i.e. has a solvable finite index subgroup),*
- *or $\Gamma$ contains a non-abelian free subgroup $F_2$.*

*Remark.* Virtually solvable subgroups of $\mathrm{GL}_d(K)$ have a subgroup of finite index which can be triangularized over the algebraic closure (Lie-Kolchin theorem).

In particular,

**Corollary II.2.** *A finitely generated linear group is amenable if and only if it is virtually solvable.*

Indeed free subgroups are non-amenable and subgroups of amenable groups are amenable.

The proof of the Tits alternative uses a technique called *"ping-pong"* used to find generators of a non-abelian free subgroup in a given group. The basic idea is to exhibit a certain geometric action of the group $\Gamma$ on a space $X$ and two elements $a, b \in \Gamma$, the *"ping-pong players"* whose action on $X$ have the following particular behavior:

**Lemma II.3.** *(Ping-pong lemma) Suppose a group $\Gamma$ acts on a set $X$ and there are two elements $a, b \in \Gamma$ and 4 disjoint (non-empty) subsets $A^+$, $A^-$, $B^+$, and $B^-$ of $X$ such that*

- *$a$ maps $Y \setminus A^-$ into $A^+$,*
- *$a^{-1}$ maps $Y \setminus A^+$ into $A^-$,*
- *$b$ maps $Y \setminus B^-$ into $B^+$, and*
- *$b^{-1}$ maps $Y \setminus B^+$, into $B^-$.*

*where $Y := A^+ \cup A^- \cup B^+ \cup B^-$. Then $a$ and $b$ are free generators of a free subgroup $\langle a, b\rangle \simeq F_2$ in $\Gamma$.*

*Proof.* The subset $A^+$ is called the attracting set for $a$ and $A^-$ the repelling set, and similarly for the other letters. Pick a reduced word $w$ in $a$ and $b$ and their inverses. Say it starts with $a$. Pick a point $p$ not in $A^+$ and not in the repelling set of the last letter of $w$ (note that there is still room to choose such a $p$) Then the above ping-pong rules show that $w \cdot p$ belongs to $A^+$ hence is not equal to $p$. In particular $w$ acts non trivially on $X$ and hence is non trivial in $\Gamma$. □

*Remark.* There are other variants of the ping-pong lemma (e.g. it is enough that there are disjoint non-empty subsets $A$ and $B$ such that any (positive or negative) power of $a$ sends $B$ inside $A$ and any power of $b$ sends $A$ inside $B$ (e.g. take $A := A^+ \cup A^-$ and $B := B^+ \cup B^-$ above). But the above is the most commonly used in practice.

*On Tits's proof.* Tits's proof uses algebraic number theory and representation theory of linear algebraic groups to construct a local field $K$, namely $\mathbb{R}$, $\mathbb{C}$ or a finite extension of $\mathbb{Q}_p$ or $\mathbb{F}_p((t))$, and an irreducible linear representation of $\Gamma$ in $GL_m(K)$ whose image is unbounded. If $\Gamma$ is not virtually solvable, one can take $m \geqslant 2$. Then he shows that one can change the representation (passing to an exterior power) and exhibit an element $\gamma$ of $\Gamma$ which is semisimple (i.e. diagonalizable in a field extension) and has the property that both $\gamma$ and $\gamma^{-1}$ have a unique eigenvalue (counting multiplicity) of maximal modulus (such elements are called proximal elements). Then one considers the action of $\Gamma$ on the projective space of the representation $X := \mathbb{P}(K^m)$ and observes that the powers $\gamma^n$, $n \in \mathbb{Z}$, have the following contracting behavior on $X$. If we decompose $K^m = Kv^+ \oplus H_\gamma$ into the direct sum of the eigenline $Kv^+$ of maximal modulus of $\gamma$ and the complementary $\gamma$-invariant subspace $H_\gamma$, we see that the positive powers $\gamma^n$, $n \geqslant 1$ push any compact subset of $\mathbb{P}(K^m)$ which is disjoint from $\mathbb{P}(H_\gamma)$ inside a small neighborhood around the point $\mathbb{P}(Kv^+)$, if $n$ is large enough. Using the irreducibility of the action, one then finds a conjugate $c\gamma c^{-1}$ of $\gamma$ such that $a := \gamma^n$ and $b = c\gamma^n c^{-1}$ exhibit the desired "ping-pong" behavior for all large enough $n$ and thus generate a free subgroup. For details, see the original article [115] or e.g. [63] and [15].

It turns out that one can give a shorter proof of the corollary, which by-passes the proof of the existence of a free subgroup. This was observed by Shalom [109] and the argument, which unlike the proof of the Tits alternative does not require the theory of algebraic groups, is as follows.

*Sketch of a direct proof of Corollary II.2.* Let us first assume that $\Gamma$ is an unbounded subgroup of $\mathrm{GL}_n(k)$, for some local field $k$, which acts strongly irreducibly on $k^n$ (i.e. it does not preserve any finite union of proper linear subspaces). If $\Gamma$ is amenable, then it must preserve a probability measure on $\mathbb{P}(k^n)$ (by Proposition I.5). However recall:

**Lemma II.4.** *(Furstenberg's Lemma) Suppose $\mu$ is a probability measure on the projective space $\mathbb{P}(k^n)$. Then the stabilizer of $\mu$ in $\mathrm{PGL}_n(k)$ is compact unless $\mu$ is degenerate in the sense that it is supported on a finite number of proper (projective) linear subspaces.*

For the proof of this lemma, see Zimmer's book [120], Furstenberg's beautifully written original note [51], or just try to prove it yourself. Clearly the stabilizer of a degenerate measure preserves a finite union of proper subspaces. This contradicts our assumption.

To complete the proof, it remains to see that if $\Gamma$ is not virtually solvable, then we can always reduce to the case above. This was proved by Tits at the start of his proof of Theorem II.1. It follows from two claims.

*Claim 1.* A linear group is not virtually solvable if and only if it has a finite index subgroup which has a linear representation in a vector space of dimension at least 2 which is absolutely strongly irreducible (i.e. it preserves no finite union of proper vector subspaces defined over any field extension).

*Claim 2.* If a finitely generated subgroup $\Gamma$ of $\mathrm{GL}_d(K)$ acts absolutely strongly irreducibly on $K^d$, $d \geqslant 2$, and $K$ is a finitely generated field, then $K$ embeds in a local field $k$ in such a way that $\Gamma$ is unbounded in $\mathrm{GL}_d(k)$. □

*Exercise.* Prove Claim 1.

The proof of Claim 2 requires some basic algebra and number theory and proceeds as follows.

*Exercise.* Prove that if a subgroup of $\mathrm{GL}_d(\overline{K})$ acts irreducibly ($\overline{K}$=algebraic closure) and all of its elements have only 1 in their spectrum (i.e. are unipotents), then $d = 1$ (hint: use Burnside's theorem that the only $\overline{K}$-subalgebra of $M_d(\overline{K})$ acting irreducibly on $\overline{K}^d$ is all of $M_d(\overline{K})$.)

*Exercise.* Show that a finitely generated field $K$ contains only finitely many roots of unity and that if $x \in K$ is not a root of unity, they there is a local field $k$ with absolute value $|\cdot|$ such that $K$ embeds in $k$ and $|x| \neq 1$ (hint: this is based on Kronecker's theorem that if a polynomial in $\mathbb{Z}[X]$ has all its roots within the unit disc, then all its roots are roots of unity; see [115, Lemma 4.1] for a full proof).

*Exercise.* Use the last two exercises to prove Claim 2.

(B). **Kazhdan's property** $(T)$**.** Let us go back to general (countable) groups and introduce another spectral property of groups, namely Kazhdan's property $(T)$. Our goal here is to give a very brief introduction to the notion first introduced by Kazhdan in [75]. Many excellent references exist on property $(T)$ starting with the 1989 Astérisque monograph by de la Harpe and Valette [64], the recent book by Bekka, de la Harpe and Valette [8] for the classical theory; see also Shalom 2006 ICM talk [111] for more recent developments.

Let $\pi$ be a unitary representation of $\Gamma$ on a Hilbert space $\mathcal{H}_\pi$. We say that $\pi$ admits (a sequence of) *almost invariant vectors* if there is a sequence of unit vectors $v_n \in \mathcal{H}_\pi$ ($||v_n|| = 1$) such that $||\pi(\gamma)v_n - v_n||$ converges to 0 as $n$ tends to $+\infty$ for every $\gamma \in \Gamma$.

**Definition II.5.** *(Kazhdan's property $(T)$) A group $\Gamma$ is said to have* Kazhdan's property $(T)$ *if every unitary representation $\pi$ admitting a sequence of almost invariant vectors admits a non-zero $\Gamma$-invariant vector.*

Groups with property $(T)$ are sometimes also called *Kazhdan groups*.

A few simple remarks are in order following this definition:

- The definition resembles that of non-amenability, except that we are now considering all unitary representations of $\Gamma$ and not just the left regular representation $\ell^2(\Gamma)$ (given by $\lambda(\gamma)f(x) := f(\gamma^{-1}x)$). Indeed Proposition I.10(3) above shows that a group is amenable if and only if the regular representation on $\ell^2(\Gamma)$ admits a sequence of almost invariant vectors.
- Property $(T)$ is inherited by quotient groups of $\Gamma$ (obvious from the definition).
- Finite groups have property $(T)$ (simply average an almost invariant unit vector over the group).
- If $\Gamma$ has property $(T)$ and is amenable, then $\Gamma$ is finite (indeed $\ell^2(\Gamma)$ has a non-zero invariant vector iff the constant function 1 is in $\ell^2(\Gamma)$ and this is iff $\Gamma$ is finite).

A first important consequence[4] of property $(T)$ is the following:

**Proposition II.6.** *Every countable group with property $(T)$ is finitely generated.*

*Proof.* Let $S_n$ be an increasing family of finite subsets of $\Gamma$ such that $\Gamma = \bigcup_n S_n$. Let $\Gamma_n := \langle S_n \rangle$ be the subgroup generated by $S_n$. We wish to show that $\Gamma_n = \Gamma$ for all large enough $n$. Consider the left action of $\Gamma$ on the coset space $\Gamma/\Gamma_n$ and the unitary representation $\pi_n$ it induces on $\ell^2$ functions on that coset space, $\ell^2(\Gamma/\Gamma_n)$. Let $\pi = \oplus_n \pi_n$ be the Hilbert direct sum of the $\ell^2(\Gamma/\Gamma_n)$'s with the natural action of $\Gamma$ on each factor. We claim that this unitary representation of $\Gamma$ admits a sequence of almost invariant vectors. Indeed let $v_n$ be the Dirac mass at $[\Gamma_n]$ in the coset space $\Gamma/\Gamma_n$. We view $v_n$ as a (unit) vector in $\pi$. Clearly for every given $\gamma \in \Gamma$, if $n$ is large enough $\gamma$ belongs to $\Gamma_n$ and hence preserves $v_n$. Hence $||\pi(\gamma)v_n - v_n||$ is equal to 0 for all large enough $n$ and the $(v_n)_n$ form a family of almost invariant vectors. By Property $(T)$, there is a non-zero invariant vector $\xi := \sum_n \xi_n$. The $\Gamma$-invariance of $\xi$ is equivalent to the $\Gamma$-invariance of all $\xi_n \in \ell^2(\Gamma/\Gamma_n)$ simultaneously. However observe that if $\xi_n \neq 0$, then $\Gamma/\Gamma_n$ must be finite (otherwise a non-zero constant function cannot be in $\ell^2$). Since there must be some $n$ such that $\xi_n \neq 0$, we conclude that some $\Gamma_n$ has finite index in $\Gamma$. But $\Gamma_n$ itself is finitely generated. It follows that $\Gamma$ is finitely generated. □

So let $\Gamma$ have property $(T)$, and let $S$ be a finite generating set for $\Gamma$. Then from the very definition we observe that there must be some $\varepsilon = \varepsilon(S) > 0$ such that for every unitary representation $\pi$ of $\Gamma$ without non-zero $\Gamma$-invariant vectors, one has:

$$\max_{s \in S} ||\pi(s)v - v|| \geqslant \varepsilon ||v||,$$

for every vector $v \in \mathcal{H}_\pi$.

And conversely it is clear that if there is a finite subset $S$ in $\Gamma$ with the above property, then every unitary representation of $\Gamma$ with almost invariant vectors has an invariant vector. Hence this is equivalent to Property $(T)$.

[4]This was partly the motivation for the introduction of property $(T)$ by Kazhdan in 1967 (at age 21). He used it to prove that non-uniform lattices in (higher rank) semisimple Lie groups are finitely generated. Nowadays new proofs exist of this fact, which are purely geometric and give good bounds on the size of the generating sets, see Gelander's lecture notes in this volume.

**Definition II.7.** *(Kazhdan constant) The (optimal) number $\varepsilon(S) > 0$ above is called a Kazhdan constant for the finite set $S$.*

Another important property of Kazhdan groups is that they have finite abelianization:

**Proposition II.8.** *Suppose $\Gamma$ is a countable group with property $(T)$. Then $\Gamma/[\Gamma,\Gamma]$ is finite.*

*Proof.* Indeed, $\Gamma/[\Gamma,\Gamma]$ is abelian hence amenable. It also has property $(T)$, being a quotient of a group with property $(T)$. Hence it is finite (see itemized remark above). $\square$

This implies in particular that the non-abelian free groups do not have property $(T)$ although they are non-amenable. In fact Property $(T)$ is a rather strong spectral property a group might have. I tend to think of it as a rather rare and special property a group might have (although in some models of random groups, almost every group has property $(T)$).

*Exercise.* Show that if $\Gamma$ has a finite index subgroup with property $(T)$, then it has property $(T)$. And conversely, if $\Gamma$ has property $(T)$, then every finite index subgroup also has property $(T)$ (hint: induce the representation from the finite index subgroup to $\Gamma$).

In fact establishing Property $(T)$ for any particular group is never a simple task. In his seminal paper in which he introduced Property $(T)$ Kazhdan proved that Property $(T)$ for simple Lie groups of rank[5] at least 2. Then he deduced (as in the above exercise) that Property $(T)$ is inherited by all discrete subgroups of finite co-volume in the Lie group $G$ (i.e. lattices).

**Theorem II.9.** *(Kazhdan 1967,* [75]*) A lattice in a simple real Lie group of real rank at least 2 has property $(T)$.*

There are several proofs of Kazhdan's result for Lie groups (see e.g. Zimmer's book [120] and Bekka-delaHarpe-Valette [8] for two slightly different proofs). They rely of proving a "relative property $(T)$" for the pair $(\mathrm{SL}_2(\mathbb{R}) \ltimes \mathbb{R}^2, \mathbb{R}^2)$. This relative property $(T)$ means that every unitary representation of the larger group with almost invariant vectors admits a non-zero vector which is invariant under the smaller group. One proof of this relative property makes use of Furstenberg's lemma above (Lemma II.4). The proof extends to simple groups defined over a local field with rank at least 2 (over this local field).

