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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 4 Eigenvalues and eigenvectors

- Pair of eigenvalue and eigenvector

We consider a vector space E of finite dimension n, and a map u from E to E: for each $x \in E$, there exists a unique vector $y = u(x)$ image of x by the map u and $y \in E$. We say that u is an endomorphism of E and we write $u \in \mathscr{L}(E)$. We remark that $u(0) = 0$. Then for each number λ, we have $u(0) = \lambda\,.\,0$.

We say that a **non-zero** vector $x \in E$ is an eigenvector of the operator u (or of the linear map u) if on one hand $x \neq 0$ and on the other hand there exists some number λ such that $u(x) = \lambda\,.x$. The number λ is called the eigenvalue associated with the eigenvector $x \neq 0$.

We say also that λ is an eigenvalue of the operator u if and only if there exists some vector $x \in E$ such that $x \neq 0$ and $u(x) = \lambda\,.x$.

For example, consider $E = P_1$ the vector space of all affine functions with the basis (f_0, f_1) defined by $\mathbb{R} \ni t \longmapsto f_0(t) = 1 \in \mathbb{R}$ and $\mathbb{R} \ni t \longmapsto f_1(t) = t \in \mathbb{R}$. The operator w from P_1 to P_1 defined by the relation $w(b\, f_0 + a\, f_1) = (2\,a + 3\,b)\, f_1$ is a linear map and $\lambda = 0$ is an eigenvalue of this operator. We have $w(b\, f_0 + a\, f_1) = 0$ if and only if $2\,a + 3\,b = 0$. Then taking $a = 3$ and $b = -2$ to fix the ideas, we have $w(r_1) = 0\,.\,r_1$ with $r_1 = -2\, f_0 + 3\, f_1$. We observe that $r_1 \neq 0$ then it can be called eigenvector of the linear map w associated with the eigenvalue $\lambda = 0$. For this very specific example, we recognize also that r_1 is a basis of the kernel $\mathrm{Ker}\, w$.

- Matrix expression

If $(e_1, \ldots, e_n)$ is a basis of the vector space E, we consider the matrix A of the linear map $u \in \mathscr{L}(E)$. A vector $x \in E$ can be decomposed in an unique way under the form $x = \sum_{j=1}^{n} x_j\, e_j$ and we can introduce the column matrix $X = (x_1, \ldots, x_n)^{\mathrm{t}}$ of its components. Then x is an eigenvector of the operator u if and only if $X \neq 0$ and if there exists an eigenvalue $\lambda \in \mathbb{R}$ (or $\lambda \in \mathbb{C}$) such that $A\,X = \lambda\,X$.

By extension of the previous definition, we say that such a non-zero column vector X is an eigenvector of the matrix A with an associated eigenvalue equal to λ when we have the relation $A\,X = \lambda\,X$ with $X \neq 0$.

François Dubois, 08 november 2022.

• Computation of the eigenvalues

We first recall that a square matrix B with n lines and n columns is invertible if and only if its determinant is not equal to zero. If there exists a non-zero column matrix X such that $BX = 0$, then the matrix B is not invertible and its determinant is equal to zero.

Denote by I the identity matrix with n lines and n columns. Then the relation $AX = \lambda X$ is equivalent to the relation $(A - \lambda \mathrm{I})\, X = 0$. If X is an eigenvector of the matrix A, the matrix $B = A - \lambda \mathrm{I}$ is not invertible and we have the relation $\det(A - \lambda \mathrm{I}) = 0$. An eigenvalue λ is a root of the polynomial $p(\lambda) \equiv \det(A - \lambda \mathrm{I})$. This polynomial is called the characteristic polynomial. It is a polynomial of degree n if the matrix A is a square matrix of order n. We have to keep in mind that the number of eigenvalues is always limited.

For the previous example in the vector space $E = P_1$, the matrix of the operator w in the basis (f_0, f_1) is equal to $A = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$. Then $\det(A - \lambda \mathrm{I}) = \begin{vmatrix} -\lambda & 0 \\ 3 & 2-\lambda \end{vmatrix} = \lambda\, (\lambda - 2)$. The operator w admits two eigenvalues: $\lambda = 0$ studied previously and $\lambda = 2$.

• Computation of an eigenvector once the eigenvalue is known.

We suppose that the eigenvalue λ is known. Then it satisfies $\det(A - \lambda \mathrm{I}) = 0$. An eigenvector $x \neq 0$ in the vector space is represented with a column vector X such that $(A - \lambda \mathrm{I})\, X = 0$. We have to find a **non-zero** solution of this set of n linear equations. It is possible since the determinant of the associated linear system is null.

With the previous example, we have $A = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$. If the eigenvector $r_2 = b\, f_0 + a\, f_1$ is associated with the eigenvalue $\lambda = 2$, it satisfies the relation $\begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \begin{pmatrix} b \\ a \end{pmatrix} = 2 \begin{pmatrix} b \\ a \end{pmatrix}$. Then we have $b = 0$ and a can be chosen *ad libitum*, except the value $a = 0$. A simple choice is $r_2 = f_1$.

• Diagonalizable operator, diagonalizable matrix

Let E be a vector space of dimension n and u a linear map, $u \in \mathscr{L}(E)$. If there exists a basis $(r_1, r_2, \ldots, r_n)$ composed by eigenvectors of the operator u, we say that the linear map u is diagonalizable. Recall that the vectors r_j are necessarily not equal to zero and moreover there exits eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ satisfying the n relations $u(r_j) = \lambda_j\, r_j$ for $1 \leq j \leq n$. With the matrix A of the operator u in a given basis, we introduce the column vector R_j composed with the coordinates of the vector r_j. We have the relations $A\, R_j = \lambda_j\, R_j$ and the conditions $R_j \neq 0$ for all indexes j satisfying $1 \leq j \leq n$.

