TL;DR – geometry of numbers, again

Important points

• The main conceptual point is that the geometry of numbers we encountered in the general setting of division algebras over \( \mathbb{Q} \) and their orders can be made explicit enough for applications in the case of number fields.

• A key player, given the ring of integers \( \mathcal{O}_K \) of some number field \( K \), is the lattice \( \mathcal{O}_K \) in the real vector space \( K \times_{\mathbb{Q}} \mathbb{R} \). It may be endowed with a natural euclidean norm.

• The euclidean norm above is defined using the natural norms on \( \mathbb{R} \) and \( \mathbb{C} \) – beware of factors of 2 at the complex places!

• The covolume of the lattice above is \( \sqrt{d_K} \), where \( d_K \) is the absolute discriminant.

• Minkowski’s theorem provides nonzero elements of an ideal in \( \mathcal{O}_K \), and provides ideals of small norms in any given class in the ideal class group. It may be used to compute the ideal class group.

• Conversely, the discriminant may be bounded through the inequalities above. We obtain two important results. First, any number field ramifies somewhere over \( \mathbb{Q} \) (Minkowski). Second, given a set of primes \( S \) in \( \mathbb{Z} \), and a degree \( d \), there are only finitely many extensions of \( \mathbb{Q} \) of degree at most \( d \), unramified outside \( S \) (Hermite).
References
Standard references apply.