TL;DR – 18-19/11/19

Important points

We fix a Dedekind ring $\mathcal{O}$ with fraction field K.

- If L is a finite separable extension of K, $\mathcal{O}_L$ its ring of integers, then $\mathcal{O}_L$ is a Dedekind ring, and a $\mathcal{O}_K$-module of finite type.
- The study of the morphism $\pi : \mathrm{Spec}\mathcal{O}_L \to \mathrm{Spec}\mathcal{O}_K$ is very important. Understanding the fiber of π over a prime $\mathfrak{p}$ is related to the factorization of $\mathfrak{p}\mathcal{O}_L$ into primes, which are those in $\pi^{-1}(\mathfrak{p})$.
- We say that a prime $\mathfrak{P}$ in $\mathcal{O}_L$ lies above $\mathfrak{p}$ if $\pi(\mathfrak{P}) = \mathfrak{p}$. The residual extension $f_{\mathfrak{P}/\mathfrak{p}}$ is the degree of the extension of residue fields $(\mathcal{O}_L/\mathfrak{P})/(\mathcal{O}_K/\mathfrak{p})$. The ramification index $e_{\mathfrak{P}/\mathfrak{p}}$ is the maximal integer e such that $\mathfrak{P}^e$ divides $\mathfrak{p}\mathcal{O}_L$.
- The sum of all products $e_{\mathfrak{P}/\mathfrak{p}} f_{\mathfrak{P}/\mathfrak{p}}$ as $\mathfrak{P}$ runs through all primes above $\mathfrak{p}$, is equal to $n = [L : K]$.
- $\mathfrak{p}$ is totally split if all e and f are 1 – $\mathfrak{p}$ is totally ramified is there is only one $\mathfrak{P}$ above $\mathfrak{p}$, with $e = n$ and $f = 1$. $\mathfrak{p}$ is inert if $\mathfrak{p}\mathcal{O}_L$ is prime.
- There are only finitely many ramified primes.
- Important construction: norm of fractional ideals. This is compatible with the norm of elements.
- Section 3.4 is important: it gives you explicit ways of working through the concepts above. Note the apparition of Eisenstein polynomials and their relationship to ramification. Note also that we may always work locally.
- Galois extensions have even more structures. All invariants e and f are constant above a given prime, and the Galois group acts transitively on the fibers of π.
- Galois extensions along decomposition groups, that map to the Galois group of residual extensions, and inertia group, that control ramification.
- We worked out some differential calculus. There is an $\mathcal{O}_L$-module $\Omega^1_{\mathcal{O}_L}/\mathcal{O}_K$ whose support is exactly the ramified primes (no Galois assumption). Its annihilator is an ideal called the different $\mathcal{D}_{L/K}$. We may compute it explicitely in some cases.
- **Proposition 3.72 was given an incorrect proof in class – check the notes for a correct proof.**

References

Everything is classical and covered in the general references, except for the relationship between differentials and the dualizing module. A beautiful discussion for this in a way more general context is given in the Stacks project, chapter 47.