Semi-abelian varieties over separably closed fields and maximal divisible subgroup

Franck Benoist (Université Paris-Sud) - Joint work with Elisabeth Bouscaren and Anand Pillay

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Motivation: the model-theoretic proof of the conjecture of Mordell-Lang

$G$ semi-abelian variety (commutative algebraic group of a special kind)
over $K$ a differentially closed field of characteristic 0 or a non perfect separably closed field of characteristic $p > 0$

**Definition 1** $G^\#$ is the smallest type-definable subgroup of $G(K)$ which is Zariski-dense in $G$.
$G \mapsto G^\#$ defines a functor.

In order to use the machinery of Zariski geometries on $G^\#$, a notion of dimension is required.
• In char 0: Morley rank \((DCF_0)\) is \(\omega\)-stable

• In char \(p > 0\): U-rank \((SCF_{p,e})\) is not superstable, but \(U(G^\#)\) is finite

Is it possible to give a uniform treatment of these two cases by using relative Morley rank for type-definable sets?
Is it true that \(G^\#\) has always a relative Morley rank in char \(p\)?
$G^\# \text{ in char } p$

$K \models SCF_{p,1}, \aleph_1$-saturated

**Proposition 1** $G^\# = p^\infty G(K) = \cap_{n \geq 0} p^n G(K)$ the biggest divisible subgroup of $G(K)$

**Definition 2** $\mathcal{C}$ subfield of constants of $K$: $\mathcal{C} = K^{p^\infty} = \cap_{n \geq 0} K^{p^n}$

$\mathcal{C}$ is a pure algebraically closed field, with relative Morley rank 1.

A special case: if $G$ is defined over $\mathcal{C}$, $G^\# = G(\mathcal{C})$, hence has relative Morley rank equal to $\dim(G)$. 
The structure of semi-abelian varieties

A semi-abelian variety \( G \) can be written inside an exact sequence

\[
0 \to T \to G \to A \to 0 \quad (*)
\]

\( T = \mathbb{G}_m^d \) torus

\( A \) abelian variety (i.e connected projective algebraic group)

**Remark:** \( T^\# \) has relative Morley rank (it is defined over \( \mathcal{C} \))

\( A^\# \) has relative Morley rank (look at the case of simple abelian varieties and use an appropriate version of Zilber’s indecomposability theorem)
Theorem 1 \( G^\# \) has relative Morley rank \( \iff \) the sequence \( 0 \rightarrow T^\# \rightarrow G^\# \rightarrow A^\# \rightarrow 0 \) induced by (\( \ast \)) is exact.

Sketch of the proof
\( \Leftarrow \): it is a general fact about relative Morley rank.
\( \Rightarrow \): the only problem may be that \( T^\# \subsetneq T \cap G^\# \).
We can show that \( T^\# \) is the connected component of \( T \cap G^\# \), with \( (T \cap G^\#)/T^\# \) torsion free because \( T \) has no \( p \)-torsion. But if \( G^\# \) has relative Morley rank, this quotient has to be finite.
Question (arbitrary characteristic)
Does the functor $G \mapsto G^\#$ preserve exact sequences?

In order to exhibit a counter-example, we prove:

**Theorem 2 (char 0 or $p$)** Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of semi-abelian varieties over $K$. We assume moreover that they are ordinary in the positive characteristic case. If the sequence of $G_i$’s is exact, and if $G_1$ and $G_3$ descend to $\mathcal{C}$ (i.e are isomorphic to something defined over $\mathcal{C}$), then $G_2$ descend to $\mathcal{C}$. 
Sketch of the proof

- Char 0: uses “D-structures” and work by Buium and Bertrand-Pillay.

- Char $p$: uses $p$-torsion

**Lemma 1** $G$ ordinary semi-abelian variety over $K$. For any $n \geq 0$, if the $p^n$-torsion $G[p^n] \subseteq G(K)$, $G$ descends to $Kp^n$.

**Corollary 1** $G$ as before. $G$ descends to $\mathcal{C}$ iff $T_pG(K) = T_pG$ (Tate-module of power of $p$ torsion points)

We obtain the theorem by the fact that, if the sequence of $G_i$'s is exact, then the sequence of $T_pG(K)$'s is exact.
Consequence (char \( p \))
There is a semi-abelian variety \( G \), written as \( 0 \to \mathbb{G}_m \to G \to A \to 0 \), with \( A \) ordinary abelian variety over \( \mathbb{C} \), such that \( G^\# \) does not have relative Morley rank.

It uses the parametrization of such extensions \( G \) by \( \hat{A} \), the dual abelian variety of \( A \): a point in \( \hat{A}(K) \setminus \hat{A}(\mathbb{C}) \) corresponds to a semi-abelian variety over \( K \) which does not descend to \( \mathbb{C} \). From the previous, the sequence \( 0 \to T^\# \to G^\# \to A^\# \to 0 \) is not exact and \( G^\# \) does not have relative Morley rank.
Some positive results

Proposition 2 (arbitrary char) Let $E$ be an elliptic curve which does not descend to $C$, and $G$ a semi-abelian variety given by the exact sequence $0 \to T \to G \to E \to 0$. Then the $\#$-functor preserves this exact sequence.

Proposition 3 (char 0) Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of abelian varieties. Then the $\#$-functor preserves this exact sequence.

Remark This last result is false in char $p$. 