TL;DR – geometry of numbers, again

Important points

- The main conceptual point is that the geometry of numbers we encountered in the general setting of division algebras over $\mathbb{Q}$ and their orders can be made explicit enough for applications in the case of number fields.
- A key player, given the ring of integers $\mathcal{O}_K$ of some number field K, is the lattice $\mathcal{O}_K$ in the real vector space $K \times_{\mathbb{Q}} \mathbb{R}$. It may be endowed with a natural euclidean norm.
- The euclidean norm above is defined using the natural norms on $\mathbb{R}$ and $\mathbb{C}$ – beware of factors of 2 at the complex places !
- The covolume of the lattice above is $\sqrt{d_K|}$, where d_K is the absolute discriminant.
- Minkowski's theorem provides nonzero elements of an ideal in $\mathcal{O}_K$, and provides ideals of small norms in any given class in the ideal class group. It may be used to compute the ideal class group.
- Conversely, the discriminant may be bounded through the inequalities above. We obtain two important results. First, any number field ramifies somewhere over $\mathbb{Q}$ (Minkowski). Second, given a set of primes S in $\mathbb{Z}$, and a degree d, there are only finitely many extensions of $\mathbb{Q}$ of degree at most d, unramified outside S (Hermite).

References

Standard references apply.