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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 5 Autoadjoint operators

- Euclidian space

We consider a vector space E of finite dimension n. A scalar product is a map defined on the product space $E \times E$: for each $x \in E$ and each $y \in E$, we associate the real number denoted by (x, y) and called the scalar product of the vectors x and y. It satisfies three properties

(i) the scalar product is bilinear

$(x+x', y) = (x, y) + (x', y), \ \forall x, x', y \in E, \quad (\lambda x, y) = \lambda (x,y), \ \forall \lambda \in \mathbb{R}, \forall x, y \in E$

$(x, y+y') = (x, y) + (x, y'), \ \forall x, y, y' \in E, \quad (x, \lambda y) = \lambda (x,y), \ \forall \lambda \in \mathbb{R}, \forall x, y \in E$

(ii) the scalar product is symmetric

$(y, x) = (x, y), \ \forall x, x', y \in E$

(iii) the scalar product is positive definite

$(x, x) \geq 0, \ \forall x \in E$

if $(x, x) = 0$, then $x = 0$.

When the vector space E is equipped with a scalar product $(.,.)$, we speak of an Euclidian space $(E, (.,.))$ or simply of the Euclidian space E when there is no ambiguity on the definition of the scalar product.

A fundamental example is the "canonical scalar product" defined in the space $\mathbb{R}^n$ by the relations $(x, y) = \sum_{j=1}^{n} x_j \, y_j$ with $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$ and $y = (y_1, \ldots, y_n) \in \mathbb{R}^n$. The bilinearity and the symmetry are easy to check. Positivity is a consequence of the fact that the x_j are real numbers: we have $(x, x) = \sum_{j=1}^{n} (x_j)^2 \geq 0$. For the definite positive property, if $(x, x) = 0$, then the previous sum of squares is equal to zero. Then each term is null and $x_1 = \ldots = x_n = 0$. In other words, $x = 0$ in the space $\mathbb{R}^n$.

- Orthogonality

Let E be an Euclidian space. The two vectors x and y in E are orthogonals and we note $x \perp y$ if and only if their scalar product (x, y) is null. We have $x \perp y \Longleftrightarrow (x, y) = 0$.

If F and G are two subspaces of the Euclidian space E, we say that F is orthogonal to G and we denote $F \perp G$ if and only if for each $x \in F$ and each $y \in G$, we have $(x, y) = 0$.

We can equip the space P_1 introduced in the previous lectures with the following scalar product: $(b \, f_0 + a \, f_1, b' \, f_0 + a' \, f_1) = b \, b' + a \, a'$. It is an exercice left to the reader that this function satisfies

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the three axioms (i), (ii) and (iii) introduced previously. Then the two basis vectors f_0 and f_1 are orthogonals. Moreover, the spaces $< f_0 >$ and $< f_1 >$ generated by f_0 and f_1 respectively are orthogonal subspaces of P_1.
If we set $\varphi_0 = f_0 + f_1$ and $\varphi_1 = f_0 - f_1$, these two vectors are also orthogonals.

- Orthogonal basis

A basis $(e_1, \dots, e_n)$ of the Euclidian space E is said to be orthogonal if and only if two different vectors of the basis are always orthogonals: if $i \neq j$, then $(e_i, e_j) = 0$.
For example, the family (φ_0, φ_1) is an orthogonal basis of the euclidien space P_1.

- Norm

The norm $\| x \|$ of the vector x in the Euclidian space E is defined by $\| x \| = \sqrt{(x, x)}$.
For example, in the Euclidian space P_1 introduced previously, we have $\| f_0 \| = \| f_1 \| = 1$ and $\| \varphi_0 \| = \| \varphi_1 \| = \sqrt{2}$.

- Pythagore theorem

Let x and y two orthogonal vectors in an Euclidian space E. Then if there are orthogonals, we have the relation betwwen the square of norms: $\| x+y \|^2 = \| x \|^2 + \| y \|^2$.
The proof consists simply in an expansion of $\| x+y \|^2 = (x+y, x+y)$ taking into account the bilinearity of the scalar product. Then taking into account the symmetry and the orthogonality hypothesis, we have $(x, y) = (y, x) = 0$. Then the conclusion is clear.

