Partial exam

30/11/18, 3h

No documents are allowed.

Exercice 1. The next questions need only brief justifications.

1. State Wedderburn's theorem. Deduce from it the classifications of central simple algebras of finite dimension over an algebraically closed field. What can you say about the center of a simple algebra of finite dimension over a field.
2. Recall the definition of a lattice in $\mathbb{R}^n$. Let L be a lattice in $\mathbb{R}^n$, and let L' be a sublattice of L. Compute the covolume of L' in terms of the covolume of L.
3. Recall the definition of decomposition and inertia groups in the context of a finite extension of number fields. If L/K is a finite extension of number fields, show that the inertia group is trivial at almost all primes $\mathfrak{P}$ of $\mathcal{O}_L$.

Exercice 2. Let K be a number field.

1. Let A be a discrete valuation ring, with fraction field K. Let B be a subring of K and assume that B is a discrete valuation ring containing A. Show that $A = B$ or $B = K$.
2. Let A be a discrete valuation ring, with fraction field K. Let L be a finite extension of K, and let B be a subring of K, local, noetherian. Assume $\Omega^1_{B/A} = 0$. Show that B is a discrete valuation ring.
3. Let L be a finite extension of K. Let B be an $\mathcal{O}_K$-order of L. If $\Omega^1_{B/\mathcal{O}_K} = 0$, show that $B = \mathcal{O}_L$.
4. Let a be an element of K, and let n be a positive integer. Let L be the finite extension $L = K(a^{1/n})$. Give, in terms of a and n, a finite number of prime ideals of $\mathcal{O}_K$ outside which L is unramified.
5. Let I be a fractional ideal of $\mathcal{O}_K$. If I^n is principal, show that there exists a such that $I\mathcal{O}_L$ is principal.
6. Let I be a fractional ideal of $\mathcal{O}_K$, of order $n > 1$ in the class group of K. Show that if L is a finite extension of K of degree prime to L, then $I\mathcal{O}_L$ is not principal.

Exercice 3. Let $d \neq 0, 1$ be a squarefree integer. Let $K = \mathbb{Q}(\sqrt{d})$.

1. Let P be a nonconstant polynomial with integer coefficients. Show that there exist infinitely many primes p such that there exists n with $P(n)$ divisible by p.
2. Show that there exist infinitely many primes p such that (p) is totally decomposed in $\mathcal{O}_K$.

3. Let p be a prime number, and assume $p\mathcal{O}_K = \mathfrak{p}\mathfrak{q}$, with $\mathfrak{p}$, $\mathfrak{q}$ distinct primes. Assume that for some integer n and some $x \in \mathcal{O}_K$, we have $\mathfrak{p}^n = (x)$. What can you say about the norm of x ? Can x belong to $\mathbb{Z}$?
4. Now assume $d < 0$. Show that $p^n \geq |d|/4$. As a consequence, find a lower bound on the class number of $\mathcal{O}_K$.
5. Assume that d is not divisible by 3. Give a necessary and sufficient condition on d for (3) to be totally decomposed in $\mathcal{O}_K$. As a consequence, show that the class numbers of the fields $\mathbb{Q}(\sqrt{d})$, $d < 0$ are unbounded.

Problem

Let k be a field. Let A be a central simple algebra of dimension n^2 over k.

1. Assume $A = \mathrm{End}(V)$, where V is a k-vector space of dimension n. Recall what is, up to isomorphism, the unique simple left A-module.
2. Let $B = \mathrm{End}(W)$, where W is a k-vector space of dimension n, and let $\phi : A \to B$ be an algebra automorphisms. Making use of the previous question, construct from ϕ an isomorphism $f : V \to W$, and compute ϕ in terms of f.
3. Let V be a k-vector space of dimension n. Show that the symmetric group $\mathfrak{S}_n$ acts linearly on $V^{\otimes n}$ in such a way that for every permutation σ and all $v_1, \ldots, v_n \in V$, we have
$$\sigma(v_1 \otimes \ldots \otimes v_n) = v_{\sigma(1)} \otimes \ldots \otimes v_{\sigma(n)}.$$
4. Let L be the subspace of $V^{\otimes n}$ consisting of those $\alpha \in L$ such that for every permutation σ, we have
$$\sigma(\alpha) = \epsilon(\sigma)\alpha,$$
where $\epsilon(\sigma) \in \pm 1$ is the signature of σ. Show that L has dimension 1.
5. Construct a natural map $p : V^{\otimes n} \to V^{\otimes n}$ with image L, such that $p \circ p = n!p$.
6. Construct a canonical isomorphism $\mathrm{End}(V)^{\otimes n} \to \mathrm{End}(V^{\otimes n})$.
7. Let H be the image of the morphism
$$\mathrm{End}(V^{\otimes n}) \to \mathrm{End}(V^{\otimes n}), f \mapsto fp.$$
Show that H is a k-vector space of dimension n^n, and that the natural map
$$\mathrm{End}(V^{\otimes n}) \to \mathrm{End}(H)$$
that you will define, is an isomorphism.

8. We now assume that A is arbitrary. Let K be a finite Galois extension of k such that there exists an isomorphism $\phi : A \otimes_k K \to \mathrm{End}(V)$ with $V = K^n$. Let G be the Galois group of K/k. For every $\sigma \in G$, the action of σ on K induces a map $i_\sigma : \mathrm{End}(V) \to \mathrm{End}(V)$ which is σ-linear. Similarly, let $j_\sigma : A \otimes_k K \to A \otimes_k K$ be the σ-linear isomorphism induced by σ on K. Construct a K-linear isomorphism $f_\sigma : V \to V$ such that the isomorphism of K-algebras
$$i_\sigma \circ \phi \circ (j_\sigma)^{-1} \circ \phi^{-1} : \mathrm{End}(V) \to \mathrm{End}(V)$$
is conjugation by f_σ.
9. If all the the f_σ are homotheties, show that A is split over k.
10. Using the construction of question 7, show that $A^{\otimes n}$ is a matrix algebra.
11. Show that any element of the Brauer group of a field of characteristic 0 is torsion.