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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 3 Changing the basis

- Linear map

We consider two vector spaces E and F and a map u from E to F: for each $x \in E$, there exists a unique vector $y = u(x)$ image of x by the map u. We say that the map u is linear if and only if the two following conditions of compatibility are satisfied: compatibility with the addition $\forall x, y \in E$, $u(x+y) = u(x) + u(y)$, and compatibility with the external multiplication $\forall \lambda \in \mathbb{R}$, $u(\lambda .x) = \lambda .u(x)$.

We use the following example constructed as follows. We denote by P_1 the vector space of all affine functions. In particular the function f_0 defined by $\mathbb{R} \ni t \longmapsto f_0(t)) = 1 \in \mathbb{R}$ and the function f_1 is such that $\mathbb{R} \ni t \longmapsto f_1(t)) = t \in \mathbb{R}$. The affine functions f_0 and f_1 are vectors in the space P_1. The family (f_0, f_1) is a basis of P_1. Each $f \in P_1$ can be decomposed in the following way: $f = b\, f_0 + a\, f_1$ and the real coefficients a and b are unique. The application w from P_1 to P_1 is defined by the relation $w(b\, f_0 + a\, f_1) = (2\,a + 3\,b)\, f_1$. It is a linear map defined on P_1 and taking its values in P_1.

- Kernel

We consider a linear map $u \in \mathscr{L}(E, F)$ between the vector spaces E and F. The kernel Ker u is a subset of E defined by the following condition: $x \in \text{Ker}\, u$ if and only if $u(x) = 0$. The kernel Ker u is a vector subspace of the space E. In particular, $\text{Ker}\, u \subset E$.

With the previous example $w \in \mathscr{L}(P_1)$ and we have
$\text{Ker}\, w = \{f \in P_1, \exists a \in \mathbb{R}, \forall t \in \mathbb{R}, f(t) = a\left(t - \frac{2}{3}\right)\} = < \varphi >$ with $\varphi(t) = t - \frac{2}{3}$.

- Image

We consider a linear map $u \in \mathscr{L}(E, F)$ between the vector spaces E and F. The Image $\text{Im}\, u$ is a subset of F defined by the condition that $y \in \text{Im}\, u$ if and only if there exists $x \in E$ such that $y = u(x)$. The image $\text{Im}\, u$ is a vector subspace of the space F and $\text{Im}\, u \subset F$.

For the previous example with $w \in \mathscr{L}(P_1)$, we have
$\text{Im}\, w = \{f \in P_1, \exists \alpha \in \mathbb{R}, f = \alpha\, f_1\} = < f_1 >$.

- Conservation of the dimension

We consider a vector space E with a finite dimension: $\dim E = n$, where n is a nonnegative integer, and we introduce also $u \in \mathscr{L}(E)$. Then the spaces $\text{Ker}\, u$ and $\text{Im}\, u$ are of finite dimensions and we have the relation $\dim \text{Ker u} + \dim \text{Im u} = \dim E$.

François Dubois, 17 october 2023.

For the previous example with $w \in \mathscr{L}(P_1)$, we have $\dim \mathrm{Ker\ w} = 1$ and $\dim \mathrm{Im\ w} = 1$ whereas $\dim P_1 = 2$ as we observed in the previous chapter.

- Matrix of a linear map relatively to a set of bases

We consider a vector space E of finite dimension n and we introduce a basis $(e_1, e_2, \ldots, e_n)$ of this space. We suppose given also a vector space F of dimension p and we introduce a basis $(f_1, f_2, \ldots, f_p)$ of the vector space F. For $j = 1, \ldots, n$, the vector $u(e_j) \in F$ can be secomposed in a unique way in the basis $(f_1, f_2, \ldots, f_p)$: there exists unique coefficients $a_{1j}, a_{2j}, \ldots, a_{pj}$ in such a way that $u(e_j) = \sum_{i=1}^{p} a_{ij} \cdot f_i$. We regroup these $n\,p$ coefficients into a matrix $M_u \equiv \big(a_{ij}\big)_{1 \leq i \leq p,\, 1 \leq j \leq n}$ with p lines and n columns. This matrix is the matrix of the linear map u relatively to the bases $(e_1, e_2, \ldots, e_n)$ of E and $(f_1, f_2, \ldots, f_p)$ of F. We can write it in the following way: $M_u = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix}$.

With the linear map $w \in \mathscr{L}(P_1)$ introduced previously, the associated matrix M_w is given by the relation $M_w = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$ relatively to the basis (f_0, f_1).

