TL;DR – 25/10/19

Important points

Let k be a field.

- We gathered a lot of properties of cyclic algebras: essentially, bilinearity in a for (σ, a), and a lot of properties concerning the dependency on σ.
- On the way, we used an important tool: Rieffel's lemma. It states the following: let A be a simple algebra, finite-dimensional over k. Let I be a nonzero left ideal of A, and let $B = \mathrm{End}_A(I)$. Then $A = \mathrm{End}_B(I)$ (yet another version of bicommutant statements !).
- To make sense of the results above, we made a detour through absolute Galois groups. The absolute Galois group of k is the Galois group of the (a priori infinite) extension $\overline{k}$ of k (here it is preferable to take $\overline{k}$ to be a separable closure of k so that it is indeed Galois).
- Infinite Galois theory can be deduced from usual Galois theory. The salient point is the introduction of the Krull topology on Galois groups, which turn them into profinite groups (limits of finite groups). Those are compact.
- The Galois correspondence gives a bijection between closed subgroups of the Galois group and intermediate extensions.
- Let G be a profinite group. The character group of G is the group

  $$\widehat{G} = \mathrm{Hom}(G, \mathbb{Q}/\mathbb{Z})$$

  of continuous homomorphisms from G to $\mathbb{Q}/\mathbb{Z}$, where the latter is endowed with either its natural topology iduced from that of $\mathbb{Q}$ or the discrete topology. The group $\widehat{G}$ is a discrete torsion group. If G is the absolute Galois group of k, we write $X(k)$ for the character group of G.
- In this setting, we may finally state and prove the main property of cyclic algebras as follows. Let χ be an element of $X(k)$. Then χ induces an isomorphism $Gal(K/k) \simeq 1/n\mathbb{Z}/\mathbb{Z}$ for some integer n and some cyclic extension K of k. Let σ be the k-automorphism of K such that $\chi(\sigma) = 1/n$. Then we set $(\chi, a) = (\sigma, a)$ for any $a \in k^*$. Now we prove that the map

  $$X(k) \times k^* \longrightarrow Br(k)$$

  defined above is bilinear.
- We started discussing orders in algebras – see next week's TLDR.

References

The first two items are as last week.

- Gille, Szamuely, *Central simple algebras and Galois cohomology*
- Draxl, *Skew fields*
- The very beginning of Serre, *Galois cohomology* contains a nice (though short) discussion of Galois groups for infinite extensions.
- Neukirch, *Algebraic Number theory*, ch.4, par. 1 and 2.