The Lie group $SL_2(\mathbb{R})$ admits a lattice isomorphic to a free group (e.g. the fundamental group of an non-compact hyperbolic surface of finite co-volume). Hence $\mathrm{SL}_2(\mathbb{R})$ does not have property $(T)$. A similar argument can be made for $SL_2(\mathbb{C})$. In 1969 Kostant [77] gave a precise description of the spherical irreducible unitary representations of an arbitrary simple real Lie group of rank one. From it he was able to prove that the

[5]In fact he proved it for rank at least 3 by reducing the proof to $\mathrm{SL}_3(\mathbb{R})$ since every simple real Lie group of rank at least 3 contains a copy of $\mathrm{SL}_3(\mathbb{R})$, but it was quickly realized by others (treating the case of $Sp_4(\mathbb{R})$) that the argument extends to groups of rank 2 as well.

rank one groups $Sp(n,1)$ and $F_4^{-20}$ have property $(T)$, while the other rank one groups $SU(n,1)$ and $SO(n,1)$ (including $SL_2(\mathbb{R})$ and $SL_2(\mathbb{C})$) do not have property $(T)$.

The discrete group $\mathrm{SL}_n(\mathbb{Z})$ is a lattice in $\mathrm{SL}_n(\mathbb{R})$ and hence has property $(T)$ by Kazhdan's theorem. Nowadays (following Burger and Shalom) they are more direct proofs that $\mathrm{SL}_n(\mathbb{Z})$ has property $(T)$ using bounded generation.

Recently property $(T)$ was established for $\mathrm{SL}_n(R)$, $n \geqslant 3$, where $R$ is an arbitrary finitely generated commutative ring with unit and for $EL_n(R)$, where $R$ is an arbitrary finitely generated associative ring with unit.

**Theorem II.10.** *(Ershov and Jaikin-Zapirain* [49]*) Let $R$ be a (non-commutative) finitely generated ring with unit and $EL_n(R)$ be the subgroup of $n \times n$ matrices generated by the elementary matrix subgroups $Id_n + RE_{ij}$. If $n \geqslant 3$, then $EL_n(R)$ has property $(T)$.*

In particular, if $\mathbb{Z}\langle x_1, \ldots, x_k\rangle$ denotes the free associative algebra on $k$ generators, $EL_n(\mathbb{Z}\langle x_1, \ldots, x_k\rangle)$ has property $(T)$ for all $k \geqslant 0$ and $n \geqslant 3$. As an other special case, the so-called *universal lattices* $EL_n(\mathbb{Z}[x_1, ..., x_k]) = \mathrm{SL}_n(\mathbb{Z}[x_1, ..., x_k])$, where $\mathbb{Z}[x_1, ..., x_k]$ is the ring of polynomials on $k$ (commutative) indeterminates has property $(T)$ when $n \geqslant 3$. This remarkable result extends earlier works of Kassabov, Nikolov, Shalom and Burger on various special cases (see [72, 73, 110, 34] and references therein) and is based on a new method for proving property $(T)$ originating in the work of Dymara and Januszkiewicz [45] (see Kassabov's beautiful paper [70] on this subject). The Kazhdan constant in the above theorem behaves asymptotically as $\frac{1}{\sqrt{n+k}}$ for large $n$ and $k$.

An important tool in some of these proofs (e.g. see Shalom ICM talk [111]) is the following characterization of property $(T)$ in terms of affine actions of Hilbert spaces.

**Theorem II.11.** *(Delorme-Guichardet) A group $\Gamma$ has property $(T)$ if and only if every action of $\Gamma$ by affine isometries on a Hilbert space must have a global fixed point.*

See [64] or [8] for a proof. Kazhdan groups enjoy many other fixed point properties (e.g. Serre showed that they cannot act on trees without a global fixed point) and related rigidity properties (see e.g. the lectures by Dave Morris in this summer school).

Although the above class of examples of groups with property $(T)$ all come from the world of linear groups, Kazhdan groups also arise geometrically, for example as hyperbolic groups through Gromov's random groups. The following holds:

**Theorem II.12.** *In the density model of random groups, if the density is $< \frac{1}{2}$, then the random group is infinite and hyperbolic with overwhelming probability. If the density is $> \frac{1}{3}$, then the random group has property $(T)$ with overwhelming probability.*

It is unknown whether $\frac{1}{3}$ is the right threshold for property $(T)$. Below $\frac{1}{12}$ random groups have small cancellation $C'(1/6)$ and Ollivier and Wise proved that below $\frac{1}{6}$ they act freely and co-compactly on a $CAT(0)$ cube complex and are Haagerup, hence they do not have property $(T)$.

For a proof of the above see Zuk [121], Ollivier [95], Gromov [61] and Ghys' Bourbaki talk [52]. In fact Zuk proved a similar result for a slightly different model of random groups (the so-called triangular model) and Ollivier sketches a reduction of the above to Zuk's theorem in [95]. The proof of this result is based on the following celebrated geometric criterion for property $(T)$.

Let $\Gamma$ be a group generated by a finite symmetric set $S$ (with $e \notin S$). Let $L(S)$ be the finite graph whose vertices are the elements of $S$ and an edge is drawn between two vertices $s_1$ and $s_2$ iff $s_1^{-1}s_2$ belongs to $S$. Suppose that $L(S)$ is connected (this is automatic if $S$ is replaced say by $S \cup S^2 \setminus \{e\}$).

**Theorem II.13.** *(local criterion for property $(T)$) Let $\Gamma$ be a group generated by a finite symmetric set $S$ (with $e \notin S$) such that the first non-zero eigenvalue of the Laplacian on the finite graph $L(S)$ is $> \frac{1}{2}$. Then $\Gamma$ has property $(T)$.*

For a short proof, see Gromov's random walks in random groups paper [61] and the end of Ghys' Bourbaki talk [52]. The criterion is due to Zuk, Ballmann-Zwiatkowski, and originated in the work of Garland, see the above references for more historical comments.

For certain groups of geometric origin, such as $Out(F_n)$ and the mapping class groups, determining whether they have property $(T)$ or not is very hard. For example it is not known whether $Out(F_n)$ has property $(T)$ for $n \geqslant 4$ (even open for $Aut(F_n)$, not true for $n = 2, 3$ though). For the mapping class group also it is problematic; see the work of Andersen.

(C). **Uniformity issues in the Tits alternative, non-amenability and Kazhdan's property $(T)$.** A well-known question of Gromov from [60] is whether the various invariants associated with an infinite group (such as the rate of exponential growth, the isoperimetric constant of a non-amenable group, the Kazhdan constant of a Kazhdan group, etc) can be made uniform over the generating set.

For example we say:

**Definition II.14.** *(uniformity) Consider the family of all finite symmetric generating sets $S$ of a given finitely generated group. $\Gamma$. We say that $\Gamma$*

- *has uniform exponential growth if $\exists \varepsilon > 0$ such that $\lim \frac{1}{n} \log |S^n| \geqslant \varepsilon$, for all $S$,*
- *is uniformly non-amenable if $\exists \varepsilon > 0$ such that $|\partial_S A| \geqslant \varepsilon |A|$ for $S$,*
- *has uniform property $(T)$ if $\exists \varepsilon > 0$ such that $\max_S ||\pi(s)v - v|| \geqslant \varepsilon ||v||$ for all $S$ and all unitary representations of $\Gamma$ with no non-zero invariant vector.*
- *satisfies the uniform Tits alternative if $\exists N \in \mathbb{N} > 0$ such that $S^N$ contains generators of a non-abelian free subgroup $F_2$.*

Note that there are some logical implications between these properties. For example if $\Gamma$ satisfies the uniform Tits alternative, or if $\Gamma$ (is infinite and) has uniform property $(T)$, then $\Gamma$ is uniformly non-amenable (exercise). Similarly if $\Gamma$ is uniformly non-amenable, then $\Gamma$ has uniform exponential growth.

Uniform exponential growth holds for linear groups of exponential growth (Eskin-Mozes-Oh [47], see also [16] in positive characteristic), for solvable groups of exponential growth (Osin), but fails for general groups: John Wilson [118] gave an example of a non-amenable group (even containing $F_2$) whose exponential growth is not uniform. In fact Bartholdi and Erschler recently proved in [7] that every countable group embeds in a finitely generated group of non-uniform exponential growth. They also show that if $G$ is any finitely generated group with exponential growth, the permutational wreath product $G_{012} \wr_X G$, where $G_{012}$ is the first Grigorchuk group (of sub-exponential growth) permuting the orbit $X$ of the right-most branch of the binary rooted tree, then $G$ has non-uniform exponential growth.

The uniform Tits alternative is known to hold for non-elementary Gromov hyperbolic groups (Koubi [78]) and for non-virtually solvable linear groups by work of Breuillard-Gelander [16]. In this case the uniformity is even stronger as one has:

**Theorem II.15.** *(Uniform Tits alternative [23]) Given $d \in \mathbb{N}$, there is $N = N(d) \in \mathbb{N}$ such that for any field $K$ and any finite symmetric set $S \subset \mathrm{GL}_d(K)$ one has $S^N$ contains two generators of a non-abelian free subgroup $F_2$ unless $\langle S \rangle$ is virtually solvable.*

The uniformity in the field in the above theorem requires some non-trivial number theory (see [22]). This result implies that the rate of exponential growth is bounded below by a positive constant depending only on $d$ (= the number of rows of the matrix) and not on the field. So the uniform exponential growth is also uniform in the field.

However this is known to hold only for non-virtually solvable groups. The solvable case remains an open problem, already for $K = \mathbb{C}$: is the rate of exponential growth uniform over all virtually solvable subgroups of $\mathrm{GL}_d(\mathbb{C})$ ? In fact even the case of solvable subgroups of $\mathrm{GL}_2(\mathbb{C})$ is open. One can show however that if this is indeed the case, then this would imply the *Lehmer conjecture* from number theory [24]. Besides, the analogous uniform Tits alternative for free semi-groups does not hold.

Although it is a result about infinite linear groups, the above uniform Tits alternative has applications to finite groups as well. It turns out that the uniformity in the field allows one to transfer information from the infinite world to the finite world (we will see more of that in the remainder of this course). For example the following can be derived from Theorem II.15

**Corollary II.16.** *There is $N = N(d) \in \mathbb{N}$ and $\varepsilon = \varepsilon(d) > 0$ such that if $S$ is a generating subset of $\mathrm{SL}_d(\mathbb{F}_p)$ ($p$ arbitrary prime number), then $S^N$ contains two elements $a, b$ which generate $\mathrm{SL}_d(\mathbb{F}_p)$ and have no relation of length $\leqslant (\log p)^\varepsilon$. In other words the Cayley graph $\mathcal{G}(\mathrm{SL}_d(\mathbb{F}_p), \{a^{\pm 1}, b^{\pm 1}\})$ has girth at least $(\log p)^\varepsilon$.*

It is an open question (connected to whether all Cayley graphs of $\mathrm{SL}_d(\mathbb{F}_p)$ are uniformly expanders) whether one can replace $(\log p)^{\varepsilon}$ with $C \log p$, for some $C > 0$, in the above result.

Uniform property $(T)$ is even more mysterious. Examples were constructed by Osin and Sonkin [96] (every infinite hyperbolic group with property $(T)$ has a quotient with uniform property $(T)$). Osin showed on the other hand that hyperbolic groups do not have uniform property $(T)$ and Gelander and Zuk showed that any countable group which maps densely in a connected Lie group, and this includes all co-compact lattices in semi-simple real Lie groups, does not have uniform property $(T)$). But it is an open problem to determine whether $\mathrm{SL}_n(\mathbb{Z})$ has uniform property $(T)$ for $n \geqslant 3$. See the article by Lubotzky and Weiss [86] for further discussion.

## III. Lecture 3: Property ($\tau$) and expanders

There are many excellent existing texts for the material in this lecture, starting with Lubotzky's monograph [82] and recent AMS survey paper [83]. For expander graphs and their use in theoretical computer science, we refer the reader to the survey by Hoory, Linial and Wigderson [67]. Computer scientists also have numerous lecture notes on expander graphs available on the web (e.g. Linial and Wigderson). We give here only a brief introduction.

### (A). **Expander graphs.** We start with a definition.

**Definition III.1.** *(Expander graph) Let $\varepsilon > 0$. A finite connected $k$-regular graph $\mathcal{G}$ is said to be an $\varepsilon$-expander if for every subset $A$ of vertices in $\mathcal{G}$, with $|A| \leqslant \frac{1}{2}|\mathcal{G}|$, one has the following isoperimetric inequality:*

$$|\partial A| \geqslant \varepsilon |A|,$$

*where $\partial A$ denotes the set of edges of $\mathcal{G}$ which connect a point in $A$ to a point in its complement $A^c$.*

The optimal $\varepsilon$ as above is sometimes called the *discrete Cheeger constant* of the graph:

$$h(\mathcal{G}) = \inf_{A \subset \mathcal{G}, |A| \leqslant \frac{1}{2}|\mathcal{G}|} \frac{|\partial A|}{|A|}, \tag{2}$$

Just as in Lecture 1, when we discussed the various equivalent definitions of amenability, it is not a surprise that this definition turns out to have a spectral interpretation.

Given a $k$-regular graph $\mathcal{G}$, one can consider the Markov operator (also called averaging operator, or sometimes Hecke operator in reference to the Hecke graph of an integer lattice) on functions on vertices on $\mathcal{G}$ defined as follows:

$$Pf(x) = \frac{1}{k} \sum_{x \sim y} f(y), \tag{3}$$

where we wrote $x \sim y$ to say that $y$ is a neighbor of $x$ in the graph.

This operator is easily seen to be self-adjoint on the finite dimensional Euclidean space $\ell^2(\mathcal{G})$. Moreover it is a contraction, namely $||Pf||_2 \leqslant ||f||_2$ and hence its spectrum is real and contained in $[-1, 1]$. We can write the eigenvalues of $P$ in decreasing order as $\mu_0 = 1 \geqslant \mu_1 \geqslant \ldots \geqslant \mu_{|\mathcal{G}|}$. The top eigenvalue $\mu_0$ must be 1, because the constant function $\mathbf{1}$ is clearly an eigenfunction of $P$, with eigenvalue 1. On the other hand, since $\mathcal{G}$ is connected $\mathbf{1}$ is the only eigenfunction (up to scalars) with eigenvalue 1. This is immediate by the maximum principle (if $Pf = f$ and $f$ achieves its maximum at $x$, then $f$ must take the same value $f(x)$ at each neighbor of $x$, and this value spreads to the entire graph). Hence the second eigenvalue $\mu_1$ is strictly less than 1.

Instead of $P$, we may equally well consider $\Delta := Id - P$, which is then a non-negative self-adjoint operator. This operator is called the *combinatorial Laplacian* in analogy with the Laplace-Beltrami operator on Riemannian manifolds.

$$\Delta f(x) := f(x) - \frac{1}{k}\sum_{x\sim y} f(y).$$

Its eigenvalues are traditionally denoted by $\lambda_0 = 0 < \lambda_1 \leqslant \ldots \leqslant \lambda_{|\mathcal{G}|}$ and :

$$\lambda_i(\mathcal{G}) = 1 - \mu_i(\mathcal{G}).$$

As promised, here is the connection between the spectral gap an the edge expansion.

**Proposition III.2.** *(Discrete Cheeger-Buser inequality) Given a connected $k$-regular graph, we have:*

$$\frac{1}{2}\lambda_1(\mathcal{G}) \leqslant \frac{1}{k}h(\mathcal{G}) \leqslant \sqrt{2\lambda_1(\mathcal{G})}$$

The proof of this proposition is basically an exercise and follows a similar line of argument as the proof we gave in Lecture 1 of the Kesten criterion relating the Folner condition and the spectral radius of the averaging operator (Proposition I.10). See Lubotzky's book [82] for a detailed derivation and the references therein for the original papers (e.g. [44]).