- Output of a given vector

With the previous notations, we regroup the components $x_1, x_2, \ldots, x_n$ of the vector $x = \sum_{j=1}^{n} x_j \cdot e_j$ in the basis $(e_1, e_2, \ldots, e_n)$ of E into a single vector X with one column and n lines: $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$.

Analogously, the coordinates $y_1, y_2, \ldots, y_p$ of the vector $y = u(x) = \sum_{i=1}^{p} y_i \cdot f_i$ in the basis $(f_1, f_2, \ldots, f_p)$ of F are presented with a vector Y with one column and p liges :
$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}$.

Then the coordinates $y_i = \sum_{j=1}^{n} a_{ij} x_j$ can be expressed with the help of the product of the matrix M_u with the vector X: $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = M_u \cdot X.$

The coordinates Y of the image vector $u(x)$ are obtained by the mutiplication of the matrice M_u of operator u by the coordinates X of the vector $x \in E$: $Y = M_u\, X$.

With the previous linear map $w \in \mathscr{L}(P_1)$ and the vector $x = 4\, f_0 - f_1$, we have $X = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

We can perform the product and $Y = M_w\, X = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} . \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$. Thus $w(x) = 10\, f_1$.

- Bijectivity

Racall that a map u from E to F is bijective if and only if for each $y \in F$, the equation $u(x) = y$ has unique solution x that belongs to the domain E.

Theorem. Let E be a vector space of finite dimension: $\dim E = n$ with $n \in \mathbb{N}$, and let u be a linear map from E to E ($u \in \mathscr{L}(E)$). Then u is bijective if and only if one of the following conditions is satisfied: (i) u is injective, (ii) $\mathrm{Ker}\, u = \{0\}$, (iii) u is surjective, (iv) $\mathrm{Im}\, u = E$, (v) u transforms a given basis of E into a new basis of E, (vi) the matrix M_u of the operator u relatively to a given basis is invertible in $\mathscr{M}_n$.

The linear map $w \in \mathscr{L}(P_1)$ introduced previously is not bijectie. We have for example $\mathrm{Ker}\, u$ of dimension 1. We remark also that the matrix M_w is clearly not invertible.

The linear map $\theta \in \mathscr{L}(P_1)$ defined by $P_1 \ni f = b\, f_0 + a\, f_1 \longmapsto \theta(f) = a\, f_0 + b\, f_1 \in P_1$ is bijective. Its matrix M_θ is equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and is invertible.

- Change of basis

Let E be the vector space $< e_1, e_2, \ldots, e_n >$ of dimension n. Then the family $(e_1, e_2, \ldots, e_n)$ is a basis of E. Each vector $x \in E$ can be decomposed as a linear combination of the vectors of this basis: $x = \sum_{j=1}^{n} x_j\, e_j$ and the coordinates x_j are uniquely defined. We introduce a new family of vectors $\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n$ defined by their decomposition in the previous basis: $\widetilde{e}_k = \sum_{j=1}^{n} P_{jk}\, e_j$. The coefficients P_{jk} for $1 \leq j, k \leq n$ compose a square matrix P with n lines and n columns, called the transfer matrix. The components of the new vector $\widetilde{e}_k$ define the kth column of the transfer matrix. We have the following result.

Theorem. The family of vectors $(\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n)$ is a basis of the space E if and only if the transfer matrix P is invertible.

If we wish to write the new coordinates $\widetilde{x}_k$ of the previous vector $x \in E$, we have the relation $P\widetilde{X} = X$ between the column vector X of the old coordinates x_j and the column vector $\widetilde{X}$ of the new coordinates $\widetilde{x}_k$: $x = \sum_{j=1}^{n} x_j\, e_j = \sum_{k=1}^{n} \widetilde{x}_k\, \widetilde{e}_k$. To explicit the coordinates in the new basis, it is necessary to solve a linear system associated with the transfer matrix.

- Change of matrix of a linear map when changing the basis of the vector space

With the standard hypothesis of a finite dimensional vector space E of dimension $n \in \mathbb{N}$, we consider a linear map $u \in \mathscr{L}(E)$ and the associated matrix M_u relatively a given basis $(e_1, e_2, \ldots, e_n)$. When we change the basis of E for a new basis $(\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n)$ of the same space, we introduce an invertible transfer matrix P. Then the matrix $\widetilde{M}_u$ of the linear map u in the new basis is related to the previous data according to the relation $\widetilde{M}_u = P^{-1}\, M_u\, P$.

