Exam

25/01/19, 3h

No notes are allowed.

Exercice 1. You only need to give a brief justification for the following.

1. Give the definition of a Dedekind ring.
2. Let L/K be a finite extension of number fields, $\mathcal{O}_K$ and $\mathcal{O}_L$ the corresponding rings of integers. Describe in terms of $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ the ramified primes of $\mathcal{O}_L$.
3. Give the definition of a local field.
4. Let K be a field. Let $\alpha \in Br(K)$, $n > 0$. Does there exist $\beta \in Br(K)$ with $n\beta = \alpha$? Give a class of fields for which this always holds.
5. Let K be a local field. How many isomorphism classes of central simple finite-dimensional algebras A with $A^{\otimes n} \simeq M_n(K)$ are there ?

Exercice 2. Let K be a number field, and let L be a finite extension of K. We denote by $P(L/K)$ the set of primes $\mathfrak{p}$ of $\mathcal{O}_K$ such that there exists in $\mathcal{O}_L$ a prime $\mathfrak{P}$ above $\mathfrak{p}$ such that $f(\mathfrak{P}/\mathfrak{p}) = 1$.

1. If L/K is Galois, recall the definition of Frobenius elements and the Cebotarev theorem for L/K.
2. We do not assume that L/K is Galois. Let N be a Galois extension of K containing L. Show that $\mathfrak{p}$ is in $P(L/K)$ if and only if the conjugacy class of the Frobenius at $\mathfrak{p}$ in $Gal(N/K)$ intersects $Gal(N/L)$.
3. Using the question above, show that the analytic density of $P(L/K)$ is at least $1/[L : K]$.
4. Show that L/K is Galois if and only if equality holds in the previous question.
5. Assume that all primes of $\mathcal{O}_K$, except a finite number, totally split in L. Show that $L = K$.
6. Show that L/K is Galois if and only if any prime in $P(L/K)$ totally splits in L.

Exercice 3. Let $n \geq 3$, ζ_n a primitive root of unity in $\mathbb{C}$.

1. Recall, for a prime number p, when the ideal (p) is ramified in $\mathbb{Q}(\zeta_n)$.
2. Describe the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, and describe the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ inside it.
3. Describe the Frobenius in the Galois group of the extension $\mathbb{Q}(\zeta_n + \zeta_n^{-1})/\mathbb{Q}$.
4. At which condition on p does the ideal (p) totally split in $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$?

Problem

Let K be a non-archimedean local field, and let L be a finite extension of K. Assume that L/K is Galois with group G. Let v_K and v_L be the discrete valuations of K and L respectively, $\mathfrak{p}$ and $\mathfrak{P}$ their prime ideals. Let p be the residual characteristic.

1. Show that there exists an integer e such that $(ev_L)_{|K} = v_K$. Give an alternate characterization of e.
2. For any integer s, let G_s be the subgroup of G

$$G_s = \{\sigma \in G | \forall a \in \mathcal{O}_L, v_L(\sigma(a) - a) \geq s + 1\}.$$

   What are the groups G_{-1}, G_0 ? What is the intersection of the G_s ?
3. For $s \geq 1$, let U_s be the group $1 + \mathfrak{P}^s$, $U_0 = \mathcal{O}_L^*$. Let ϖ be a uniformizing parameter of L. Show that the morphism

$$G_s/G_{s+1} \to U_s/U_{s+1}, \sigma \mapsto \frac{\sigma(\varpi)}{\varpi}$$

   is well-defined, independent of ϖ.
4. Reducing to the case where L/K is totally ramified, show that the morphism above is injective.
5. What can you say about the groups G_s/G_{s+1} and their cardinality ?
6. Assume now that K contains a primitive p-th root of unity. Let $L = K(\varpi^{1/p})$. Show that L is Galois, cyclic, totally ramified.
7. Let σ be a generator of $Gal(L/K)$. Find the largest s such that $\sigma \in G_s$, as a function of e and p.
8. Assume K is arbitrary. Choose n prime to p with $n < pv_K(p)/(p-1)$.. Let $\alpha \in K$ be an alement with valuation $-n$, $L = K[X]/(X^p - X - \alpha)$. Show that L/K is Galois with groupe $\mathbb{Z}/p\mathbb{Z}$ (you might express all the roots of the equation as a function of one of them).
9. Compute the groupes G_s in the previous case.
10. Compute the groups G_s for the extension $\mathbb{Q}_p(\zeta_{p^n})$.