We note in passing that, since $P$ is self-adjoint, the following holds:

$$||P||_{\ell_0^2} = \max_{i\neq 0} |\mu_i|$$

where $\ell_0^2$ is the space of functions on $\mathcal{G}$ with zero average, and

$$\mu_1 = \sup\{\frac{\langle Pf, f\rangle}{||f||_2^2}; \sum_{x\in\mathcal{G}} f(x) = 0\}$$

and hence

$$\lambda_1 = \inf\{\frac{\langle \Delta f, f\rangle}{||f||_2^2}; \sum_{x\in\mathcal{G}} f(x) = 0\} = \frac{1}{k}\inf\{\frac{||\nabla f||_2^2}{||f||_2^2}; \sum_{x\in\mathcal{G}} f(x) = 0\}. \quad (4)$$

Expander graphs very important to theoretical computer science (e.g. in the construction of good error correcting codes, see [67]). Typically one wants to have a graph of (small) bounded degree (i.e. $k$ is bounded) but whose number of vertices is very large. For this it is convenient to use the following definition:

**Definition III.3.** *(family of expanders) Let $k \geqslant 3$. A family $(\mathcal{G}_n)_n$ of $k$-regular graphs is said to be a family of expanders if the number of vertices $|\mathcal{G}_n|$ tends to $+\infty$ and if there is $\varepsilon > 0$ independent of $n$ such that for all $n$*

$$\lambda_1(\mathcal{G}_n) \geqslant \varepsilon.$$

Although almost every random $k$-regular graph is an expander (Pinsker 1972), the first explicit construction of an infinite family of expander graphs was given using Kazhdan's property $(T)$ and is due to Margulis [90] (see below Proposition III.7).

Clearly an $\varepsilon$-expander graph of size $N$ has diameter at most $O(\frac{1}{\varepsilon}\log|\mathcal{G}|)$. But more is true. A very important feature of expander graphs is the fact that the (lazy) simple random walk on such a graph equidistributes as fast as could possibly be towards the uniform probability distribution. This is made precise in the following proposition:

**Proposition III.4.** *(Random walk characterization of expanders) Suppose $\mathcal{G}$ is a $k$-regular graph and let $Q = \alpha Id + (1-\alpha)P$ be the lazy averaging operator on $\ell^2(\mathcal{G})$ (here $\alpha \in (0,1)$ and $P$ is the original averaging operator defined in* (3)*). Assume that $\lambda_1(\mathcal{G}) \geqslant \varepsilon$, then there is $C = C(\varepsilon, k, \alpha) > 0$ such that if $n \geqslant C\log|\mathcal{G}|$ then*

$$\max_{x,y\in\mathcal{G}} |\langle Q^n\delta_x, \delta_y\rangle - \frac{1}{|\mathcal{G}|}| \leqslant \frac{1}{|\mathcal{G}|^{10}}.$$

*Conversely for every $C > 0$ there is $\varepsilon = \varepsilon(C, k, \alpha) > 0$ such that if the $k$-regular graph $\mathcal{G}$ satisfies*

$$\max_{x\in\mathcal{G}} |\langle Q^n\delta_x, \delta_x\rangle - \frac{1}{|\mathcal{G}|}| \leqslant \frac{1}{|\mathcal{G}|^{10}},$$

*for some $n \leqslant C\log|\mathcal{G}|$, then $\mathcal{G}$ is an $\varepsilon$-expander.*

It is important in this proposition to consider the lazy walk, that is the operator $Q$ with $\alpha > 0$, rather than just the simple walk with operator $P$, because of possible issues with negative eigenvalues of $P$ close to $-1$ (see exercise below). The spectrum of $Q$ is given by $\alpha + (1-\alpha)Spec(P)$. In particular the eigenvalues of $Q$ are all $\geqslant -1+2\alpha$, so

$$1-(1+\alpha)\lambda_1(\mathcal{G}) \leqslant ||Q||_{\ell^2_0(\mathcal{G})} \leqslant \max\{1-2\alpha, 1-(1+\alpha)\lambda_1(\mathcal{G})\}. \tag{5}$$

Here $\langle Q^n\delta_x, \delta_y\rangle$ can be interpreted in probabilistic terms as the transition probability from $x$ to $y$ at time $n$, namely the probability that the $\alpha$-lazy simple random walk (i.e. the walk that either stays put with probability $\alpha$ or jumps to a nearest neighbor with equal probability $(1-\alpha)/k$) starting at $x$ visits $y$ at time $n$. Note that if $\mathcal{G}$ is a Cayley graph, then $\langle Q^n\delta_x, \delta_y\rangle$ depends only on $yx^{-1}$, and in particular $\langle Q^n\delta_x, \delta_x\rangle = \langle Q^n\delta_e, \delta_e\rangle$.

The exponent 10 in the remainder term is nothing special and can be replaced by any exponent $> 1$ (at the cost of increasing $C$).

*Proof of Proposition III.4.* The function $f_x := \delta_x - \frac{1}{|\mathcal{G}|}\mathbf{1}$ has zero mean on $\mathcal{G}$, hence

$$|\langle Q^n\delta_x, \delta_y\rangle - \frac{1}{|\mathcal{G}|}| = |\langle Q^n f_x, \delta_y\rangle| \leqslant ||Q||^n ||f_x|| ||\delta_y|| \leqslant \sqrt{2}||Q||^n.$$

Now this is at most $1/|\mathcal{G}|$ as some as $n \geqslant C_\varepsilon \log|\mathcal{G}|$ for some $C_\varepsilon > 0$.

Conversely observe that $trace(Q^n) = \sum_{x\in\mathcal{G}}\langle Q^n\delta_x, \delta_x\rangle$, and hence summing the estimates for $\langle Q^n\delta_x, \delta_x\rangle$, we obtain

$$|trace(Q^n) - 1| \leqslant \frac{1}{|\mathcal{G}|^9},$$

But on the other hand $trace(Q^n) = 1 + \mu_1^n + \ldots + \mu_{|\mathcal{G}|}^n$, where the $\mu_i$'s are the eigenvalues of $Q$, hence

$$\max_{i \neq 0} |\mu_i|^n \leqslant \mu_1^n + \ldots + \mu_{|\mathcal{G}|}^n \leqslant \frac{1}{|\mathcal{G}|^9},$$

thus recalling that $|\mathcal{G}|^{1/\log|\mathcal{G}|} = e$, we obtain the desired upper bound on $||Q||_{\ell_0^2(\mathcal{G})} = \max_{i \neq 0} |\mu_i|$, hence the lower bound on $\lambda_1(\mathcal{G})$ via (5). □

This fast equidistribution property is usually considered as a feature of expander graphs, a consequence of the spectral gap. We will see in the last lecture, when explaining the Bourgain-Gamburd method, that the proposition can also be used in the reverse direction, that is to establish the spectral gap.

*Exercise.* Show that the eigenvalue $-1$ appears in the spectrum of the averaging operator $P$ (defined in 3) if and only if the graph $\mathcal{G}$ is bi-partite, that is $|\mathcal{G}| = 2|A|$ for some subset of vertices $A \subset \mathcal{G}$ and every edge has one end point in $A$ and the other in the complement $\mathcal{G} \setminus A$. If $\mathcal{G}$ is a Cayley graph with generating set $S$, this happens iff $G$ has an index two subgroup $H$ such that $S \cap H = \varnothing$.

In fact more is true: non bi-partite expander Cayley graphs are characterized by the fast equidistribution of their simple random walk (not just the lazy one). Indeed:

*Exercise.*(There are no almost bi-partite expander Cayley graphs) If $\mathcal{G}$ is a Cayley graph with generating set $S$ of size $k$, which is not bi-partite, then the smallest eigenvalue $\mu_{|\mathcal{G}|}$ of the averaging operator $P$ satisfies

$$\mu_{|\mathcal{G}|} \geqslant -1 + c_k \lambda_1(\mathcal{G})^2,$$

for some constant $c = c_k$ depending on $k$ only.

So, unless $G$ has an index 2 subgroup, a lower bound on $\lambda_1$ implies an upper bound on the norm of $P$. In particular Proposition III.4 holds with $Q = P$ (i.e. $\alpha = 0$) when $\mathcal{G}$ is a Cayley graph which is not bi-partite.

Here is a hint for the exercise: use Proposition III.2 to show the existence of a subset $A$ of size roughly $|G|/2$ such that $|ss'A\Delta A| = o(|G|)$, $\forall s, s' \in S$, then use expansion to show that each right translate $Ag$ is very close to either $A$ or $sA$). See the last appendix in [20] for a proof.

Next we describe another spectral estimate, which is special to Cayley graphs. The Cheeger constant (see (2)) of a $k$-regular graph $\mathcal{G}$ is obviously at least $2k/|\mathcal{G}|$. By the Cheeger inequality (right hand side of Proposition III.2) this gives a lower bound for the first eigenvalue of the Laplacian in $1/|\mathcal{G}|^2$ up to constants. It turns out that when the graph $\mathcal{G}$ is a Cayley graph, one can improve this bound and replace the size of the graph $|\mathcal{G}|$ by the diameter $D(\mathcal{G})$. Indeed we have (see e.g. Diaconis and Saloff-Coste [43, Cor. 1]):

**Proposition III.5.** *(spectral gap from diameter) Suppose $\mathcal{G}$ is a Cayley graph of a finite group $G$ associated to a finite symmetric generating set of size $k$. Then*

$$\lambda_1(\mathcal{G}) \geqslant \frac{2}{k \cdot D(\mathcal{G})^2},$$

*where $D(\mathcal{G})$ is the diameter of the graph $\mathcal{G}$.*

*Proof.* Write $D = D(\mathcal{G})$. Any $y \in G$ can be written $y = s_1 \cdot \ldots \cdot s_D$, there each $s_i$ is either 1 or one of the $k$ generators. For any function $f$ on $\mathcal{G}$ apply Cauchy-Schwarz and get

$$|f(x) - f(xy)|^2 \leqslant D \sum_{i=1}^{D} |f(xw_i) - f(xw_{i-1})|^2$$

where $w_0 = 1$ and $w_i = s_1 \cdot \ldots \cdot s_i$. Summing over $x$ we get

$$\sum_{x \in \mathcal{G}} |f(x) - f(xy)|^2 \leqslant D \sum_{i=1}^{D} \sum_{x \in \mathcal{G}} |f(xs_i) - f(x)|^2 \leqslant D^2 ||\nabla f||_2^2.$$

Now assuming that $\sum_x f(x) = 0$ and summing over $y$, we obtain

$$2|\mathcal{G}| \cdot ||f||_2^2 \leqslant D^2 |\mathcal{G}| \cdot ||\nabla f||_2^2$$

The desired inequality follows from the variational characterization of $\lambda_1(\mathcal{G})$ (i.e. (4)). □

The above lower bound on $\lambda_1$ is not enough to prove that $\mathcal{G}$ is an expander, but it is already useful in some applications (e.g. in [46]). For Riemannian manifolds there is a rich theory relating the $\lambda_1$ to the volume or the diameter of the manifold in presence of curvature bounds. See for instance Cheng's paper [41] for an analogue of the above proposition and Chavel's books [39, 40] for a comprehensive introduction. Many of these results have graph theoretic analogues. For more about random walks on finite graphs and groups and the speed of equidistribution, the cut-off phenomenon, etc, see the survey by Saloff-Coste [105].

*Exercise.* Show that the above proposition fails for general finite $k$-regular graphs (hint: connect two large blubs by an edge).

For Cayley graphs the following simple reformulation of the expander property is very useful (see below Proposition III.9 and Theorem IV.2). Given a Cayley graph $\mathcal{G}(G, S)$ of a finite group $G$, let $\alpha(\mathcal{G})$ be the infimum of all values $\alpha > 0$ such that the following holds. For every unitary representation $V$ of $G$ and every vector $v \in V$ one has

$$\max_{g \in G} ||\rho(g)v - v|| \leqslant \alpha \max_{s \in S} ||\rho(s)v - v||.$$

**Proposition III.6.** *(Representation theoretic reformulation) If $\mathcal{G} = \mathcal{G}(G, S)$ is a $k$-regular Cayley graph of a finite group $G$, then we have*

$$\frac{\lambda_1(\mathcal{G})}{2} \leqslant \alpha(\mathcal{G})^{-1} \leqslant \sqrt{2k\lambda_1(\mathcal{G})}$$

*Exercise.* Prove the above proposition (hint: every linear representation of $G$ decomposes into irreducible components, each of which appears in $\ell^2(G)$).

In particular a family of $k$-regular Cayley graphs $\mathcal{G}_n$ is a family of expanders, if and only if the values $\alpha(\mathcal{G}_n)$ are uniformly bounded.

(B). **Property** $(\tau)$**.** Margulis [90] was the first to construct an explicit family of $k$-regular expander graphs. For this he used property $(T)$ through the following observation:

**Proposition III.7.** *($(T)$ implies $(\tau)$) Suppose $\Gamma$ is a group with Kazhdan's property $(T)$ and $S$ is a symmetric set of generators of $\Gamma$ of size $k = |S|$. Let $\Gamma_n \leqslant \Gamma$ be a family of finite index subgroups such that the index $[\Gamma : \Gamma_n]$ tends to $+\infty$ with $n$. Then the family of $k$-regular Schreier graphs $\mathcal{G}(\Gamma/\Gamma_n, S)$ forms a family of expanders.*

Recall that the *Schreier graph* $\mathcal{G}(\Gamma/\Gamma_0, S)$ of a coset space $\Gamma/\Gamma_0$ associated to a finite symmetric generating set $S$ of $\Gamma$ is the graph whose vertices are the left cosets of $\Gamma_0$ in $\Gamma$ and one connects $g\Gamma_0$ to $h\Gamma_0$ if there is $s \in S$ such that $g\Gamma_0 = sh\Gamma_0$.

*Proof.* The group $\Gamma$ acts on the finite dimensional Euclidean space $\ell^2_0(\Gamma/\Gamma_n)$ of $\ell^2$ functions with zero average on the finite set $\Gamma/\Gamma_n$. Denote the resulting unitary representation of $\Gamma$ by $\pi_n$. Property $(T)$ for $\Gamma$ gives us the existence of a Kazhdan constant $\varepsilon = \varepsilon(S) > 0$ such that $\max_{s \in S} ||\pi(s)v - v|| \geqslant \varepsilon ||v||$ for every unitary representation $\pi$ of $\Gamma$ without invariant vectors. In particular, this applies to the $\pi_n$ since they have no non-zero $\Gamma$-invariant vector. This implies that the graphs $\mathcal{G}_n := \mathcal{S}(\Gamma/\Gamma_n, S)$ are $\varepsilon$-expanders, because if $A \subset \mathcal{G}_n$ has size at most half of the graph, then $v := 1_A - \frac{|A|}{|\mathcal{G}|}\mathbf{1}$ is a vector in $\ell^2_0(\Gamma/\Gamma_n)$ and $||\pi_n(s)v - v||^2 = ||\pi_n(s)1_A - 1_A||^2 = |sA\Delta A|$, while $||v||^2 = 2|A|(1 - \frac{|A|}{|\mathcal{G}_n|}) \geqslant |A|$. In particular $|\partial A| \geqslant \varepsilon^2 |A|$. □

So we see that Cayley graphs (or more generally Schreier graphs) of finite quotients of finitely generated groups can yield families of expanders. This is the case for the family of Cayley graphs of $\mathrm{SL}_3(\mathbb{Z}/m\mathbb{Z})$ associated to the reduction mod $m$ of a fixed generating set $S$ in $\mathrm{SL}_3(\mathbb{Z})$. To characterize this property, Lubotzky introduced the following terminology:

**Definition III.8.** *(Property $(\tau)$) A finitely generated group $\Gamma$ with finite symmetric generating set $S$ is said to have* property $(\tau)$ *with respect to a family of finite index normal subgroups $(\Gamma_n)_n$ if the family of Cayley graphs $\mathcal{G}(\Gamma/\Gamma_n, S_n)$, where $S_n = S\Gamma_n/\Gamma_n$ is the projection of $S$ to $\Gamma/\Gamma_n$, is a family of expanders. If the family $(\Gamma_n)_n$ runs over all finite index normal subgroups of $\Gamma$, then we say that $\Gamma$ has* property $(\tau)$.