Exercices

- A change of basis in the space of affine functions

We denote by P_1 the space of affine functions. The basis functions f_0 and f_1 are defined by the relations $f_0(t) = 1$ and $f_1(t) = t$ for any arbitrary $t \in \mathbb{R}$. We consider the two new functions φ_0 and φ_1 defined by the relations $\varphi_0(t) = 1+t$ and $\varphi_1(t) = 1-t$ for an arbitrary $t \in \mathbb{R}$.

a) Express the two vectors φ_0 and φ_1 as linear combinations of f_0 and f_1.

b) What is the transfer matrix P between the family (f_0, f_1) and the new family (φ_0, φ_1) ?

c) Prove that the family (φ_0, φ_1) is a basis of the space P_1.

d) What are the coordinates of the affine function f defined by $f(t) = at+b$ (for an arbitrary real number $t \in \mathbb{R}$) in the basis (φ_0, φ_1) ?

- Changing the basis of a linear map

We still denote by P_1 the space of affine functions and by (f_0, f_1) and (φ_0, φ_1) the bases defined previously. The operator w (or the linear map w) is defined by the relation
$w(b\,f_0 + a\,f_1) = (2\,a + 3\,b)\,f_1$.

a) Recall the value of the matrix M_w of the linear map w relatively to the basis (f_0, f_1).

b) With a relation introduced in this chapter, precise the value of the matrix $\widetilde{M}_w$ in the new basis (φ_0, φ_1).

c) Express the vectors $w(\varphi_0)$ and $w(\varphi_1)$ in the basis (φ_0, φ_1) and recover the result of the previous question.

- Changing the basis for an other linear map

We still denote by P_1 the space of affine functions and by (f_0, f_1) and (φ_0, φ_1) the bases introduced during the first exercice. The operator θ is defined by the relation
$\theta(b\,f_0 + a\,f_1) = a\,f_0 + b\,f_1$.

a) Recall the value of the matrix M_θ of the linear map w relatively to the basis (f_0, f_1).

b) Prove that the map θ is a bijection from P_1 on the space P_1.

c) With an algebraic relation introduced in this chapter, precise the value of the matrix $\widetilde{M}_\theta$ in the new basis (φ_0, φ_1).

d) Express the vectors $\theta(\varphi_0)$ and $\theta(\varphi_1)$ in the basis (φ_0, φ_1) and recover the result of the previous question.

- Determinant of a linear map

Let E be of dimension n, $u \in \mathscr{L}(E)$ a linear map from E to E, M_u the matrix of this map u relatively to a given basis and P the transfer matrix from the given basis and a new basis of E. We denote by $\widetilde{M}_u$ the matrix of u relatively the new basis.

a) Propose an algebraic relation between the matrices P, M_u and $\widetilde{M}_u$.

b) Prove that the determinant does not depend on the choice of the basis: $\det \widetilde{M}_u = \det M_u$.

- A singular matrix

a) Solve the linear system $\left\{ \begin{array}{l} 2x - y + 3z = 1 \\ x + 2y - z = 2 \\ 3x + y + 2z = 1 \end{array} \right.$.

b) Determine the kernel of the matrix $A = \begin{pmatrix} 2 & -1 & 3 \\ 1 & 2 & -1 \\ 3 & 1 & 2 \end{pmatrix}$,
id est the set of column vectors X with 3 lines such that $AX = 0$.

c) Check your results.

- Changing the bases in both departure and arrival spaces

Let $u \in \mathscr{L}(E,F)$ with $\dim E = n$ and $\dim F = p$. We introduce the matrix A of the linear map u relatively to a set of two "old bases" in E and F repectively. A vector $x \in E$ is represented by the column X of its coordinates in the basis of E.

a) What is the number of lines and the number of columns of the matrix A?

b) What is the expression of the column vector Y composed by the coordinates of the vector $y = u(x) \in F$?

When we change the basis in the vector space E, we introduce the transfer matrix P.

b) What is the number of lines and the number of columns of the matrix P?

d) How the new coordinates $\widetilde{X}$ of the vector $x \in E$ are related to the previous coordinates X and to the matrix P?

When we change the basis in the vector space F, we introduce the transfer matrix Q.

e) What is the number of lines and the number of columns of the matrix Q?

f) How the new coordinates $\widetilde{Y}$ of the vector $y \in F$ are related to the previous coordinates Y and to the matrix Q?

g) Express the matrix $\widetilde{A}$ of the linear map u relatively to the new bases in E and in F, with the help of the matrices A, P and Q. $[\widetilde{A} = Q^{-1}AP]$