Exemples de méthodes d'éléments finis discontinus

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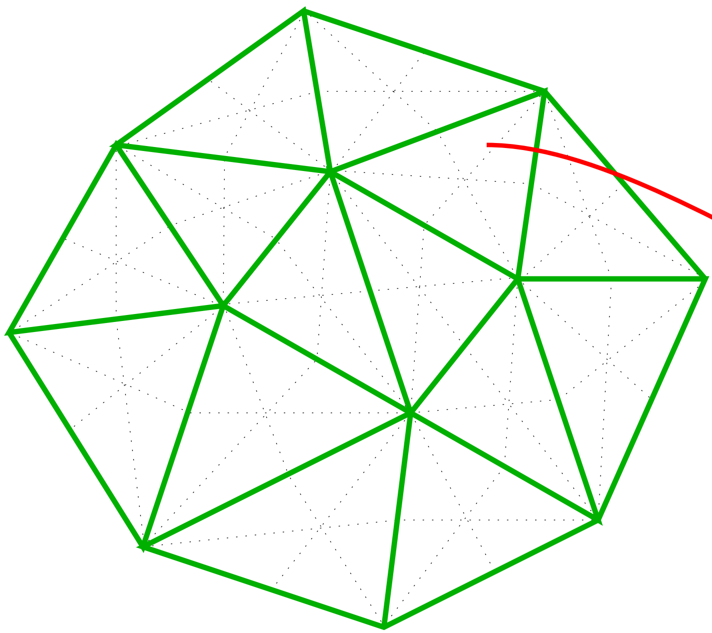
Introduction

Elliptic model problem on Ω polygonal domain

$$-\nabla \cdot A\nabla u = f \quad \text{in } \Omega \quad , \quad u = g_D \text{ on } \Gamma_D \quad , \quad A\nabla u \cdot n = g_N \text{ on } \Gamma_N \tag{1}$$



Piecewise polynomial approximation

$$V_h^r = \prod_{K \in \mathcal{T}_h} P_{k_K}(K) \qquad r = 1 + \max_{K \in \mathcal{T}_h} k_K \geq 2$$

$\mathcal{T}_h$: Family of triangulations that is assumed
- to be quasi-uniform
- to satisfy the minimum angle cond.
- segments d=1, triangles d=2, tetrahedra d=3.
- not necessarily conform

Construction of a DGM (1/2)

$E_h = \prod_{K \in \mathcal{T}_h} W^{2,p}(K)$ if we assume $u \in W^{2,p}(\Omega)$ with $\frac{2d}{d+1} \leq p \leq 2$ and $d = 2$ or 3

$E_h = \prod_{K \in \mathcal{T}_h} H^s(K)$ if we assume $u \in H^s(\Omega)$ with $s \geq 2$

$\forall v \in E_h$, IBP $\Longrightarrow \sum_{K \in \mathcal{T}_h} \int_K \nabla u \nabla v - \sum_{K \in \mathcal{T}_h} \int_{\partial K} (n_K \cdot \nabla u) v = \sum_{K \in \mathcal{T}_h} \int_K f v$

Jumps and average on $e = K^+ \cap K^- \in \mathcal{E}^I$. Let $n = n^+$ oriented from K^+ to K^- :

$$[u] = u^+ - u^- \quad , \quad \{v\} = \frac{1}{2}\left(v^+ + v^-\right)$$

$$\{\partial_n v\} \equiv \{n \cdot \nabla v\} = \frac{1}{2}\left(\frac{\partial v^+}{\partial n^+} + \frac{\partial v^-}{\partial n^+}\right) \quad , \quad [\partial_n u] \equiv [n \cdot \nabla u] = \frac{\partial v^+}{\partial n^+} - \frac{\partial v^-}{\partial n^+}$$

$$\begin{aligned} \sum_{K \in \mathcal{T}_h} \int_{\partial K} (n_K \cdot \nabla u) v &= \sum_{e \in \mathcal{E}^B} \int_e (n \cdot \nabla u) v + \sum_{e \in \mathcal{E}^I} \int_e \{n \cdot \nabla u\} [v] + [n \cdot \nabla u] \{v\} \\ &= \sum_{e \in \mathcal{E}^I \cup \mathcal{E}^B} \int_e \{n \cdot \nabla u\} [v] \quad \text{since } [n \cdot \nabla u] = 0 \end{aligned}$$

