TL;DR – local class field theory

Important points

Let K be a local field (usually we assume that K has characteristic zero).

- The first point is the computation of the Brauer group of K.
- Central simple algebras over K are always cyclic algebras associated to an unramified extension of K.
- The result above allows us to construct a natural isomorphism, the Hasse invariant
$$inv_K : Br(K) \longrightarrow \mathbb{Q}/\mathbb{Z}.$$
It is functorial in K.
- Through Pontryagin duality, the Hasse invariant gives rise to the Artin reciprocity map
$$\rho_K : K^* \longrightarrow G_K^{ab}$$
where G_K is the absolute Galois group of K, and G_K^{ab} its abelianization as a profinite group, i.e the limit of all abelian finite quotients of G_K. Then G_K^{ab} is the Galois group of the maximal abelian extension of K.
- ρ_K is characterized by the formula
$$inv_K(\chi, a) = \chi(\rho_K(a))$$
for $\chi \in X(K)$, $a \in K^*$.
- The main theorem of class field theory describes the behaviour of ρ_K: it is functorial in K, the composition
$$K^* \longrightarrow G_K^{ab} \longrightarrow \widehat{\mathbb{Z}}$$
given by restricting to the maximal unramified extension is the valuation map. If L is a finite abelian extension of K, then the composition
$$K^* \longrightarrow G_K^{ab} \longrightarrow Gal(L/K)$$
induces an isomorphism $K^*/N_{L/K}(L^*) \simeq Gal(L/K)$, and ρ_K induces an isomorphism
$$\rho_K : \widehat{K^*} \longrightarrow G_K^{ab}.$$
- The interesting part of ρ_K is its restriction to $\mathcal{O}_K^*$, which maps isomorphically to I_K, the Galois group of a maximal totally ramified, abelian, extension of K. The difficult part is proving injectivity, which amounts to finding enough totally ramified abelian extensions of K. This is the existence theorem.

- If $K = \mathbb{Q}_p$, an explicit computation shows that cyclotomic extensions give enough extensions to prove the existence theorem. This is then enough to prove the local and global versions of the Kronecker-Weber theorem: any finite abelian extension of $\mathbb{Q}_p$ (resp. $\mathbb{Q}$) is contained in a cyclotomic extension.

- In the general case, cyclotomic extensions are not enough (as K^* is larger – it has dimension > 1 as a p-adic Lie group, and cyclotomic extensions only account for a $\mathbb{Z}_p$-worth of extensions). We gave an argument adding roots of unity and considering Kummer extensions. Another argument constructs the needed extensions through Lubin-Tate theory.

References We followed the presentation of Kato-Kurokawa-Saito. Serre's treatment in local fields follows a cohomological approach which is more efficient but more abstract – it is more standard nowadays. You can find a beautiful presentation of Lubin-Tate theory in Cassels-Fröhlich. For a very involved modern proof of local class field theory through geometric techniques, see L. Fargues, *Simple connexité des fibres d'une application d'Abel-Jacobi et corps de classe local*, ASENS to appear.