Proposition III.7 above shows that every group with property $(T)$ has property $(\tau)$. The converse is not true and property $(\tau)$ is in general a weaker property which holds more often. For example Lubotzky and Zimmer [87] showed that an irreducible lattice $\Gamma$ in a semisimple real Lie group $G$ without compact factors has property $(\tau)$ as soon as one of the simple factors of the ambient semisimple Lie group has property $(T)$.

Note however that for $\Gamma$ to have property $(T)$ it is necessary that all factors of $G$ have property $(T)$.

*Exercise.* Show that if $\Gamma$ is amenable and has property $(\tau)$, then $\Gamma$ has only finitely many finite index subgroups.

*Exercise.* Recall that the regular representation of a finite group $G$ contains an isomorphic copy of each irreducible representation of $G$ (see e.g. [108]). Let $H \leqslant G$ a subgroup and $S$ a symmetric generating set for $G$. Show that $\lambda_1(\mathcal{G}(G/H, S)) \geqslant \lambda_1(\mathcal{G}(G, S))$, where $\mathcal{G}(G/H, S)$ is the associated Schreier graph. Deduce that if $\Gamma$ has property $(\tau)$, then the family of all Schreier graphs $\mathcal{G}(\Gamma/\Gamma_n, S)$, where $\Gamma_n$ ranges over all finite index (not necessarily normal) subgroups of $\Gamma$, is a family of expanders.

Property $(\tau)$ is stable under quotients (obviously). In particular groups with property $(\tau)$ have finite abelianization, just as Kazhdan's groups.

As property $(T)$, property $(\tau)$ is also stable and under passing to and from a finite index subgroup:

**Proposition III.9.** *Suppose $\Gamma' \leqslant \Gamma$ is a subgroup of finite index in $\Gamma$. Then $\Gamma$ has property $(\tau)$ if and only if $\Gamma'$ has property $(\tau)$.*

*Proof.* (sketch) Let $S$ a finite symmetric generating set for $\Gamma$ and $S' \subset S^{2[\Gamma:\Gamma']}$ be a finite generating set for $\Gamma'$, which is obtained as usual from the Reidemeister-Schreier rewriting process, so that if $\{\gamma_i\}_i$ is a set of representatives of the cosets of $\Gamma'$ in $\Gamma$ contained in $S^{[\Gamma:\Gamma']}$, then for every $i$ and $s \in S$ there is $j$ and $s' \in S'$ such that $s\gamma_i = \gamma_j s'$ (see [89, sec 2.3]).

Suppose $\Gamma'$ has $(\tau)$. Then thanks to Proposition III.6, there is an upper bound on $\alpha(\mathcal{G}'_n)$ for all Cayley graphs associated to finite quotients of $\Gamma'$. It is easy to check that this upper bound lifts to an upper bound on $\alpha(\mathcal{G}_n)$ for the Cayley graphs of all finite quotients of $\Gamma$, therefore $\Gamma$ has $(\tau)$.

In the converse direction, suppose $\Gamma$ has $(\tau)$ and let $(\rho, V)$ be a linear representation of a finite quotient of $\Gamma'$. Then the induced representation $(\rho', W)$ to $\Gamma$ is a linear representation of a finite quotient of $\Gamma$. Its ambient space $W$ is the set of functions on $\Gamma$ to $V$ such that $f(x\gamma') = \rho(\gamma')f(x)$ for all $x \in \Gamma$, $\gamma' \in \Gamma$, and $\Gamma$ acts by left translations. Now suppose that $v \in V$ is almost fixed by the elements $s' \in S'$, then the function $f \in W$ defined by $f(\gamma_i) = v$ for each $i$ is almost fixed by every element $s \in S$, hence almost fixed by all of $\Gamma$, by property $(\tau)$ and Proposition III.6. It follows that $v$ is almost fixed by all $\gamma' \in \Gamma'$ and we are done. For more details see [86, Prop. 3.9.]. □

Arithmetic lattices in semisimple algebraic groups defined over $\mathbb{Q}$ admit property $(\tau)$ with respect to the family of all congruence subgroups. Namely:

**Theorem III.10.** *(Selberg, Burger-Sarnak, Clozel) Let $\mathcal{G} \subset \mathrm{GL}_d$ is a semisimple algebraic $\mathbb{Q}$-group, $\Gamma = \mathcal{G}(\mathbb{Z}) := \mathcal{G}(\mathbb{Q}) \cap \mathrm{GL}_d(\mathbb{Z})$ and $\Gamma_m = \Gamma \cap \ker(\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Z}/m\mathbb{Z}))$, then $\Gamma$ has property $(\tau)$ with respect to the $\Gamma_m$'s.*

This property is also called the *Selberg property* because in the case of $\mathcal{G} = \mathrm{SL}_2$ it is a consequence (as we will see below) of a celebrated theorem of Selberg [107], the $\frac{3}{16}$ theorem, which asserts that the non-zero eigenvalues of the Laplace-Beltrami laplacian on the hyperbolic surfaces of finite co-volume $\mathbb{H}^2/\ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/m\mathbb{Z}))$ are bounded below by a positive constant independent of $m$ (in fact $\frac{3}{16}$). The general case was established by Burger-Sarnak [35] and Clozel [42].

This connects property $(\tau)$ for lattices with another interesting feature of some lattices, namely the *congruence subgroup property.* This property of an arithmetic lattice asks that every finite index subgroup contains a congruence subgroup (i.e. a subgroup of the form $\mathcal{G}(\mathbb{Z}) \cap \ker(\mathrm{GL}_d(\mathbb{Z}) \to \mathrm{GL}_d(\mathbb{Z}/m\mathbb{Z}))$.

*Exercise.* Show that if $\mathcal{G}(\mathbb{Z})$ has both the Selberg property and the congruence subgroup property, then it has property $(\tau)$ with respect to all of its finite index subgroups (hint: see the second exercise after Definition III.8).

An interesting open problem in this direction is to determine whether or not lattices in $SO(n, 1)$ can have property $(\tau)$ or not. Lubotzky and Sarnak conjecture that they do not, and this would also follow from Thurston's conjecture that such lattices have a subgroup of finite index with infinite abelianization (now proved in dimension 3 !).

The link between Selberg's $\frac{3}{16}$ theorem and property $(\tau)$ is provided by the following general fact, which relates the combinatorial spectral gap of a Cayley (or Schreier) graph of finite quotients of the fundamental group of a manifold with the spectral gap for the analytic Laplace-Beltrami operator on the Riemannian manifold.

Recall that given a connected Riemannian manifold $M$ the Laplace-Beltrami operator is a non-negative self-adjoint operator on $L^2$ functions ($L^2$ with respect to the Riemannian volume measure). If $M$ is compact, the spectrum of this operator is discrete $\lambda_0(M) = 0 < \lambda_1(M) \leqslant \ldots$ (e.g. see [9]).

The fundamental group $\Gamma = \pi_1(M)$ acts freely and co-compactly on the universal cover $\widetilde{M}$ by isometries (for the lifted Riemannian metric on $\widetilde{M}$). Given a base point $x_0 \in \widetilde{M}$, the set

$$\mathcal{F}_M = \{x \in \widetilde{M}; d(x, x_0) < d(x, \gamma \cdot x_0) \ \forall \gamma \in \Gamma \setminus \{1\}\} \tag{6}$$

is a (Dirichlet) fundamental domain for the action of $\Gamma$ on $\widetilde{M}$. Moreover the group $\Gamma$ is generated by the finite symmetric set $S := \{\gamma \in \Gamma; \gamma\overline{\mathcal{F}_M} \cap \overline{\mathcal{F}_M} \neq \varnothing\}$. We can now state:

**Theorem III.11.** *(Brooks* [27]*, Burger* [32]*) Let $M$ be a compact Riemannian manifold with fundamental group $\Gamma = \pi_1(M)$. Let $S$ be the finite symmetric generating set of $\Gamma$ obtained from a Dirichlet fundamental domain $\mathcal{F}_M$ as above. Then there are constants $c_1, c_2 > 0$ depending on $M$ only such that for every finite cover $M_0$ of $M$*

$$c_1\lambda_1(M_0) \leqslant \lambda_1(\mathcal{G}(\Gamma/\Gamma_0, S)) \leqslant c_2\lambda_1(M_0),$$

*where* $\Gamma_0$ *is the fundamental group of* $M_0$ *and* $\mathcal{G}(\Gamma/\Gamma_0, S))$ *the Schreier graph of the finite coset space* $\Gamma/\Gamma_0$ *associated to the generating set* $S$*.*

We deduce immediately:

**Corollary III.12.** *Suppose* $(M_n)_n$ *is a sequence of finite covers of* $M$*. Then there is a uniform lower bound on* $\lambda_1(M_n)$ *if and only if* $\Gamma := \pi_1(M)$ *has property* $(\tau)$ *with respect to the sequence of finite index subgroups* $\Gamma_n := \pi_1(M_n)$*.*

The proof consists in observing that the Schreier graph can be drawn on the manifold $M_0$ as a dual graph to the decomposition of $M_0$ into translates of the fundamental domain $\mathcal{F}_M$. The geometry of this Schreier graph closely resembles that of the cover $M$ up to a bounded disturbance depending on $M_0$ only. For the proof of this Brooks-Burger transfer principle, we refer the reader to Appendix V, where we give a complete treatment and further discussion. The result also extends to non-compact hyperbolic manifolds of finite co-covolume (see [10, Section 2] and [46, Appendix]).

In a similar spirit, with similar proof, Brooks showed:

**Proposition III.13** (Brooks [26])**.** *If* $M_0$ *is a compact Riemannian manifold and* $M$ *a normal cover of* $M_0$ *with Galois group* $\Gamma$*. Then* $\lambda_0(M) = 0$ *if and only if* $\Gamma$ *is amenable.*

For more on property $(\tau)$ we refer the reader to the forthcoming book by Lubotzky and Zuk [88].

## IV. Lecture 4: Approximate groups and the Bourgain-Gamburd method

(A). **What finite groups can be turned into expanders ?** In [86], Lubotzky and Weiss asked the question of whether the property of being an expander is a group property. Namely given a sequence of finite groups $(G_n)_n$ generated by a fixed number of elements $k \geqslant 1$, is it true that if one can find a sequence of $k$-regular Cayley graphs of the $G_n$'s which is an expander family, then the family of all Cayley graphs of the $G_n$'s on $k$ generators is an expander family ? In other words is being an expander independent of the choice of the generating set ?

It turned out that the answer to this question is no in general. An example was produced in [4] using the so-called *zig-zag* product construction. However a more natural example was given later by Martin Kassabov, who, in a remarkable breakthrough [71], managed to turn the family of symmetric groups $\mathcal{S}_n$ into a family of expanders with generating sets of bounded size. On the other hand $\mathcal{S}_n$ is not a family of expanders when generated by the transposition (12) and the long cycle $(12\dots n)$ because the diameter of the associated Cayley graph is at least $n^2 \gg \log|\mathcal{S}_n|$ (see [82, Prop. 8.1.6.]), while as we already observed the diameter of an expander graph $\mathcal{G}$ is always at most $O(\log|\mathcal{G}|)$. See also [72] for other examples involving $\mathrm{SL}_n(\mathbb{F}_p)$ for fixed $p$.

However there are classes of groups for which an answer is known or at least is expected. To begin with the following observation of Lubotzky and Weiss [86] shows that solvable groups of fixed derived length cannot be turned into expanders:

**Proposition IV.1** (Finite solvable groups are never expanders)**.** *Fix $\ell, k \in \mathbb{N}$. Suppose $\mathcal{G}_n$ is a family of $k$-regular Cayley graphs of finite solvable groups $G_n$ with derived length $\leqslant \ell$. Then $\{\mathcal{G}_n\}$ is not a family of expanders.*

*Proof.* The free solvable group $\Gamma_\ell := F_k/D^\ell(F_k)$ on $k$ generators is a finitely generated solvable group. Hence amenable. By Kesten's criterion (Theorem I.9) the probability of return $\mu^{2m}(e)$ to the identity of the simple (or lazy simple) random walk on $\Gamma_\ell$ decays subexponentially in $m$. Each $G_n$ is a homomorphic image of $\Gamma_\ell$ in such a way that the Cayley graph $\mathcal{G}_n$ is a quotient of that of $\Gamma_\ell$. Hence the probability of return of the simple (or lazy simple) random walk on $\mathcal{G}_n$ is always at least $\mu^{2m}(e)$. However if the $\mathcal{G}_n$ form an expander family, then Proposition III.4 ensures that there is $c, \rho < 1$ such that $\mu^{2m}(e) < \rho^m$ if $m \leqslant c\log|G_n|$. A contradiction if $|G_n| \to +\infty$. □

In fact further arguments (see [86, Theorem 3.6]) show that one needs at least $\log^{(\ell)}(|G|)$ generators to turn a finite $\ell$-solvable group $G$ into an expander graph. For $\ell = 1$, i.e. for abelian groups, we thus need $\log|G|$ generators to make a Cayley graph with first eigenvalue of the Laplacian bounded away from 0. It is interesting to observe that Alon and Roichman [5] showed that for an arbitrary finite group $G$, $O(\log|G|)$ generators are always sufficient to produce to expanding Cayley graph, in fact a random set of $O(\log|G|)$ elements in $G$ generates an expanding Cayley graph with high probability as $|G| \to +\infty$. See [79] for a simple proof of this fact using the Ahlswede-Winter large deviation bounds for sums of independent non-negative self-adjoint operators on a Hilbert space.

If solvable groups cannot be made into expanders, what about simple groups ? As we saw above, the answer depends on the generating set. But remarkably the following holds:

**Theorem IV.2.** *(Simple groups as expanders) There is $k \geqslant 2$ and $\varepsilon > 0$ such that every finite simple group $G$ has a $k$-regular Cayley graph which is an $\varepsilon$-expander.*

The proof of this result is due to Kassabov, Lubotzky and Nikolov [74] who proved it except for the family of Suzuki groups $\{Suz(2^{2n+1})\}_n$, which was later settled in [19]. Of course the proof is based on the classification of finite simple groups. The case of the sub-family $\mathrm{PSL}_2(\mathbb{F}_q)$ was proved by Lubotzky [84], building on the $\mathrm{PSL}_2(\mathbb{F}_p)$ case (i.e. Theorem III.10, which boils down in this case to Selberg's $\frac{3}{16}$ theorem via the Brooks-Burger transfer principle), and using further deep facts from the theory of automorphic forms, but as we will see shortly (in Theorem IV.11 below) there are now new methods to settle this case and indeed all finite simple groups of Lie type and bounded rank together.