Construction of a DGM (2/2)

u_h^γ approximation in $\boxed{V_h^r = \prod_{K \in \mathcal{T}_h} P_{k_K}(K)}$ solution of

$$a_h^\gamma(u_h^\gamma, v) = F(v) \quad \forall v \in V_h^r \tag{2}$$

$$a_h^\gamma(u,v) = \sum_{K \in \mathcal{T}_h} (\nabla u, \nabla v)_K - \sum_{e \in \mathcal{E}^I \cup \mathcal{E}_D^B} \Big[\langle \{\partial_n u\}, [v] \rangle_e + s \, \langle \{\partial_n v\}, [u] \rangle_e - \gamma h_e^{-1} \langle [u], [v] \rangle_e \Big]$$

$$F(v) = (f, v) - \sum_{e \in \mathcal{E}_D^B} \big\langle g_D, s\partial_n v - \gamma h_e^{-1} v \big\rangle_e + \sum_{e \in \mathcal{E}_N^B} \langle v, g_N \rangle_e$$

$s = 1$	&	$\gamma \geq \gamma_0 > 0$	:	SIPG method
$s = -1$	&	$\gamma > 0$	:	NIPG method
$s = -1$	&	$\gamma = 0$	:	OBB DG method
$s = 0$	&	$\gamma \geq \gamma_0 > 0$	:	IIPG method

Mathematical tools

- Trace inequality : D a regular and starlike domain with $\mu =$ diam (D)

$$\|v\|^2_{\partial D} \leq c_{tr}(\mu^{-1}\|v\|^2_D + \|v\|_D\|\nabla v\|_D), \quad \forall v \in H^1(D)$$
$$\|v\|^2_{\partial D} \leq c_{tr}(\mu^{-1}\|v\|^2_D + \|v\|_{L^{\frac{p}{p-1}}(D)}\|\nabla v\|_{L^p(D)}), \quad \forall v \in W^{1,p}(D).$$

- Inverse inequality :

$$\|\nabla v\|_D \leq c_{inv}\frac{k^2}{\mu}\|v\|_D, \; \forall v \in P_k(D)$$

- Approximation properties in $H^s(K)$: $\mu = \min(s, k+1)$

$\exists \chi \in P_k(K)$ satisfying : $|u-\chi|_{j,K} \leq c h_K^{\mu-j}|u|_{s,K}$ with $0 \leq s$, $0 \leq j \leq s$

- Approximation properties in $W^{2,p}(K)$:

$\exists \chi \in P_1(K)$ satisfying : $\|u-\chi\|_{W^{j,p}(K)} \leq c h_K^{2-j}|u|_{W^{2,p}(K)}$ with $0 \leq j \leq 2$.

SIPG / NIPG : Properties

- Consistency :
  Terms are well defined and a_h^γ is consistent with the Laplacian in $E_h = \prod_{K\in\mathcal{T}_h} W^{2,p}(K)$

- Orthogonality :
  $a_h^\gamma(u - u_h^\gamma, v) = 0 \quad \forall v \in V_h^r$.

- Energy norm on E_h
  $$\|v\|_{1,h}^2 = \sum_{K\in\mathcal{T}_h} ||\nabla v||_K^2 + \sum_{e\in\mathcal{E}^I\cup\mathcal{E}_D^B} \left[h_e\,|\{\partial_n v\}|_e^2 + h_e^{-1}|[v]|_e^2\right]$$

- Continuity in E_h :
  $|a_h^\gamma(u,v)| \leq (1+\gamma)\|u\|_{1,h}\|v\|_{1,h} \quad \forall u,v \in E_h$

- Coercivity inequality in V_h^r :
  $$\exists \gamma_0, c, / \, \gamma \geq \gamma_0 \quad a_h^\gamma(v,v) \geq c\|v\|_{1,h}^2 + (\gamma - \gamma_0) \sum_{e\in\mathcal{E}^I\cup\mathcal{E}_D^B} h_e^{-1}|[v]|_e^2 \quad \forall v \in V_h^r$$

SIPG / NIPG : CV when $\gamma \longrightarrow +\infty$

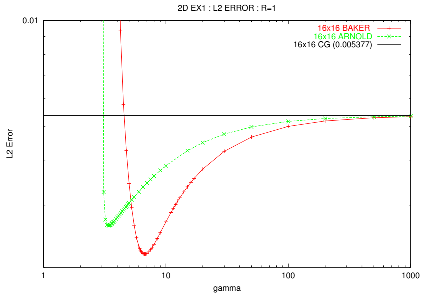
Let $u_h^G \in V_h^r \cap H^1_{g,D}(\Omega)$ be the continuous Galerkin method solution of

$$(\nabla u_h^G, \nabla \chi) = (f, \chi) + \sum_{e \in \mathcal{E}_N^B} \langle \chi, g_N \rangle_e \,, \quad \forall \chi \in V_h^r \cap H^1_{0,D}(\Omega)$$

