TL;DR – 8/11/19

Important points

In this summary we consider only $\mathbb{Z}$-orders.

- Let A be a finite dimensional algebra over $\mathbb{Q}$. An order in A is a $\mathbb{Z}$-algebra $\mathcal{O}$ in A with $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Q} = A$.
- Examples: $M_n(\mathbb{Z}) \subset M_n(\mathbb{Q})$, $Z[i] \subset \mathbb{Q}[i]$, $\mathbb{Z}[G] \subset \mathbb{Q}[G]$ for G a finite group.
- Maximal orders might exist or not, they might not be unique. They always exist in simple algebras. They are unique in the commutative case: they are the subalgebra of integral elements.
- There is a natural equivalence relation on (left) $\mathcal{O}$-module: after extending the scalar to $\mathbb{Q}$, they become isomorphic. This gives the usual equivalence relation on left ideals.
- Jordan-Zassenhaus : if A is a semisimple finite-dimensional $\mathbb{Q}$ algebra, $\mathcal{O}$ is an order in A, M_A a left A-module of finite type, there are only finitely many finitely generated left $\mathcal{O}$-modules M with $M \otimes \mathbb{Q} = M$.
- If D is a finite-dimensional division algebra over $\mathbb{Q}$ and $\mathcal{O}$ is an order, then $\mathcal{O}^*$ is cocompact in the set of $x \in D_{\mathbb{R}}$ with norm $\pm 1/$
- These two theorems generalize finiteness of class number for number fields and the Dirichlet unit theorem. The discussion of the latter and the description of $K_{\mathbb{R}}$, K a number field, is very important.
- Other applications: integral representations of finite groups, construction of Riemann surfaces.
- Key ingredient of the proofs: geometry of numbers. This studies the geometry of points in a lattice (cocompact, discrete subgroup) in a real vector space.
- Minkowski: if the covolume of a lattice is small enough, there are small points in the lattice.

References

Most of this is classical, see e.g. Neukirch. Also:

- Curtis-Reiner, *Methods in representation theory*, ch. 3.