The case of $\mathrm{PSL}_n(\mathbb{F}_q)$ with $n$ going to infinity and $q$ arbitrary is due to Kassabov [72], and can now be seen as a consequence of Theorem II.10 proving property $(T)$ for the non-commutative universal lattices $EL_3(\mathbb{Z}\langle x_1, \ldots, x_k\rangle)$. Recall that for a ring $R$, $EL_d(R)$ denotes the subgroup of $GL_d(R)$ generated by the elementary matrices $Id + RE_{ij}$. Kassabov's beautiful idea is to take advantage of the following straightforward observation:

$$EL_3(Mat_{n\times n}(\mathbb{F}_q)) \simeq EL_{3n}(\mathbb{F}_q)$$

and argue that $Mat_{n\times n}(\mathbb{F}_q)$ can be generated as an associative ring by 2 elements, and hence is a quotient of the free associative algebra $\mathbb{Z}\langle x_1, x_2\rangle$. The universal lattice $EL_3(\mathbb{Z}\langle x_1, x_2\rangle)$ is finitely generated by the elementary matrices $Id \pm E_{ij}$ and $Id \pm x_m E_{ij}$. That makes 36 generators. Now Theorem II.10 says that this group has property $(T)$ and we thus conclude (as in Proposition III.7) that the quotients $EL_{3n}(\mathbb{F}_q) \simeq SL_{3n}(\mathbb{F}_q)$ are uniformly expanders, for all $n \geqslant 1$ and all prime powers $q$, with respect to the corresponding projected generating set (still with 36 generators).

In order to go from $\mathrm{PSL}_n(\mathbb{F}_q)$ with $n$ divisible by 3 to $\mathrm{PSL}_n(\mathbb{F}_q)$ for all $n$ and finally to $G(\mathbb{F}_q)$ for every group of Lie type $G$, one uses bounded generation. Nikolov [92] shows that every $G(\mathbb{F}_q)$ can be written as a product of a bounded number of groups isomorphic to $\mathrm{PSL}_{3n}(\mathbb{F}_{q'})$ (up to the center). Now Proposition III.6, which is a representation theoretic reformulation of the expander property easily implies the following: if a group $G$ can be written as $G = H_1 \cdot \ldots \cdot H_n$, with $n$ bounded (we say that $G$ is boundedly generated by the $H_i$'s), and each $H_i$ has a generating set $\Sigma_i$ of bounded size with respect to which it is an $\varepsilon$-expanding Cayley graph, then $G$ too has a generating set of bounded size (the union of the $\Sigma_i$'s) with respect to which it is $\varepsilon'$-expanding for some $\varepsilon'$ depending only on $\varepsilon$ and $n$. This settles the remaining cases for Theorem IV.2 for simple groups of Lie type.

Finally the case of alternating groups $A_n$ was settled by Kassabov in a tour-de-force paper [71] which blends some of the above ideas, in particular by embedding large powers of $\mathrm{SL}_k(\mathbb{F}_2)$ inside $A_n$ and using the above expansion result for $\mathrm{SL}_k(\mathbb{F}_2)$, together with some ideas of Roichman [102, 103] involving character bounds for certain representations of $A_n$. See [74] for a sketch.

(B). **The Bourgain-Gamburd method.** Up until the Bourgain-Gamburd breakthrough [11] in 2005, the only known ways to turn $\mathrm{SL}_d(\mathbb{F}_p)$ into an expander graph (i.e. to find a generating set of small size whose associated Cayley graph has a good spectral gap) was either through property $(T)$ (as in the Margulis construction, i.e. Proposition III.7) when $d \geqslant 3$ or through the Selberg property when $d = 2$ via the Brooks-Burger transfer principle between combinatorial expansion of the Cayley graphs and the spectral gap for the Laplace-Beltrami Laplacian on towers of covers of hyperbolic manifolds (see Proposition III.11 and the appendix) .

This poor state of affairs was particularly well illustrated by the embarrassingly open question of Lubotzky, the *Lubotzky* 1-2-3 *problem*, which asked whether the subgroups $\Gamma_i := \langle S_i \rangle \leqslant \mathrm{SL}_2(\mathbb{Z})$ for $i = 1, 2$ and 3 given by

$$S_i = \{ \begin{pmatrix} 1 & \pm i \\ 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 0 \\ \pm i & 1 \end{pmatrix} \}$$

have property $(\tau)$ with respect to the family of congruence subgroups $\Gamma_i \cap \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}))$ as $p$ varies among the primes. The answer for $i = 1$ and 2 follows as before from Selberg's $\frac{3}{16}$ theorem, because both $\Gamma_1$ and $\Gamma_2$ are subgroups of finite index in $\mathrm{SL}(2, \mathbb{Z})$ (even $\Gamma_1 = \mathrm{SL}_2(\mathbb{Z})$). However $\Gamma_3$ has infinite index in $\mathrm{SL}_2(\mathbb{Z})$ (its limit set on the projective line $\mathbb{P}(\mathbb{R}^2)$ is a Cantor set) and therefore none of these methods apply.

Bourgain and Gamburd changed the perspective by coming up with a more head-on attack on the problem showing fast equidistribution of the simple random walk directly by more analytic and combinatorial means. As we saw in Proposition III.4 this is enough to yield a spectral gap. One of these combinatorial ingredients was the notion of an approximate group (defined below) which was subsequently studied for its own sake and lead in return to many more applications about property $(\tau)$ and expanders as we are about to describe.

Let us now state the Bourgain-Gamburd theorem:

**Theorem IV.3.** *(Bourgain-Gamburd* [11]*) Given $k \geqslant 1$ and $\tau > 0$ there is $\varepsilon = \varepsilon(k, \tau) > 0$ such that every Cayley graph $\mathcal{G}(\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}), S)$ of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ with symmetric generating set $S$ of size $2k$ and girth at least $\tau \log p$ is an $\varepsilon$-expander.*

We recall that the *girth* of a graph is the length of the shortest loop in the graph. Conjecturally all Cayley graphs of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ are $\varepsilon$-expanders for a uniform $\varepsilon$, and this was later established for almost all primes in Breuillard-Gamburd [14] using the Uniform Tits alternative. But the Bourgain-Gamburd theorem is the first instance of a result on expanders where a purely geometric property, such as large girth, is shown to imply a spectral gap.

The Bourgain-Gamburd result answers positively the Lubotzky 1-2-3 problem:

**Corollary IV.4.** *Every non-virtually solvable subgroup $\Gamma$ in $\mathrm{SL}_2(\mathbb{Z})$ has property $(\tau)$ with respect to the congruence subgroups $\Gamma_p := \Gamma \cap \ker(\mathrm{SL}_2(\mathbb{Z}) \to \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}))$ as $p$ varies among the primes.*

*Proof.* Let $S$ be a symmetric generating set for $\Gamma$. By the Tits alternative (or using the fact that $\mathrm{SL}_2(\mathbb{Z})$ is virtually free), there is $N = N(\Gamma) > 0$ such that $S^N$ contains two generators of a free group $a, b$. Now in order to prove the spectral gap for the action of $S$ on $\ell^2(\Gamma/\Gamma_p)$ it is enough to prove a spectral gap for the action of $a$ and $b$. Indeed suppose there is $f \in \ell^2_0(\Gamma/\Gamma_p)$ such that $\max_{s \in S} ||s \cdot f - f|| \leqslant \varepsilon ||f||$. Then writing $a$ and $b$ as words in $S$ of length at most $N$, we conclude that $||a \cdot f - f|| \leqslant N\varepsilon||f||$ and $||b \cdot f - f|| \leqslant N\varepsilon||f||$. Since $N$ depends only on $\Gamma$ and not on $p$ we have reduced the problem to proving spectral gap for $\langle a, b\rangle$ and we can thus assume that $\Gamma = \langle a, b \rangle$ is a 2-generated free subgroup of $\mathrm{SL}_2(\mathbb{Z})$.

Then it is easy to verify that the logarithmic girth condition holds for this new $\Gamma$. Indeed the size of the matrices $w(a, b)$, where $w$ is a word of length $n$ do not exceed $\max\{||a^{\pm 1}||, ||b^{\pm 1}||\}^n$, hence $w(a, b)$ is not killed modulo $p$ if $p$ is larger that $\max\{||a^{\pm 1}||, ||b^{\pm 1}||\}^n$, that is if $n$ is smaller that $\tau \log p$ for some $\tau = \tau(a, b) > 0$. We can then apply the theorem and we are done. □

Before we go further, let us recall the following:

**Theorem IV.5.** *(Strong Approximation Theorem, Nori* [94]*, Matthews-Vasserstein-Weisfeiler* [93, 117]*) Let $\Gamma$ be a Zariski-dense subgroup of $\mathrm{SL}_d(\mathbb{Z})$. Then its projection modulo $p$ via the map $\mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/p\mathbb{Z})$ is surjective for all but finitely many primes $p$.*

This is a deep result. The proof of Matthews-Vasserstein-Weisfeiler is based on finite group theory and uses the classification of finite simple groups. Nori's approach is different, via algebraic geometry. There are also alternate proofs by Hrushovski-Pillay [68] via model theory and by Larsen-Pink [81]. Those proofs avoid the classification of finite simple groups and have a broader scope. However in the special case of $\mathrm{SL}_2(\mathbb{Z})$ this result is just an exercise (once one observes that the only large subgroups of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ are dihedral, diagonal, or upper triangular). It will be important for us, because it says that $\Gamma/\Gamma_p = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ as soon as $p$ is large enough, and we will use several key features of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ in the proof of Theorem IV.3.

We are now ready for a sketch of the Bourgain-Gamburd theorem.

Let $\nu = \frac{1}{|S|}\sum_{s\in S}\delta_s$ be the symmetric probability measure supported on the generating set $S$. Our first task will be to make explicit the connection between the decay of the probability of return to the identity and the spectral gap, pretty much as we did in Lecture 3. We may write:

$$\nu^{2n}(e) = \langle P_\nu^{2n}\delta_e, \delta_e\rangle = \frac{1}{|G_p|}\sum_{x\in G_p}\langle P_\nu^{2n}\delta_x, \delta_x\rangle$$

where we have used the fact that the Cayley graph is homogeneous (i.e. vertex transitive) and hence the probability of return to the $e$ starting from the $e$ is the same as the one of returning to $x$ starting from $x$, whatever $x \in G_p$ may be, so $\langle P_\nu^{2n}\delta_e, \delta_e\rangle = \langle P_\nu^{2n}\delta_x, \delta_x\rangle$.

A key ingredient here is that we will make use of an important property of finite simple groups of Lie type (such as $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$) which is that they have no non-trivial finite dimensional complex representation of small dimension. This goes back to Frobenius for $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ and is due to Landazuri and Seitz [80] for arbitrary finite simple groups of Lie type. For $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ this says the following:

**Lemma IV.6.** *(Quasi-randomness) The dimension of a non-trivial irreducible (complex) representation of* $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ *is at least* $\frac{p-1}{2}$.

*Proof.* Let $V$ be a non-trivial irreducible $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ complex linear representation, and view it as a representation of the upper-triangular unipotent subgroup $U$. The subgroup $U$ and its conjugates generate $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, so conjugating if necessary, we may assume that the $U$ action on $V$ is non-trivial. Then $V$ splits into $U$-invariant isotypic components $V_\chi$, each corresponding to a character $\chi \in U^*$. These components are permuted by the normalizer $N(U)$. However the conjugation action of $N(U)$ on $U$ is isomorphic to the action of the subgroup of squares $\{x^2; x \in (\mathbb{Z}/p\mathbb{Z}) \setminus \{0\}\}$ by multiplication on $U \simeq (\mathbb{Z}/p\mathbb{Z}, +)$. Besides $\{0\}$ it has just two orbits of size $\frac{p-1}{2}$. Hence all $V_\chi$ for each $\chi$ in one of the non-trivial orbits must occur in $V$. It follows that $\dim V \geqslant \frac{p-1}{2}$. □

A finite group is called "quasi-random" if it has no non trivial irreducible character of small dimension (how small depends on the quality of the desired quasirandomness). The term *quasirandomness* is derived from a paper of Gowers [53] in which he describes some combinatorial consequences of this property.

A consequence of this fact is the following *high multiplicity trick*: the eigenvalues of $P_\nu$ on $\ell_0^2(\Gamma/\Gamma_p)$ all appear with multiplicity at least $\frac{p-1}{2}$. This is indeed true, because when $p$ is large enough, then $\Gamma/\Gamma_p = G_p := \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, by the above strong approximation theorem. And the regular representation $\ell^2(\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z}))$ can be decomposed into irreducible (complex) linear representations, each of which appears with a multiplicity equal to its dimension[6]. The operator $P_\nu$ preserves each one of these invariant subspaces, and hence its non-trivial eigenvalues appear with a multiplicity at least equal to $\frac{p-1}{2}$ by Lemma IV.6 above. Since $\frac{p-1}{2} \simeq |G_p|^{\frac{1}{3}}$, we get

[6]This is a standard fact from the representation theory of finite groups, see e.g. Serre [108].

$$\nu^{2n}(e) = \frac{1}{|G_p|}(\mu_0^{2n} + \mu_1^{2n} + \ldots + \mu_{|G_p|-1}^{2n}) \gg \mu_1^{2n}\frac{|G_p|^{\frac{1}{3}}}{|G_p|}$$

where the $\mu_i$'s are the eigenvalues of $P_\nu$, $\mu_0 = 1$ and $G_p = \mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$, and $\gg$ means larger than up to a positive multiplicative constant.

Hence

$$\mu_1^{2n} \ll \nu(e)^{2n}|G|^{\frac{2}{3}}$$

So if we knew that

$$\nu^{2n}(e) \ll \frac{1}{|G_p|^{1-\beta}}$$

for some small $\beta < \frac{1}{3}$ and for $n$ of size say at most $C \log |G_p|$ for some constant $C > 0$, we would deduce the following spectral gap:

$$\mu_1 \leqslant e^{-\frac{1/3-\beta}{C}} < 1$$

(recall that $|G_p|^{\frac{1}{\log |G_p|}}$ equals $e$ and thus is independent of $|G_p|$...)

Therefore, thanks to this high multiplicity trick, proving a spectral gap boils down to establishing rapid decay of the probability of return to the identity in a weaker sense than that of Proposition III.4 from Lecture 3, namely it is enough to establish that

$$\nu^{2n}(e) \ll \frac{1}{|G_p|^{1-\beta}} \tag{7}$$

for some $n \leqslant C \log |G_p|$ and some $\beta > 0$, where $C$ and $\beta$ are constants independent of $p$.