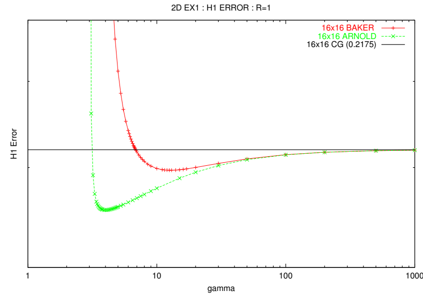
- $\lim_{\gamma \to \infty} u_h^\gamma = u_h^G$ If g_D is assumed to be restriction of a continuous fct of V_h^r
- Rate of convergence (if $\Gamma_D = \partial\Omega$)
  - $$\sum_{e \in \mathcal{E}^I} h_e^{-1} |[u_h^\gamma]|_e^2 + \sum_{e \in \mathcal{E}_D^B} h_e^{-1} |g_D - u_h^\gamma|_e^2 \leq \frac{cst(h,u)}{(\gamma - \gamma_0)^2}$$
  - $$\|u_h^G - u_h^\gamma\|_{1,h} \leq \frac{cst(h,u)}{(\gamma - \gamma_0)}$$

SIPG / NIPG : Penalty parameter

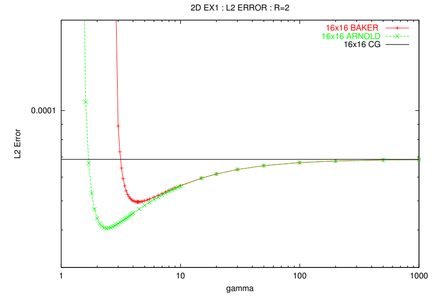
$SIPG \quad r = 2 \quad \|u_h^\gamma - u\|$ vs γ



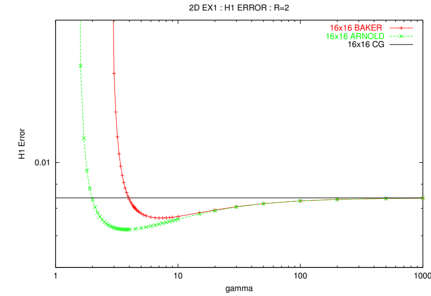
$\|u_h^\gamma - u\|_{1,h}$ vs γ



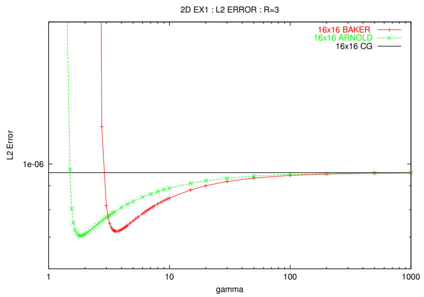
$r = 3 \quad \|u_h^\gamma - u\|$ vs γ



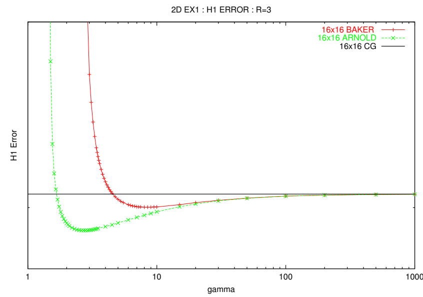
$\|u_h^\gamma - u\|_{1,h}$ vs γ



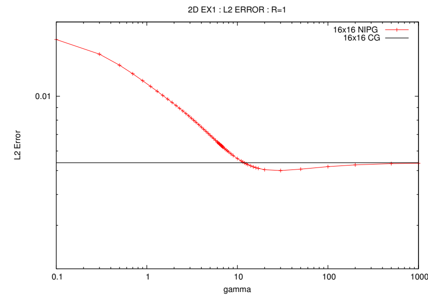
$r = 4 \quad \|u_h^\gamma - u\|$ vs γ



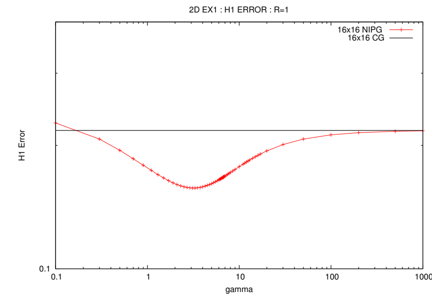
$\|u_h^\gamma - u\|_{1,h}$ vs γ



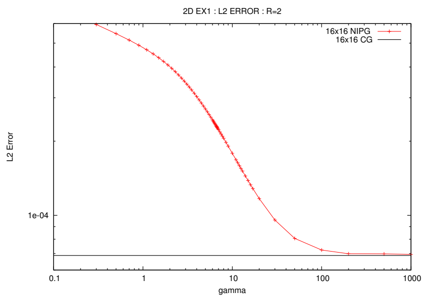
$NIPG \quad r = 2 \quad \|u_h^\gamma - u\|$ vs γ



