Exercice 1. You only need to give a brief justification for the following.

1. Give the definition of a Dedekind ring.

2. Let $L/K$ be a finite extension of number fields, $\mathcal{O}_K$ and $\mathcal{O}_L$ the corresponding rings of integers. Describe in terms of $\Omega^1_{\mathcal{O}_L/\mathcal{O}_K}$ the ramified primes of $\mathcal{O}_L$.

3. Give the definition of a local field.

4. Let $K$ be a field. Let $\alpha \in Br(K)$, $n > 0$. Does there exist $\beta \in Br(K)$ with $n\beta = \alpha$? Give a class of fields for which this always holds.

5. Let $K$ be a local field. How many isomorphism classes of central simple finite-dimensional algebras $A$ with $A^{\otimes n} \simeq M_n(K)$ are there?

Exercice 2. Let $K$ be a number field, and let $L$ be a finite extension of $K$. We denote by $P(L/K)$ the set of primes $p$ of $\mathcal{O}_K$ such that there exists in $\mathcal{O}_L$ a prime $\mathfrak{P}$ above $p$ such that $f(\mathfrak{P}/p) = 1$.

1. If $L/K$ is Galois, recall the definition of Frobenius elements and the Cebotarev theorem for $L/K$.

2. We do not assume that $L/K$ is Galois. Let $N$ be a Galois extension of $K$ containing $L$. Show that $p$ is in $P(L/K)$ if and only if the conjugacy class of the Frobenius at $p$ in $Gal(N/K)$ intersects $Gal(N/L)$.

3. Using the question above, show that the analytic density of $P(L/K)$ is at least $1/[L : K]$.

4. Show that $L/K$ is Galois if and only if equality holds in the previous question.

5. Assume that all primes of $\mathcal{O}_K$, except a finite number, totally split in $L$. Show that $L = K$.

6. Show that $L/K$ is Galois if and only if any prime in $P(L/K)$ totally splits in $L$.

Exercice 3. Let $n \geq 3$, $\zeta_n$ a primitive root of unity in $\mathbb{C}$.

1. Recall, for a prime number $p$, when the ideal $(p)$ is ramified in $\mathbb{Q}(\zeta_n)$.

2. Describe the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}$, and describe the Galois group of $\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n + \zeta_n^{-1})$ inside it.

3. Describe the Frobenius in the Galois group of the extension $\mathbb{Q}(\zeta_n + \zeta_n^{-1})/\mathbb{Q}$.

4. At which condition on $p$ does the ideal $(p)$ totally split in $\mathbb{Q}(\zeta_n + \zeta_n^{-1})$?
Problem

Let $K$ be a non-archimedean local field, and let $L$ be a finite extension of $K$. Assume that $L/K$ is Galois with group $G$. Let $v_K$ and $v_L$ be the discrete valuations of $K$ and $L$ respectively, $p$ and $\mathfrak{p}$ their prime ideals. Let $p$ be the residual characteristic.

1. Show that there exists an integer $e$ such that $(ev_L)|_K = v_K$. Give an alternate characterization of $e$.

2. For any integer $s$, let $G_s$ be the subgroup of $G$

$$G_s = \{\sigma \in G | \forall a \in \mathcal{O}_L, v_L(\sigma(a) - a) \geq s + 1\}.$$ 

What are the groups $G_{-1}, G_0$? What is the intersection of the $G_s$?

3. For $s \geq 1$, let $U_s$ be the group $1 + \mathfrak{p}^s$, $U_0 = \mathcal{O}_L^*$. Let $\varpi$ be a uniformizing parameter of $L$. Show that the morphism

$$G_s/G_{s+1} \rightarrow U_s/U_{s+1}, \sigma \mapsto \frac{\sigma(\varpi)}{\varpi}$$

is well-defined, independent of $\varpi$.

4. Reducing to the case where $L/K$ is totally ramified, show that the morphism above is injective.

5. What can you say about the groups $G_s/G_{s+1}$ and their cardinality?

6. Assume now that $K$ contains a primitive $p$-th root of unity. Let $L = K(\varpi^{1/p})$. Show that $L$ is Galois, cyclic, totally ramified.

7. Let $\sigma$ be a generator of $Gal(L/K)$. Find the largest $s$ such that $\sigma \in G_s$, as a function of $e$ and $p$.

8. Assume $K$ is arbitrary. Choose $n$ prime to $p$ with $n < pv_K(p)/(p - 1)$. Let $\alpha \in K$ be an element with valuation $-n$, $L = K[X]/(X^p - X - \alpha)$. Show that $L/K$ is Galois with group $\mathbb{Z}/p\mathbb{Z}$ (you might express all the roots of the equation as a function of one of them).

9. Compute the groups $G_s$ in the previous case.

10. Compute the groups $G_s$ for the extension $\mathbb{Q}_p(\zeta_{p^n})$. 