Note that we have not used the girth assumption yet. We will do so now (and will use it one more time towards the end of the argument). This tells us that the Cayley graph looks like a tree (a $2k$-regular homogeneous tree) on any ball of radius $< \tau \log p$ (note that the Cayley graph is vertex transitive, so it looks the same when viewed from any point). In particular the random walk behaves exactly like a random walk on a free group on $k$-generators at least for times $n < \tau \log p$. However, we saw in Lecture 1, that

$$\nu^{2n}(e) \leqslant \rho(\nu)^{2n}$$

for every $n$, where $\rho(\nu)$ is the spectral radius of the random walk. For the simple random walk on a free group $F_k$, the value of the spectral radius is well-known. It is

$$\rho = e^{-C_k} := \frac{\sqrt{2k-1}}{k} < 1,$$

as was computed by Kesten, see [76]. Hence for $n \simeq \tau \log p \simeq \frac{\tau}{3} \log |G_p|$ we have:

$$\nu^{2n}(e) \ll \frac{1}{|G_p|^{\alpha}} \tag{8}$$

where $\alpha = \alpha(\tau) = C_k\tau/3 > 0$.

However $\alpha(\tau)$ will typically be small, and our task is now to bridge the gap between (8), which holds at time $n \simeq \tau \log p$ and (7), which we want to hold before $C \log p$ for some constant $C$ independent of $p$.

Hence we need $\nu^{2n}(e)$ to keep decaying at a certain controlled rate for the time period $\tau \log p \leqslant n \leqslant C \log p$. This decay will be slower than the exponential rate taking place at the beginning thanks to the girth condition, but still significant. And this is where approximate groups come into the game.

(C). **Approximate groups.** Approximate groups were introduced around 2005 by T. Tao, who was motivated both by their appearance in the Bourgain-Gamburd theorem and because they form a natural generalization to the non-commutative setting of the objects studied in additive combinatorics such as finite sets of integers with bounded doubling (i.e. sets $A \subset \mathbb{Z}$ such that $|A+A| \leqslant K|A|$ for some fixed parameter $K \geqslant 1$).

**Definition IV.7.** *(Approximate subgroup) Let $G$ be a group and $K \geqslant 1$ a parameter. A finite subset $A \subset G$ is called a $K$-approximate subgroup of $G$ if the following holds:*

- *$A^{-1} = A$, $1 \in A$,*
- *there is $X \subset G$ with $X = X^{-1}$, $|X| \leqslant K$, such that $AA \subset XA$.*

Here $K$ should be thought as being much smaller than $|A|$. In practice it will be important to keep track of the dependence in $K$. If $K = 1$, then $A$ is the same thing as a finite subgroup. Another typical example of an approximate group is an interval $[-N, N] \in \mathbb{Z}$, or any homomorphic image of it. More generally any homomorphic image of a word ball in the free nilpotent group of rank $r$ and step $s$ is a $C(r,s)$-approximate group (a nilprogression). A natural question regarding approximate groups is to classify them and Tao coined this the "non-commutative inverse Freiman problem" (in honor of G. Freiman who classified approximate subgroups of $\mathbb{Z}$ back in the 60's, see [113]). Recently Breuillard-Green-Tao proved such a classification theorem [18] for arbitrary approximate groups showing that they are essentially built as extensions of a finite subgroup by a nilprogression.

For linear groups and groups of Lie type such as $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ a much stronger classification theorem can be derived:

**Theorem IV.8.** *(Pyber-Szabo* [100]*, Breuillard-Green-Tao* [17]*) Suppose $\mathcal{G}$ is a simple algebraic group of dimension $d$ defined over a finite field $\mathbb{F}_q$ (such as $\mathrm{SL}_n(\mathbb{F}_q)$). Let $A$ be a $K$-approximate subgroup of $\mathcal{G}(\mathbb{F}_q)$. Then*

- *either $A$ is contained in a proper subgroup of $\mathcal{G}(\mathbb{F}_q)$,*
- *or $|A| \leqslant K^C$,*
- *or $|A| \geqslant |\mathcal{G}(\mathbb{F}_q)|/K^C$.*

*where $C = C(d) > 0$ is a constant independent of $q$.*

This result can be interpreted by saying that there are no non-trivial approximate subgroups of simple algebraic groups (disregarding the case when $A$ is contained in a proper subgroup).

Theorem IV.8 was first proved by H. Helfgott [66] for $\mathrm{SL}_2(\mathbb{F}_p)$, $p$ prime, by combinatorial means (using the Bourgain-Katz-Tao *sum-product theorem* [12]). The general case was later established independently by Pyber-Szabo and Breuillard-Green-Tao using tools from algebraic geometry and the structure theory of simple algebraic groups. For a sketch of their argument, we refer the reader to the survey papers [25] and [101].

Let us now go back to the proof of the Bourgain-Gamburd theorem. The connection with approximate groups appears in the following lemma:

**Lemma IV.9.** *($\ell^2$-flattening lemma) Suppose $\mu$ is a probability measure on a group $G$ and $K \geqslant 1$ is such that*

$$||\mu * \mu||_2 \geqslant \frac{1}{K}||\mu||_2.$$

*Then there is a $K^C$-approximate subgroup $A$ of $G$ such that*

- $\mu(A) \gg \frac{1}{K^C}$
- $|A| \ll K^C ||\mu||_2^{-2}$,

*where $C$ and the implied constants are absolute constants.*

For the proof of this lemma, see the original paper of Bourgain-Gamburd [11] or [116, Lemma 15]. It is based on a remarkable graph theoretic lemma, the Balog-Szemeredi-Gowers lemma, which allows one to show the existence of an approximate group whenever we have a set which is an approximate group only in a weak statistical sense. Namely if $A \subset G$ is such that the probability that $ab$ belongs to $A$ for a random choice (with uniform distribution) of $a$ and $b$ in $A$ is larger than say $\frac{1}{K}$, then $A$ has large intersection with some $K^C$-approximate group of comparable size.

The above lemma combined with Theorem IV.8 implies the desired controlled decay of $\nu^{2n}(e)$ in the range $\tau \log p \leqslant n \leqslant C \log p$, namely (recall that $\nu^{2n}(e) = ||\nu^n||_2^2$):

**Corollary IV.10.** *There is a constant $\varepsilon > 0$ such that*

$$||\nu^n * \nu^n||_2 \leqslant ||\nu^n||_2^{1+\varepsilon}$$

*for all $n \geqslant \tau \log p$ and as long as $||\nu^n||_2^2 \geqslant \frac{1}{|G_p|^{1-\frac{1}{10}}}$ say.*

Indeed, if the lower bound failed to hold at some stage, then by the $\ell^2$-flattening lemma, there would then exist an $p^\varepsilon$-approximate subgroup $A$ of $G_p$ of size $< |G_p|^{1-\frac{1}{10}}$ such that $\nu^n(A) \geqslant \frac{1}{p^{C\varepsilon}}$. By the classification theorem, Theorem IV.8, $A$ must be a contained in a proper subgroup of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$. But those all have a solvable subgroup of bounded index. In fact proper subgroups of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ are completely known (see e.g. [11, Theorem 4.1.1] and the references therein) and besides a handful of bounded subgroups, they are contained either in the normalizer of the diagonal subgroup, or in a Borel subgroup (upper triangular matrices). Hence there is 2-step solvable subgroup

$A$ of $G_p$ such that $\nu^n(A) \geqslant \frac{1}{p^{C\varepsilon}}$ for some $n$ between $\tau \log p$ and $C \log p$. But $\nu^n(A)$ is essentially non-increasing, that is $\nu^n(A) = \sum_x \nu^m(x^{-1})\nu^{n-m}(xA) \leqslant \max \nu^{n-m}(xA)$ and so $\nu^{2(n-m)}(A) \geqslant \nu^{n-m}(xA)^2 \geqslant (\nu^n(A))^2 \geqslant \frac{1}{p^{2C\varepsilon}}$ for all $m$. In particular there is $n_0 = n - m < \frac{\tau}{10} \log p$ for which $\nu^{n_0}(A) \geqslant \frac{1}{p^{C\varepsilon}}$. However at time $n_0$, we are before the girth bound and the random walk is still in the tree. But in a free group the only 2-step solvable subgroups are cyclic subgroups, so subsets of elements whose second commutator vanish must in fact commute and thus be contained in a cyclic subgroup: they occupy a very tiny part of the free group ball of radius $n_0$. This contradicts the lower bound $\frac{1}{p^{C\varepsilon}}$. See [11, Lemma 3] for more details.

The proof is now complete as we have now a device, namely Corollary IV.10, to go from (8) to (7) by applying this upper bound iteratively a bounded number of times. This ends the proof of Theorem IV.3 and we are done.

(D). **Random generators and the uniformity conjecture.** The Bourgain-Gamburd method has been used and refined by many authors in the past few years. For example, it is powerful enough to prove that a random Cayley graph of $\mathrm{SL}_2(\mathbb{Z}/p\mathbb{Z})$ is expanding. This is already contained in the original paper by Bourgain and Gamburd [11]. Recently the method has been pushed to yield the following:

**Theorem IV.11.** *(Random Cayley graphs, Breuillard-Green-Guralnick-Tao* [20]*) Given $k \geqslant 2$ and $d \geqslant 1$, there is $\varepsilon, \gamma > 0$, such that the probability that $k$ elements chosen at random in $\mathcal{G}(\mathbb{F}_q)$ generate $\mathcal{G}(\mathbb{F}_q)$ and turn it into an $\varepsilon$-expander is at least* $1 - O(\frac{1}{|\mathcal{G}(\mathbb{F}_q)|^\gamma})$*. Here $\mathcal{G}$ is any simple algebraic group of dimension at most $d$ over* $\mathbb{F}_q$.

In particular, this yields another proof of Lubotzky's result in [84], which produced an expanding generating set of fixed size in $\mathrm{PSL}_2(\mathbb{F}_q)$. The proof of Theorem IV.11 follows the Bourgain-Gamburd method outlined above via Theorem IV.8. Besides the classification of approximate groups, the main new difficulty compare to the $\mathrm{SL}_2$ case is to prove that the random walk does not concentrate too much on proper subgroups. Proving the non-concentration estimate (i.e. the final stage of the Bourgain-Gamburd method) requires showing that certain word varieties are non trivial in $\mathcal{G}$ and this is performed by establishing the existence of *strongly dense* free subgroups of $\mathcal{G}(\overline{\mathbb{F}_p})$, namely free subgroups all of whose non-abelian subgroups are Zariski-dense, see [21]

Theorem IV.11 can be seen as an approximation towards the following conjecture:

**Conjecture IV.12.** *(Uniformity conjecture) Given $k \geqslant 2$ and $d \geqslant 1$, there is $\varepsilon > 0$, such that, for every prime power $q$ and every simple algebraic group of dimension at most $d$ over $\mathbb{F}_q$, all $k$-regular Cayley graphs of $\mathcal{G}(\mathbb{F}_q)$ are $\varepsilon$-expander graphs.*

In other words the Lubotzky-Weiss independence problem mentioned at the beginning of this lecture is expected to have an affirmative answer in the case of bounded rank finite simple groups. And indeed all examples so far of sequences of finite simple groups with both expanding and non-expanding generating sets have unbounded rank.

The only progress to date towards the above is in [14], where it is shown how the uniform Tits alternative (Theorem II.15) can be used to prove that the family $\{\mathrm{SL}_2(\mathbb{F}_p)\}_{p\in\mathcal{P}}$ for some infinite family $\mathcal{P}$ (indeed of density one) of prime numbers $p$, is uniformly expanding.

(E). **Super-strong approximation.** The Bourgain-Gamburd method has also been very successful in establishing the Selberg property (i.e. property $(\tau)$ with respect to congruence subgroups) for new examples of finitely generated linear groups. In particular all *thin groups*, that is discrete Zariski-dense subgroups $\Gamma$ of semisimple Lie groups $\mathcal{G}$ which are not lattices, are expected to have the Selberg property. One speaks of *super-strong approximation*, in reference to Theorem IV.5, because not only are the congruence quotients of $\Gamma$ generating those of $\mathcal{G}$, but their associated Cayley graphs are expanders. This is still conjectural in full generality, but here is a representative example of what is known:

**Theorem IV.13.** *(Super-strong approximation, Bourgain-Varju [13]) If $\Gamma \leqslant \mathrm{SL}_d(\mathbb{Z})$ is a Zariski-dense subgroup, then it has property $(\tau)$ with respect to the family of congruence subgroups $\Gamma \cap \ker(\mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/n\mathbb{Z}))$, where $n$ is an arbitrary integer.*

This theorem can be viewed as a vast generalization of Selberg's theorem, and indeed it gives a different proof (via the Brooks-Burger dictionary mentioned in Lecture 3) of the uniform spectral gap for the first eigenvalue of the Laplacian on the congruence covers of the modular surface $\mathbb{H}^2/\mathrm{SL}_2(\mathbb{Z})$ (although not such a good bound as $\frac{3}{16}$ of course). Despite its resemblance with Corollary IV.4, the proof of this theorem is much more involved, in particular the passage from $n$ prime to arbitrary $n$ requires much more work. See already Varju's thesis [116] for the special case of square free $n$. In a similar spirit, one has the following extension for perfect groups:

**Theorem IV.14.** *(Salehi-Golsefidy, Varju [104]) A finitely generated subgroup of $\mathrm{GL}_d(\mathbb{Q})$ has property $(\tau)$ with respect to congruence quotients modulo square free integers if and only if the connected component of its Zariski closure is perfect.*

This result has had several interesting applications to sieving in orbits (e.g. [10]) and other counting problems in groups (e.g. [85]). We refer the reader to the articles in the recent MSRI proceedings volume devoted to super-strong approximation, starting by Sarnak's overview [106] for more information on these recent developments.

## V. Appendix: The Brooks-Burger transfer

The goal of this appendix is to give a derivation of the Brooks-Burger transfer principle, which relates the first eigenvalue of the Laplacian on a cover of a compact manifold $M$ with the first eigenvalue of the combinatorial Laplacian on the Cayley-Schreier graph associated to this cover via a fixed choice of generators of the fundamental group of $M$. A similar statement holds for the Cheeger constant and we will briefly sketch the proof of that as well.

Let $M$ be a compact connected Riemannian manifold. Let $\Gamma$ be the fundamental group of $M$ defined with respect to some base point. It acts by isometries on the universal cover $\widetilde{M}$. We denote by $\mathcal{F}$ the Dirichlet fundamental domain based at $p_0 \in \widetilde{M}$ for the $\Gamma$ action on $\widetilde{M}$, namely

$$\mathcal{F} := \{x \in \widetilde{M}; d(x, p_0) < d(x, \gamma \cdot p_0), \text{for all } \gamma \in \Gamma\}.$$

If $p = \gamma \cdot p_0$ is in the $\Gamma$-orbit of $p_0$, then we denote by $\mathcal{F}(p) = \gamma \cdot \mathcal{F}$ the corresponding Dirichlet fundamental domain based at $p$.

Associated to this fundamental domain is the finite symmetric (i.e. $S = S^{-1}$) set $S$ of those elements $s \in \Gamma$ such the distance between $\overline{\mathcal{F}(s \cdot p_0)}$ and $\overline{\mathcal{F}(p_0)}$ in $\widetilde{M}$ is at most equal to the diameter of $M$. It is a generating set for $\Gamma$.

We denote by $\mathcal{G}(\Gamma, S)$ the associated Cayley graph. Given a subgroup $\Gamma' \leqslant \Gamma$, we let $\mathcal{G}(\Gamma/\Gamma', S)$ be the corresponding Cayley-Schreier graph (a quotient of $\mathcal{G}(\Gamma, S)$).