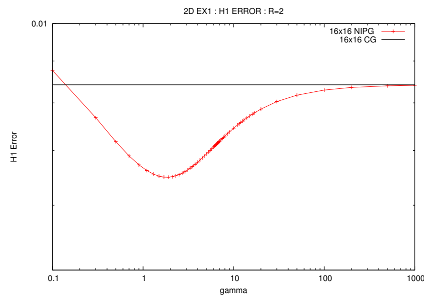
$\|u_h^\gamma - u\|_{1,h}$ vs γ



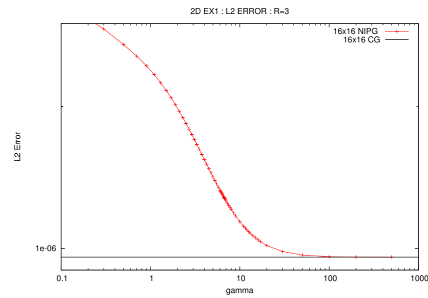
$r = 3 \quad \|u_h^\gamma - u\|$ vs γ



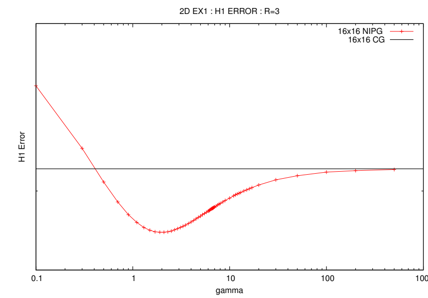
$\|u_h^\gamma - u\|_{1,h}$ vs γ



$r = 4 \quad \|u_h^\gamma - u\|$ vs γ



$\|u_h^\gamma - u\|_{1,h}$ vs γ



SIPG /NIPG : A priori estimations (Theory)

- SIPG / NIPG Energy-norm estimations :
  $\|u - u_h^\gamma\|_{1,h} \leq c h^{1-d\frac{2-p}{2p}} |u|_{W^{2,p}(\Omega)}$
- SIPG / NIPG Energy-norm estimations if $u \in H^s$ with $s \geq 2$:
  $\|u - u_h^\gamma\|_{1,h} \leq c h^{\mu-1} |u|_{s,\Omega}$ with $\mu = \min(s, r)$

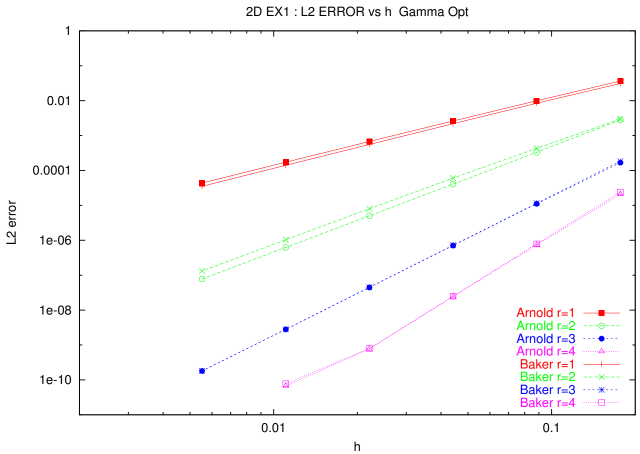
- SIPG L^2-norm estimations (W^{2,p_0} regularity of the homogeneous boundary pb)
  $\|u - u_h^\gamma\| \leq c h^{2-\frac{d}{2}\left(\frac{2-p}{p}+\frac{2-p_0}{p_0}\right)} |u|_{W^{2,p}(\Omega)}$
- SIPG L^2-norm estimations if $u \in H^s$ with $s \geq 2$ and Ω convex:
  $\|u - u_h^\gamma\| \leq c h^{\mu} |u|_{s,\Omega}$

- NIPG Sub optimal L^2-norm estimations if $u \in H^s$ with $s \geq 2$ and Ω convex:
  $\|u - u_h^\gamma\| \leq c h^{\mu-1} |u|_{s,\Omega}$