It is immediate from the relations $u(r_j) = \lambda_j\, r_j$ that the matrix of the operator u in the basis $(r_1, r_2, \ldots, r_n)$ is a diagonal matrix Λ: $\Lambda_{ij} = 0$ if $i \neq j$. Moreover, the jth diagonal coefficient of the matrix Λ is exactly the eigenvalue λ_j. We can write $\Lambda_{ij} = \lambda_j\, \delta_{ij}$ with the Kroneker symbol δ_{ij}. We remark also that if P is the transfer matrix between the initial basis $(e_1, e_2, \ldots, e_n)$ and the basis $(r_1, r_2, \ldots, r_n)$ of eigenvectors, we have the relation $P^{-1}\, A\, P = \Lambda$. The matrix A has been changed into a diagonal matrix; we have diagonalized the operator $u \in \mathscr{L}(E)$.

By extension, we say that a given matrix A is diagonalizable if there exists an invertible matrix P and a diagonal matrix Λ such that $P^{-1}\, A\, P = \Lambda$. In this case, the columns R_j of the transfer matrix P are non zero column vectors and if $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2, \ldots, \lambda_n)$, we have the relations $A\,R_j = \lambda_j R_j$ for all the indices j.

With our example $E = P_1$ and $A = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$, we have $\Lambda = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix}$ in the basis (r_1, r_2) with $r_1 = -2\,f_0 + 3\,f_1$ and $r_2 = f_1$ introduced previously.

- An important result

If a linear operator u admits n distinct eigenvalues, *id est* $\lambda_i \neq \lambda_j$ if $i \neq j$, then the linear map u is diagonalizable

It is the case for our example $E = P_1$ with $n = 2$ associated with the matrix $A = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$. The two eigenvalues, 0 and 2, are distinct.

- There exists non-diagonalizable operators

We introduce the following example in $E = P_1$. We consider the basis (f_0, f_1) and we define a linear map $\zeta \in \mathscr{L}(P_1)$ by the relations $\zeta(f_0) = 0$ and $\zeta(f_1) = f_0$. In this basis, the operator ζ has an associated matrix $J = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. We observe that this matrix is not equal to the zero matrix in $\mathscr{M}_2(\mathbb{R})$ due to the number 1 at the top right position. The calculus of the eigenvalues is easy and we observe that $\lambda = 0$ is the unique (double) eigenvalue of the characteristic polynomial $p(\lambda) = \det\,(J - \lambda\,\mathrm{I}) \equiv \lambda^2$.

We say that the operator ζ is not diagonalizable: we can not find a basis of the vector space P_1 composed uniquely with eigenvectors of ζ. Indeed, if ζ is diagonalizable, we must find an invertible matrix P such that $P^{-1}\,JA\,P = \Lambda$. In this case, the matrix Λ is equal to zero, the null matrix, because the two eigenvalues are both equal to zero. Then we must have $J = 0$ because the transfer matrix P is invertible. We are in front of a contradiction since we know that the matrix J is not the null matrix. In consequence, our hypothesis of diagonalizability is false and the associated operator ζ is **not** diagonalizable.

Exercices

- Basic diagonalization

We set $A = \begin{pmatrix} 2 & 8 & -7 \\ 3 & -3 & 3 \\ -2 & -2 & 7 \end{pmatrix}$.

a) What are the eigenvalues of this matrix ? [−6, 3, 9]

b) Suggest values for the eigenvectors, with expressions as simple as possible.

c) Check the previous computations through an elementary calculus.

d) Prove that the matrix A is diagonalizable.

e) What is the result matrix if we consider the associated operator in a basis of eigenvectors ?

f) Same questions with the matrix $B = \begin{pmatrix} -1 & -4 & 11 \\ -4 & 14 & -4 \\ 11 & -4 & -1 \end{pmatrix}$.

- Diagonalization with complex numbers

We suppose given two real numbers a and b. We set $A = \begin{pmatrix} a & -b \\ b & a \end{pmatrix}$.

a) Show that if $b = 0$, this matrix is diagonalizable on the field $\mathbb{R}$.

b) Prove that if $b \neq 0$, the matrix A is not diagonalizable on $\mathbb{R}$. We make this hypothesis $b \neq 0$ for all subsequent questions.

c) What ae the complex eigenvalues of the matrix A ?

d) Propose a set of complex eigenvectors for the matrix A.

e) If P is the square matrix whose columns are composed with the two eigenvectors of the matrix A, show without any calculation the value of the matrix $\widetilde{A} = P^{-1} A\, P$.

- A parameterized problem

For any real number a, we set $A = \begin{pmatrix} 1 & -1 & -1 \\ -1 & a^2 & 0 \\ -1 & 0 & a^2 \end{pmatrix}$.

a) Determine the eigenvalues and eigenvectors of the matrix A when $a = 0$.

b) Same question if $a = 1$.

c) Same question in all the other cases.

- Cayley-Hamilton theorem

We consider the two matrices $A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

a) What are the characteristic polynomials of these two matrices ?

b) Verify that the Cayley-Hamilton theorem is satisfied: each of these matrices annul its characteristic polynomial.