**Theorem V.1.** *There are constants $c_1, \ldots, c_4 > 0$ depending on $M$ only, such that for every finite degree cover $M'$ of $M$, with fundamental group $\Gamma' \leqslant \Gamma$, we have*

$$c_1 \lambda_1(M') \leqslant \lambda_1(\mathcal{G}(\Gamma/\Gamma', S)) \leqslant c_2 \lambda_1(M'), \tag{9}$$

*and similarly for the Cheeger constant:*

$$c_3 h(M') \leqslant h(\mathcal{G}(\Gamma/\Gamma', S)) \leqslant c_4 h(M'). \tag{10}$$

(A). **Discussion.** The Cheeger constant result is due to Brooks, while the Laplace eigenvalue statement is essentially due to Burger. Although all of the ideas to prove (10) are present in [26, 28] the statement first appears in [29]. As for (9) it is part of Marc Burger's EPFL thesis [31], where a proof of the lower bound for $\lambda_1(M')$ can be found (see also [33]). Burger also proved the upper bound for $\lambda_1(M')$ in (9) in the special case when $M$ is a rank-one locally symmetric space using the harmonic analysis of spherical functions (see [32]). In the generality of Theorem V.1 the inequalities (9) were first stated in [30] and the argument given below for the lower bound follows a suggestion from [30]. We hope that this appendix will help record these arguments in one place and give the right amount of detail.

Note that statements (9) and (10) are not immediately equivalent: indeed the Cheeger-Buser inequalities (see III.2) relating $h$ and $\lambda_1$ are not strong enough: it allows to upper bound $\lambda_1$ by (a constant times) $h$, but the lower bound is in terms of $h^2$, and could therefore be much smaller than $h$. What the Cheeger-Buser inequalities allow you to deduce from statement (10) is the fact that $\lambda_1(M'_n)$ tends to zero if and only if $\lambda_1(\mathcal{G}(\Gamma/\Gamma'_n, S))$ tends to zero for any sequence of finite covers $M'_n$ with fundamental groups $\Gamma_n$ whose index in $\Gamma$ grow to infinity. This is the version stated by Lubotzky and Zimmer in [87] and in Lubotzky's book [82, Theorem 4.3.2].

From a philosophical point of view, Theorem V.1 is not very surprising, because it is clear that the large scale geometry of a finite cover $M'$ of high degree of $M$ is basically governed by that of the associated Cayley-Schreier graph, which serves the purpose of a skeleton for the covering space $M'$. What is a bit less clear is that we have this nice control of the quantities by fixed multiplicative constants.

The constants $c_i$'s are effective and we will give bounds on them, which can be explicitly computed in terms of the geometry and spectrum of the fundamental domain $\mathcal{F}$ only. Note in particular that if one rescales the metric on the base manifold $M$ by a factor $T$, then $\lambda_1(M)$ is changed into by a factor $\frac{1}{T^2}$. This affects the constants $c_i$'s accordingly, while the Cayley-Schreier graph $\mathcal{G}(\Gamma'\backslash\Gamma)$ remains unchanged. We may thus assume without loss of generality that the diameter of $M$ is equal to 1 say. See the discussion after the proof.

We also note that the choice of the generating set in the above statement is somewhat arbitrary and it remains valid (albeit with different constants $c_i$'s) for any other finite symmetric generating set. See the discussion after the proof.

(B). **Notation.** We denote by $||\cdot||_2$ the $L^2$ norm on $M$, $M'$ or $\Gamma/\Gamma'$ alike. The Rayleigh quotient of a function $f$ on $M'$ is the quantity $\frac{||\nabla f||_2^2}{||f||_2^2}$. Recall that the first eigenvalue of the Laplace operator on $M'$ admits the following variational characterization:

$$\lambda_1(M') := \inf\{\frac{||\nabla f||_2^2}{||f||_2^2}, \int_{M'} f = 0\}$$

Moreover there exists a non zero eigenfunction of the Laplace operator on $M'$ for the eigenvalue $\lambda_1(M')$ and its Rayleigh quotient realizes the above infimum.

The same holds for functions on the Cayley-Schreier graph $\mathcal{G}(\Gamma/\Gamma', S)$, namely (see Lecture 3)

$$\lambda_1(\mathcal{G}(\Gamma/\Gamma', S)) := \inf\{\frac{||\nabla F||_2^2}{||F||_2^2}, \sum_{x\in\Gamma/\Gamma'} f(x) = 0\}$$

Here the nabla sign $\nabla$ turns a function $F$ on $\Gamma/\Gamma'$ into a function on the edges of the Cayley-Schreier graph, namely $\nabla F(e) = |F(p) - F(q)|$, if $e$ is an edge with end points $p$ and $q$.

Let *vol* be the Riemannian measure on $M$ and $\widetilde{M}$. For $p = \gamma \cdot p_0$, $\gamma \in \Gamma$, a point in the orbit of the base point $p_0 \in \widetilde{M}$, let $\mathcal{F}(p) = \gamma \cdot \mathcal{F}(p_0)$ be the Dirichlet fundamental domain based at $p$. The $\mathcal{F}(p)$ are all isometric to each other. They are connected (even star-shaped around $p$) open submanifolds of $\widetilde{M}$ with piecewise smooth boundary of measure zero. They form a tesselation of the universal cover $\widetilde{M}$.

We denote $V$ the volume of $M$ and by $N_V$ the valency of the Cayley-Schreier graph, namely the cardinality of the generating set $S$.

(C). **Proof the lower bound for $\lambda_1(M')$.** We use a variant of Burger's argument given in [33]. For a Lipschitz function $f$ on $M'$ with zero average, we need to build a function $F$ on $\Gamma'\backslash\Gamma$, with zero average and with comparable Rayleigh quotient. This is done by setting:

$$F(p) = \frac{1}{vol(\mathcal{F})} \int_{\mathcal{F}(p)} f.$$

Obviously $\sum_p F(p) = 0$. We need to upper bound $||\nabla F||_2$ in terms of $||\nabla f||_2$. For this we introduce the first Neumann eigenvalue $\mu_1(\mathcal{F})$ of the fundamental domain $\mathcal{F} \subset \widetilde{M}$. It has the following variational characterization (see [39, Chapter 1])

$$\mu_1(\mathcal{F}) := \inf\{\frac{\int_{\mathcal{F}} |\nabla u|^2}{\int_{\mathcal{F}} u^2}; \int_{\mathcal{F}} u = 0\}, \tag{11}$$

where $u$ is an arbitrary Lipschitz function on $\overline{\mathcal{F}}$ (in particular it does not have to descend to a continuous function on the base manifold $M$).

The basic idea of Burger's argument is that if $\lambda_1(M')$ were very small, in particular much smaller than $\mu_1(\mathcal{F})$, then any eigenfunction $f$ of the Laplace operator on $M'$ with eigenvalue $\lambda_1(M')$ would be almost constant on each one of the fundamental domains $\mathcal{F}(p)$, making $F$ and $f$ very close. Note that these are all isometric to $\mathcal{F}$. We now formalize this idea and pass to the details.

Given a pair of adjacent vertices $p \sim q$ in the Cayley of $\Gamma$, the union of the two associated fundamental domains $\mathcal{F}(p) \cup \mathcal{F}(q)$ inside the universal cover $\widetilde{M}$ is a bounded connected subset with piecewise smooth boundary. Hence its first Neumann eigenvalue $\mu_1(\mathcal{F}(p) \cup \mathcal{F}(q))$ is positive. Set

$$\mu := \min_{q \sim p}\{\mu_1(\mathcal{F}(p)), \mu_1(\mathcal{F}(p) \cup \mathcal{F}(q))\} > 0$$

Since $\Gamma$ acts transitively on the fundamentals domains $\mathcal{F}(p)$ in $\widetilde{M}$, the quantity $\mu$ just defined is independent of $p$.

We now choose an eigenfunction $f$ for the Laplace operator on $M'$ with eigenvalue $\lambda_1(M')$. From the variational characterization of the first Neumann eigenvalue we have the following Poincaré inequality:

$$\int_{\mathcal{F}(p)} (f - F(p))^2 \leqslant \frac{1}{\mu} \int_{\mathcal{F}(p)} |\nabla f|^2$$

and

$$\int_{\mathcal{F}(p) \cup \mathcal{F}(q)} (f - \frac{F(p) + F(q)}{2})^2 \leqslant \frac{1}{\mu} \int_{\mathcal{F}(p) \cup \mathcal{F}(q)} |\nabla f|^2$$

where $\mathcal{F}(p)$ is now considered inside $M'$. Hence, writing $(\frac{F(p)-F(q)}{2})^2 \leqslant 2((f-F(p))^2 + (f-\frac{F(p)+F(q)}{2})^2)$ on $\mathcal{F}(p)$ and similarly on $\mathcal{F}(q)$, we get (recall $V = vol(M)$)

$$V(\frac{F(p)-F(q)}{2})^2 \leqslant \frac{2}{\mu}\int_{\mathcal{F}(p)\cup\mathcal{F}(q)} |\nabla f|^2$$

hence taking squares and summing over neighbors

$$V \cdot \sum_{p\sim q} |F(p)-F(q)|^2 \leqslant \frac{16N_V}{\mu}\int_{M'} |\nabla f|^2, \tag{12}$$

where $N_V$ is the valency of the graph. On the other hand we may decompose $f$ orthogonally on each $\mathcal{F}(p)$ as

$$\int_{\mathcal{F}(p)} f^2 = V \cdot F(p)^2 + \int_{\mathcal{F}(p)} \left(f - F(p)\right)^2$$

hence, summing over $p$

$$\big|V||F||_2^2 - ||f||_2^2\big| \leqslant \frac{1}{\mu}||\nabla f||_2^2 = \frac{\lambda_1(M')}{\mu}||f||_2^2. \tag{13}$$

In particular $F$ is not identically zero if $\lambda_1(M') < \mu$. Combining (13) and (12), we get

$$\frac{||\nabla F||_2^2}{||F||_2^2} \leqslant \frac{16N_V}{\mu}\frac{\lambda_1(M')}{1-\frac{\lambda_1(M')}{\mu}}.$$

Therefore if $\lambda_1(M') \leqslant \frac{\mu}{2}$, we obtain the desired bound $\lambda_1(\mathcal{G}(\Gamma/\Gamma', S)) \leqslant c_2\lambda_1(M')$ with $c_2 = \frac{32N_V}{\mu}$. On the other hand, if $\lambda_1(M') \geqslant \frac{\mu}{2}$, then, since at any case $\lambda_1(\mathcal{G}(\Gamma/\Gamma', S)) \leqslant 2N_V$, we obviously have the desired bound in that case too.

(D). **Proof the upper bound for $\lambda_1(M')$.** Starting with any function $f$ on the vertex set $\Gamma/\Gamma'$ with zero average, we need to build a function $F$ on $M'$ with zero average and comparable Rayleigh quotient.

Given $\varepsilon > 0$, let $\mathcal{C}_{M,\varepsilon}(p)$ be the set of points $x \in \mathcal{F}(p)$ such that the distance between $x$ and the complement $(\mathcal{F}(p))^c$ is at least $\varepsilon$. As $\varepsilon \to 0$, the measure $vol(\mathcal{C}_{M,\varepsilon}(p))$ tends to $V := vol(\mathcal{F}(p))$. Let $\mathcal{F} = \mathcal{F}(p_0)$ for some base point $p_0$.

Without loss of generality we may normalize the Riemannian metric on $M$ so that the diameter of $M$ is equal to 1.

Let $f$ be a $\Gamma'$-invariant function on $\Gamma$ with zero average on $\Gamma'\backslash\Gamma$. We now define a $\Gamma'$-invariant Lipschitz continuous function $F_\varepsilon$ on $\widetilde{M}$ by setting its value on each $\mathcal{C}_{M,\varepsilon}(p)$ to be $f(p)$ and by filling in using a weighted average as follows:

$$F_\varepsilon(x) = \frac{1}{\sum_p d_p(x)}\sum_p f(p)d_p(x), \tag{14}$$

where $d_p(x) = \frac{(\frac{1}{2}-d(x,\mathcal{C}_{M,\varepsilon}(p)))^+}{d(x,\mathcal{C}_{M,\varepsilon}(p))}$ when $x$ lies outside $\mathcal{C}_{M,\varepsilon}(p)$ (we denoted $y^+ := \max\{0, y\}$). The function $F$ above is defined outside the union of all $\mathcal{C}_{M,\varepsilon}(p)$'s but it clearly extends by continuity to a continuous function on all of $\widetilde{M}$ equal to $f(p)$ inside each $\mathcal{C}_{M,\varepsilon}(p)$.

Observe finally that in the sum defining $F_\varepsilon$ at $x$, only a bounded number of terms are non-zero, namely the number of tiles intersecting the ball of radius $\frac{1}{2}$ at $x$, and this number is bounded independently of $x$. It is clear that $F_\varepsilon$ is $\Gamma'$-invariant.

We now choose $\varepsilon > 0$. The crucial point is that $\varepsilon$ has to be chosen independently of the cover $M'$. We do so by picking $\varepsilon \in (0, \frac{1}{4})$ small enough so that

$$vol(\mathcal{C}_{M,\varepsilon}) - N_V vol(\mathcal{F} \setminus \mathcal{C}_{M,\varepsilon}) \geqslant \frac{1}{2} vol(\mathcal{F}) \tag{15}$$

where $N_V$ (the valency of our Cayley-Schreier graph) is the number of fundamental domains $\mathcal{F}(p)$ containing a point at distance at most 1 from $\mathcal{F}$. This is possible since $vol(\mathcal{F} \setminus \mathcal{C}_{M,\varepsilon})$ tends to zero as $\varepsilon \to 0$.

**Lemma V.2.** *Setting $G_\varepsilon = F_\varepsilon - \frac{1}{vol(M')}\int_{M'} F_\varepsilon$ the associated zero mean function on $M'$, we have*

$$||G_\varepsilon||^2_{L^2(M')} \geqslant \frac{1}{2}||f||^2 vol(\mathcal{F}), \tag{16}$$

$$|\nabla G_\varepsilon(x)| \leqslant \frac{4}{\varepsilon}\sum_{p,q} \mathbf{1}_{d(x,\mathcal{F}(p))\leqslant\frac{1}{2}} \mathbf{1}_{d(x,\mathcal{F}(q))\leqslant\frac{1}{2}} |f(p) - f(q)| \tag{17}$$

Before proving the above lemma, let us explain first how to conclude the proof of the left hand side inequality in part (2) of Theorem V.1. Let $N \leqslant N_V$ be the maximal number of fundamental domains intersecting a ball of radius $\frac{1}{2}$ in $\widetilde{M}$. In the second displayed equation above, for any given $x$, the sum is restricted to at most $N^2$ couples $(p, q)$. Moreover these couples are neighbors ($p \sim q$) in the Cayley-Schreier graph, because $\mathcal{F}(q)$ has a point at distance at most 1 from $\mathcal{F}(p)$. Hence by Cauchy-Schwarz:

$$|\nabla G_\varepsilon(x)|^2 \leqslant \frac{16}{\varepsilon^2} N^2 \sum_{p\sim q} |f(p) - f(q)|^2 \mathbf{1}_{d(x,\mathcal{F}(p))\leqslant\frac{1}{2}} \mathbf{1}_{d(x,\mathcal{F}(q))\leqslant\frac{1}{2}}$$

Integrating we obtain:

$$||\nabla G_\varepsilon||^2_{L^2(M')} \leqslant \frac{16}{\varepsilon^2} N^2 \sum_{p\sim q} |f(p) - f(q)|^2 N_V vol(\mathcal{F}) = \frac{8N^2 N_V}{\varepsilon^2} vol(\mathcal{F})||\nabla f||^2_2$$

From these two estimates, it readily follows that

$$\frac{||\nabla G_\varepsilon||^2_{L^2(M')}}{||G_\varepsilon||^2_{L^2(M')}} \leqslant \frac{1}{c_1} \frac{||\nabla f||^2_{L^2(\Gamma'\backslash\Gamma)}}{||f||^2_{L^2(\Gamma'\backslash\Gamma)}}$$

where $c_1 := \frac{\varepsilon^2}{16 N_V N^2}$. We thus obtain the desired lower bound $c_1\lambda_1(M') \leqslant \lambda_1(\mathcal{G}(\Gamma/\Gamma', S))$.

