le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Student presentation 7 (course 12)

- An elliptic problem in one space dimension

We propose to determine explicitly with an integral representation the solution of the Dirichlet problem composed on one hand by the differential equation $-\frac{d^2u}{dx^2} = f$ in $\Omega \equiv]0, 1[$, where f is an arbitrary scalar function defined on Ω, and on the other hand by the boundary conditions $u(0) = u(1) = 0$.

We define the Green function $G(x, \xi)$ for $x \in \Omega$ and $\xi \in \Omega$ by the relations $G(x, \xi) = \xi\,(1-x)$ if $\xi \leq x$ and $G(x, \xi) = (1-\xi)\,x$ if $\xi \geq x$.

a) For an arbitrary $x \in]0, 1[$, represent the graph of the function $[0, 1] \ni \xi \longmapsto G(x, \xi)$.

b) Show that the funtion $u(x)$ defined for $0 \leq x \leq 1$ by the relation $u(x) = \int_0^1 G(x, \xi)\, f(\xi)\, d\xi$ is a solution of the problem $-\frac{d^2u}{dx^2} = f$.

c) Show that moreover, the function u introduced in the question b) satisfies the homogeneous Dirichlet boundary conditions $u(0) = u(1) = 0$.

d) Deduce from the previous points that we can explicit a solution of the Dirichlet problem of finding a function $u : \Omega \longrightarrow \mathbb{R}$ such that $-\frac{d^2u}{dx^2} = f$ in Ω and satisfying the boundary conditions $u(0) = u(1) = 0$ with the representation formula $u(x) = \int_0^1 G(x, \xi)\, f(\xi)\, d\xi$.

François Dubois, september 2023.