TL;DR – 18/10/19

Important points

Let k be a field.

- FIelds with trivial Brauer groups: algebraically closed fields, finite fields. This is Wedderburn's theorem that any finite division algebra is commutative. The proof uses the Chevalley-Warning theorem to show that in any central simple algebra over a finite field there exists a nonzero element with vanishing reduced norm.

- We can finally construct examples of 4-dimensional central simple algebras. These are the quaternions algebras (a, b), whose behaviour is related to the equation $ax^2 + by^2 = 1$. The morphism

$$k^* \times k^* \longrightarrow Br(k), (a, b) \mapsto [(a, b)]$$

  is bilinear, and maps to the 2-torsion subgroup.

- The Brauer group of $\mathbb{R}$ is $\mathbb{Z}/2\mathbb{Z}$. It is generated by the class of the 4-dimensional algebra of Hamilton quaternions.

- One of the main results to come is the determination of the Brauer group of a p-adic field (we will define those). We will use this to get information on the absolute Galois group of a p-adic field (local class field theory).

- The link between Galois groups and Brauer groups is given by the construction of cyclic algebras. Given a cyclic extension of K of degree n with a generator σ of $Gal(K/k)$, we may construct a cyclic algebra (σ, a) for any $a \in k^*$. This a central simple algebra of dimension n^2 containing K as a subfield, and it is split by k. Those generalise quaternions algebras.

- Cyclic algebras as below are much simpler if n is invertible in k and k contains all n-th roots of unity.

- Given σ, $a \mapsto [(\sigma, a)]$ is a group morphism.

- (σ, a) is a matrix algebra if and only if $a \in N_{K/k}K^*$. This gives a link between Brauer groups and certain equations related to norms. In particular, $[(\sigma, a)]$ has order n in $Br(k)$ and we get an injective morphism

$$k^*/N_{K/k}K^* \longrightarrow Br(k).$$

- How (σ, a) depends on σ is more difficult – we started with some computations.

- On the way, we proved a very important result on central simple algebras: the Skolem-Noether theorem. If B is simple and A is central simple, then any two morphisms $B \to A$ are conjugate.

References

References are mostly the same as last week. An important theorem that we did not prove is the double centralizer theorem, which is very useful. See references below.

- Gille, Szamuely, *Central simple algebras and Galois cohomology*, Ch. 2. Note that 4.5 also contains a discussion of period, index, and splitting fields, that is important but we don't have time to cover.
- Stacks project, ch.11 contains all results and detailed proofs on central simple algebras (more than we covered, and allows for a more streamlined presentations).
- Draxl, *Skew fields*, par. 7, 10, 11 contains similar material as the class.
- It might be interesting to read the beginning of Serre, *Cours d'arithmétique* with quaternion algebras in mind.