It only remains to prove the lemma.

*Proof of Lemma V.2.* Consider the first estimate. Since $f$ has zero average on $\Gamma'\backslash\Gamma$ we have

$$\int_{M'} F_\varepsilon = \int_{M' \setminus \cup_p \mathcal{C}_{M,\varepsilon}(p)} F_\varepsilon$$

and thus

$$|\int_{M'} F_\varepsilon| \leqslant \int_{M' \setminus \cup_p \mathcal{C}_{M,\varepsilon}(p)} \max\{f(p), d(x, \mathcal{F}(p)) \leqslant \frac{1}{2}\}$$

which becomes by Cauchy-Schwarz

$$\frac{1}{vol(M')}\Big(\int_{M'} F_\varepsilon\Big)^2 \leqslant \frac{vol(M' \setminus \cup_p \mathcal{C}_{M,\varepsilon}(p))}{vol(M')} \int_{M'} \sum_{d(x,\mathcal{F}(p)) \leqslant \frac{1}{2}} f(p)^2 \leqslant vol(\mathcal{F} \setminus \mathcal{C}_{M,\varepsilon})||f||_2^2 N_V$$

where $N_V$ is the number of fundamental domains $\mathcal{F}(p)$ at distance at most 1 from $\mathcal{F}$.

On the other hand clearly

$$||F_\varepsilon||^2 \geqslant \sum_p f(p)^2 vol(\mathcal{C}_{M,\varepsilon}(p)) = vol(\mathcal{C}_{M,\varepsilon})||f||_2^2$$

So combining the last inequality with (18) we obtain:

$$||G_\varepsilon||_2^2 = ||F_\varepsilon||^2 - \frac{1}{vol(M')}\Big(\int_{M'} F_\varepsilon\Big)^2 \geqslant \frac{1}{2} vol(\mathcal{F})||f||_2^2$$

as desired, as soon as

$$vol(\mathcal{C}_{M,\varepsilon}) - N_V vol(\mathcal{F} \setminus \mathcal{C}_{M,\varepsilon}) \geqslant \frac{1}{2} vol(\mathcal{F}).$$

This yields the first estimate in Lemma V.2 and we now turn to the second estimate. We compute $\nabla G_\varepsilon = \nabla F_\varepsilon$ as

$$\begin{aligned} \nabla F_\varepsilon &= \frac{1}{(\sum_p d_p)^2}\big((\sum_p f(p)\nabla d_p)(\sum_p d_p) - (\sum_p f(p) d_p)(\sum_p \nabla d_p)\big) \\ &= \frac{1}{(\sum_p d_p)^2} \sum_{p,q} (f(p) - f(q))(\nabla d_p) d_q \end{aligned}$$

From the definition of $d_p(x)$ it is a simple matter to verify the following bound from which the desired estimate in Lemma V.2 follow directly:

$$|\frac{d_q \nabla d_p}{\big(\sum_m d_m\big)^2}| \leqslant \frac{4}{\varepsilon} \mathbf{1}_{d(x,\mathcal{F}(p)) \leqslant \frac{1}{2}} \mathbf{1}_{d(x,\mathcal{F}(q)) \leqslant \frac{1}{2}}. \tag{18}$$

Indeed first note that

$$|\nabla d_p| \leqslant \frac{\mathbf{1}_{d(x,\mathcal{C}_{M,\varepsilon}(p)) \leqslant \frac{1}{2}}}{2d(x, \mathcal{C}_{M,\varepsilon}(p))^2}$$

While if $d(x, \mathcal{C}_{M,\varepsilon}(p)) \leqslant \varepsilon$, then $d(x, \mathcal{C}_{M,\varepsilon}(q)) \geqslant \varepsilon$ for every $q \neq p$, and thus $d_q(x) \leqslant \frac{1}{2\varepsilon}$. On the other hand $\sum_m d_m(x) d(x, \mathcal{C}_{M,\varepsilon}(p)) \geqslant d_p(x) d(x, \mathcal{C}_{M,\varepsilon}(p)) \geqslant \frac{1}{2} - \varepsilon$ and putting this together yields (18), when $d(x, \mathcal{C}_{M,\varepsilon}(p)) \leqslant \varepsilon$.

If on the other hand $d(x, \mathcal{C}_{M,\varepsilon}(p)) \geqslant \varepsilon$, then $|\nabla d_p| \leqslant \frac{1}{2\varepsilon^2}$. But every $x \in M' \setminus \cup_m \mathcal{C}_{M,\varepsilon}(m)$ belongs to at least one $\overline{\mathcal{F}(m)}$, and hence has $d_m(x) \geqslant \frac{\frac{1}{2}-\varepsilon}{\varepsilon} \geqslant \frac{1}{4\varepsilon}$, as $\varepsilon < \frac{1}{4}$. So we always have $\sum_m d_m \geqslant \frac{1}{4\varepsilon}$. It follows that

$$|\frac{d_q \nabla d_p}{\left(\sum_m d_m\right)^2}| \leqslant \frac{1}{2\varepsilon^2} |\frac{d_q}{\left(\sum_m d_m\right)}| 4\varepsilon \leqslant \frac{2}{\varepsilon}.$$

So we do get (18) in all cases and this ends the proof the lemma and of part (2) of Theorem V.1. □

(E). **Dependence of the constants on the geometry of $M$.** The constants $c_i$'s, $i = 1, \ldots, 4$ in Theorem V.1 depend only on the geometry of the fundamental domain $\mathcal{F}(p)$. Recall that for $c_1$ and $c_2$, we had found:

$$c_1 = \frac{\varepsilon^2}{16 N_V^3}, c_2 = \frac{32 N_V}{\mu},$$

Here $N_V$ is the number of fundamental domains at distance at most 1 from the fundamental domain $\mathcal{F}(p_0)$ associated to a base point $p_0$. We denoted by $\mu$ a positive lower bound for the non-zero Neumann eigenvalues of $\mathcal{F}(p)$ and $\mathcal{F}(p) \cup \mathcal{F}(q)$ as defined by (11). The constant $\varepsilon = \varepsilon(M)$ is defined by (15).

Due to Gromov's compactness theorem (see e.g. [58] and [97]) the set of Riemannian metrics on a compact manifold $M$ with bounded diameter, bounded curvature and a lower bound on the injectivity radius, is pre-compact. As a consequence there is a uniform bound $C = C(n, D, \kappa, r) > 0$ such that if $M$ is a compact $n$-dimensional Riemannian manifold with diameter at most $D$, sectional curvature $|K_M| \leqslant \kappa$ and injectivity radius at least $r > 0$, then the constants $c_i$'s from Theorem V.1 lie in $[\frac{1}{C}, C]$.

Using the standard comparison theorems in Riemannian geometry, it is easy to get an explicit control of $N_V$ in terms of the parameters $(n, D, \kappa, r)$. For $\mu$ one can use Cheeger's inequality (cf. [36]) and the estimate of the Cheeger constant for Dirichlet domains obtained in [37, Lemma 5.1.]. Controlling $\varepsilon$ explicitly seems a bit more challenging however (although note that one is allowed to regularize the boundary of $\mathcal{F}$ without altering the associated graph.)

(F). **A sketch of the proof of the Cheeger constant inequalities.** For the bound $c_3 h(M') \leqslant h(\mathcal{G}(\Gamma/\Gamma', S))$, take a set $A$ of at most half of the vertices of the graph $\mathcal{G}(\Gamma/\Gamma', S)$ which almost realizes the combinatorial Cheeger constant, i.e. $|\partial A_0|/|A_0| \leqslant 2h(\mathcal{G}(\Gamma/\Gamma', S))$ say. Then consider the hypersurface defined as the boundary of the union of the Dirichlet fundamental domains $\mathcal{F}(p)$ with $p \in A_0$. The area of the hypersurface is clearly bounded by $|\partial A_0| vol_{n-1}(\partial \mathcal{F})$, while the volume enclosed is $|A_0| vol(\mathcal{F})$. This yield the desired inequality with $c_3 = \frac{vol_{n-1}(\partial \mathcal{F})}{2 vol(\mathcal{F})}$.

The proof of the opposite inequality is more delicate. The difficulty (dubbed "the problem of hairs" in [37]) is that we have little information on the hypersurfaces that may realize or almost realize the Cheeger constant: in particular there is no guarantee that the hypersurface does not intersect every single fundamental domain $\mathcal{F}(p)$.

There are two possible strategies to overcome this difficulty. The first is the one chosen by Brooks [28], p100–102. It consists in looking for a minimizing hypersurface for the Cheeger constant. Typically no smooth minimizer exists, but one can use a non-trivial result from geometric measure theory according to which there is a minimizing integral current $T$ with some strong regularity properties and constant mean curvature, such that the Cheeger constant is realized for $T$. The fact that the (Ricci or sectional) curvature of the covers $M'$ are uniformly bounded implies that the mean curvature of the current is uniformly bounded. In turn this implies that the intersection of $T$ with any fixed small ball has controlled area, and thus the area of $T$ is controlled by the number of domains $\mathcal{F}(p)$ intersecting it, and we are done.

The second strategy avoids the use of currents and geometric measure theory and instead uses the standard comparison theorems in Riemannian geometry. The idea is due to Buser [37, Sec. 4] who used it for a slightly different purpose. We consider an almost minimizing smooth hypersurface $X = \partial A = \partial B$ separating $M'$ into two disjoint connected pieces $A$ and $B$, with $vol_{n-1}(X) \leqslant 2h(M') \min\{vol(A), vol(B)\}$ say. And we modify it by setting

$$\widetilde{A} := \{x \in M'; vol(A \cap B(x,r)) > \frac{1}{2} vol(B(x,r))\}$$

$$\widetilde{B} := \{x \in M'; vol(A \cap B(x,r)) < \frac{1}{2} vol(B(x,r))\},$$

where $r > 0$ is a number defined a posteriori. The two sets are again disjoint, and their boundary is $\widetilde{X} = \{x \in M'; vol(A \cap B(x,r)) = \frac{1}{2} vol(B(x,r))\}$. Now pick a maximal $r$-separated set $\{x_i\}_i$ in $\widetilde{X}$ and write

$$\begin{aligned} vol(X \cap B(x_i,r)) &\geqslant h(B(x_i,r)) \min\{vol(A \cap B(x_i,r), vol(B \cap B(x_i,r))\} \\ &\geqslant \frac{h(B(x_i,r))}{2} vol(B(x_i,r)), \end{aligned}$$

where $h(B(x_i,r))$ is the Cheeger constant of this ball, which can be bounded below using the bounded curvature assumption ([37, Lemma 5.1]). Summing over $i$, one gets

$$vol(\widetilde{X}_{2r}) \ll_r h(M') \min\{vol(A), vol(B)\}.$$

To conclude, let $C_0$ (resp. $A_0$, $B_0$) be the subset of fundamental domains $\mathcal{F}(p)$ which intersect $\widetilde{X}_{2r}$ non trivially (resp. are contained in $\widetilde{A} \setminus \widetilde{X}_{2r}$, $\widetilde{B} \setminus \widetilde{X}_{2r}$,). Then $|C|$ ($|A_0|$, $|B_0|$) are controlled by $vol(\widetilde{X}_{2r})$ (resp. $vol(A)$, $vol(B)$). The three sets are disjoint, in particular $|\partial A_0|, |\partial B_0| \leqslant N_V |C_0|$ and the result follows. For more details, see [37, Sec. 4].

(G). **Additional remarks.**

**Remark V.3** (On the precise definition of the Schreier graph)**.** *There is a certain amount of indeterminacy in the very definition of a Schreier graph of a finite quotient*

*$\Gamma/\Gamma'$. We chose to define it as the quotient of the Cayley graph associated to $S$ with respect to the action of $\Gamma'$. This produces a graph which may have some loops and double edges (note that already the Cayley graph may have double edges). However we may just as well consider the graph obtained from this one by removing all loops and keeping only one edge in case of multiple edges. The resulting graph is then a graph in the common sense of the word. The inequalities of Theorem V.1 remain valid (albeit with slightly different constants) for this new graph. In the case of the Laplace eigenvalues inequalities, this is because the Dirichlet forms $D(f,g) = \sum_{p\sim q} |f(q) - f(p)|^2$ of the two graphs are comparable up to multiplicative constants. And hence $\frac{1}{C}\lambda_1(\mathfrak{g}) \leqslant \lambda_1(\mathfrak{g}') \leqslant C\lambda_1(\mathfrak{g})$ for some $C > 0$, where $\mathfrak{g}$ and $\mathfrak{g}'$ are the old and new graph. Similarly one has $\frac{1}{C}h(\mathfrak{g}) \leqslant h(\mathfrak{g}') \leqslant Ch(\mathfrak{g})$ for the Cheeger constant.*

**Remark V.4** (Change of generating set). *In the statement of Theorem V.1, we could have taken any other* fixed *generating set $S$ for the Cayley graph of $\Gamma$ at the expense of modifying the constants $c_i$'s. If $S'$ is another generating set, then there is an integer $N$ such that $S' \subset S^N$ and also $S \subset S'^N$. This implies that the Dirichlet forms $D(f,g) = \sum_{p\sim q} |f(q) - f(p)|^2$ associated to two the Cayley-Schreier graphs $\mathcal{G}(\Gamma'\backslash\Gamma)$ relative to $S$ and $S'$ are comparable up to multiplicative constants independent of $\Gamma'$, and hence so are their $\frac{1}{C}\lambda_1(\mathcal{G}(\Gamma'\backslash\Gamma, S) \leqslant \lambda_1(\mathcal{G}(\Gamma'\backslash\Gamma, S') \leqslant C\lambda_1(\mathcal{G}(\Gamma'\backslash\Gamma, S)$, where $C = C(S,S') > 0$ is independent of $\Gamma'$.*

*It is worth pointing out however that one must keep the same generating for all finite index subgroups $\Gamma'$, that is the generating set cannot be allowed to vary with the finite index subgroup $\Gamma'$ without violating the uniformity of the constants $c_i$'s. See the discussion of uniform property $(T)$ and $(\tau)$ in Lecture 3.*

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Laboratoire de Mathématiques, Bâtiment 425, Université Paris Sud 11, 91405 Orsay, FRANCE

*E-mail address*: emmanuel.breuillard@math.u-psud.fr