- Orthonormal basis

An orthogonal basis $(e_1, \dots, e_n)$ of the Euclidian space E is said to be orthonormal if and only if the orthogonal vectors e_j have all a norm equal to unity. We then have $(e_i, e_j) = \delta_{ij}$, with δ_{ij} the Kronecker symbol equal to 1 if $i = j$ and to zero in the other cases.

- Expression of the scalar product

We consider an Euclidian space E and an orthonormal basis $(e_1, \dots, e_n)$ of this space. Arbitrary vectors x and y can be decomposed in this basis: $x = \sum_{j=1}^n x_j \, e_j$ and $y = \sum_{k=1}^n y_k \, e_k$. We can also introduce the column vectors of the components of x and y: $X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$ $Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{pmatrix}$.
Then $(x, y) = \sum_{j=1}^n x_j \, y_j = X^{\rm t} \, Y = Y^{\rm t} \, X$.

- Orthogonal operators

Let $u \in \mathscr{L}(E)$ a linear operator in the Euclidian space E. We say that u is orthogonal if it conserves the scalar produc of two arbitrary vectors: $\forall x \in E$, $\forall y \in E$, $(u(x), u(y)) = (x, y)$.
An example of a family of orthogonal operators ρ_θ is given in the euclidian space P_1 defined previously by the conditions $\rho_\theta \in \mathscr{L}(P_1)$, $\rho_\theta(f_0) = \cos\theta \, f_0 + \sin\theta \, f_1$ and
$\rho_\theta(f_1) = -\sin\theta \, f_0 + \cos\theta \, f_1$.

- Orthogonal matrices

Let $u \in \mathscr{L}(E)$ an orthogonal operator in the Euclidian space E and consider an orthonormal basis $(e_1, \dots, e_n)$ of this space. Then the matrix R of the operator u relatively to the basis

$(e_1, \ldots, e_n)$ satisfies the condition $R^{\rm t} R = {\rm I}$. In other terms, the matrix R is invertible and its inverse is equal to its transpose.

- Autoadjoint operator

Let $u \in \mathscr{L}(E)$ a linear operator in the Euclidian space E. We say that u is autoadjoint if we have the relation $(u(x), y) = (x, u(y))$ for each pair of vectors $x \in E$ and $y \in E$.
For example, in the Euclidian space P_1 the linear operator θ defined by the two conditions $\theta(f_0) = f_1$ and $\theta(f_1) = f_0$ defines an autoadjoint operator.

- Matrix of an autoadjoint operator in an orthonormal basis

Let $u \in \mathscr{L}(E)$ an autoadjoint operator in the Euclidian space E as previously. Consider an orthonormal basis $(e_1, \ldots, e_n)$ of the space E and the matrix A of the operator u relatively to this basis. Then A is a symmetric matrix, equal to its transpose: $A^{\rm t} = A$.

- Spectral structure of an autoadjoint operator

Let $u \in \mathscr{L}(E)$ be an autoadjoint operator in the Euclidian space E. Then we have the following "spectral theorem": the space E admits an orthogonal basis $(r_1, \ldots, r_n)$ composed by eigenvectors of the linear map u. We have $u(r_j) = \lambda_j r_j$ for appropriate eigenvalues λ_j and the orthogonality of eigenvectors $(r_i, r_j) = 0$ when $i \neq j$.
Replacing r_j by the normed vector $e_j = \frac{1}{\|r_j\|} r_j$, we have moreover the existence of an orthonormal basis of the Euclidian space E uniquely composed with eigenvectors of the autoadjoint operator u.

- Diagonalization of symmetric matrices

If the matrix A is symmetric ($A^{\rm t} = A$), then there exists an orthogonal matrix R ($R^{-1} = R^{\rm t}$) and a diagonal matrix Λ such that $R^{\rm t} A R = \Lambda$. Every symmetric matrix is diagonalizable in an orthonormal basis. This result express in terms of matrices the spectral theorem presented at the previous point.
We can *e.g.* explicit the eigenvectors of the matrix $A = \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}$ and verify that these eigenvectors are orthogonals.

- Symmetric positive definite matrices

We consider a symmetric matrix $A \in \mathscr{M}_n(\mathbb{R})$. This matrix is said to be positive definite if we have the two conditions: for each column vector X we have the inequality $X^{\rm t} A X \geq 0$ and if $X^{\rm t} A X = 0$, then $X = 0$.
In other terms, the function $(X, Y) \longmapsto X^{\rm t} A Y$ is a scalar product in the vector space $\mathscr{M}_{n1}$ of columns vectors.

Exercices

- Orthogonal operators

In the space P_1 with the basis (f_0, f_1), we define the scaler product by the relations $(b\, f_0 + a\, f_1, b'\, f_0 + a'\, f_1) = b\, b' + a\, a'$. Let $\rho_\theta \in \mathcal{L}(P_1)$ a family of linear operators defined by the conditions $\rho_\theta(f_0) = \cos\theta\, f_0 + \sin\theta\, f_1$ and $\rho_\theta(f_1) = -\sin\theta\, f_0 + \cos\theta\, f_1$.

a) What is the matrix R_θ of the linear operator ρ_θ relatively to the basis (f_0, f_1) ?

b) Prove that for an arbitrary $\theta \in \mathbb{R}$, the operator ρ_θ is an orthogonal operator in the Euclidian space P_1.

- A symmetric real matrix

We consider the following matrix $A = \begin{pmatrix} -1 & -2 & 1 \\ -2 & 2 & -2 \\ 1 & -2 & -1 \end{pmatrix}$.

a) Why the matrix A is diagonalizable ?

b) Determine the eigenvalues of the matrix A. [4 simple and -2 double]

c) Determine an orthogonal basis composed with eigenvectors of the matrix A.

d) Check your results!

- Orthogonal symmetries in $\mathbb{R}^2$

For $\theta \in \mathbb{R}$, we define $S(\theta) = \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$.

a) Show that $S(\theta) \in \mathrm{O}(2, \mathbb{R})$.

b) What is the value of $\det S(\theta)$?

c) What is the value of $S(\theta)^2$?

d) What are the eigenvalues of the matrix $S(\theta)$?

e) Explicit a basis of eigenvectors of the matrix $S(\theta)$.

f) Show that the two eigenspaces are orthogonal.

g) Show that the matrix $S(\theta)$ is the matrix of an orthonal symmetry and precise the geometric characteristics of this transformation.

- An orthogonal projector in $\mathbb{R}^3$

For $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ and $y = (y_1, y_2, y_3) \in \mathbb{R}^3$, the canonical scalar product is defined by the relation $(x, y) = x_1\, y_1 + x_2\, y_2 + x_3\, y_3$. We introduce also the subspace Q of $\mathbb{R}^3$ of all vectors $x = (x_1, x_2, x_3) \in \mathbb{R}^3$ such that $x_1 + x_2 + x_3 = 0$.

a) Propose en orthonal basis of the linear space Q.

b) What is the dimension of the subspace Q?

c) Show that the orthogonal $Q^\perp$ of Q is a subspace of $\mathbb{R}^3$ of dimension 1.

d) Propose a basis of the subspace $Q^\perp$.

e) If $x \in \mathbb{R}^3$, explicit the vectors $y \in Q$ and $z \in Q^\perp$ such that $x = y + z$.

f) Si $x \in \mathbb{R}^3$, explicit the expression of $P\,x$, orthogonal projection of vector x on the space Q.

g) What is the matrix M of the projector P relatively to the basis of $\mathbb{R}^3$ composed by a basis of Q and a basis of $Q^\perp$ considered in the previous questions.

h) What is the matrix M_P of the projector P relatively to the canonical basis of $\mathbb{R}^3$?

i) What are the eigenvalues and the eigenvectors of the matrix M_P ?