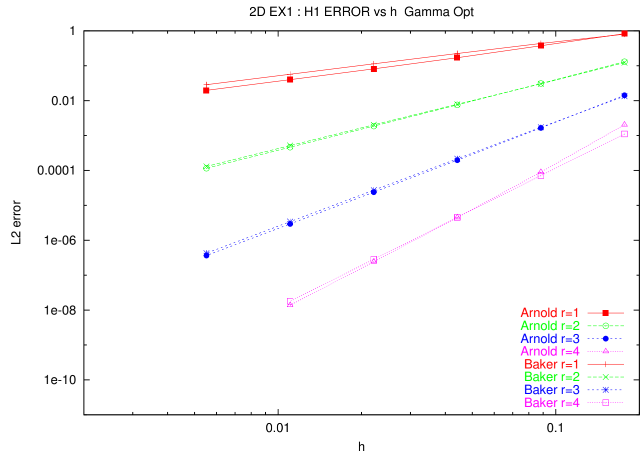
SIPG : A priori estimations (Numerics)

2d results : $u(x) = \sin(\pi x)\sin(\pi y)$

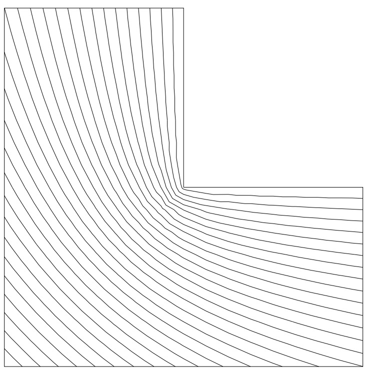
$\|u_h^\gamma - u\|$ vs $1/h$



$|u_h^\gamma - u|_{1,h}$ vs $1/h$



$u(x) = r^{\frac{2}{3}} \sin(\frac{2}{3}\theta)$ with $-\frac{3\pi}{2} < \theta < 0$



$\partial = 1$		$\partial = 1$		$\partial = 3$		$\partial = 4$	
ord L^2	ord H^1	ord L^2	ord H^1	ord L^2	ord H^1	ord L^2	ord H^1
1.3194	0.6179	1.4973	0.6661	1.5048	0.6665	1.6409	0.6666
1.3475	0.6378	1.4560	0.6662	1.4663	0.6670	1.6280	0.6669
1.3579	0.6490	1.4218	0.6667	1.4311	0.6667	1.6088	0.6663
1.3584	0.6551	1.3950	0.6666	1.4033	0.6668	1.5831	0.6667
1.3558	0.6602						

SIPG : Choice of the basis

(HB) : $r = 1\ :\ 1, y, x$;

$r = 2\ :\ 1, y, y^2, x, xy, x^2$

(WB) : $r = 1\ :\ 1, 3y-1, 3x-1$;

$r = 2\ :\ 1, 3y-1, (3y-1)^2, 3x-1, (3x-1)(3y-1), (3x-1)^2$

(RB) : Hierarchical shape basis fct (Demkowicz)

(LB) : Lagrangian nodal basis

ne	r	(HB)	(WB)	(RB)	(LB)	(HB)	(WB)	(RB)	(LB)
		Iter CG vs basis, ∂, h				Iter PCG vs basis, ∂, h			
64	4	nc	4109	5142	1157	489	472	466	467
64	3	5779	2117	1349	839	353	358	351	351
64	2	1097	1222	703	612	233	238	231	232
64	1	552	438	394	394	161	162	158	158
32	4	37907	3143	5257	855	321	347	310	311
32	3	5807	1559	1311	579	239	253	232	232
32	2	921	774	486	405	149	155	146	146
32	1	361	289	264	264	95	96	93	83
16	4	37774	2223	5145	633	191	220	185	186
16	3	5896	977	1278	390	143	149	138	139
16	2	936	473	375	243	107	111	105	105
16	1	254	164	159	159	60	60	59	59

Efficient preconditionning :

Babuška, Craig, Mandel, Pitkäranta / DD and multigrid Feng & Karakashian

OBB DG : Properties

- Consistency :
  Terms are well defined and a_h^γ is consistent with the Laplacian in $E_h = \prod_{K \in \mathcal{T}_h} W^{2,p}(K)$
- Orthogonality :
  $a_h^\gamma(u - u_h^\gamma, v) = 0 \quad \forall v \in V_h^r$.
- Continuity in E_h :
  $|a_h^\gamma(u,v)| \leq \|u\|_{1,h}\|v\|_{1,h} \quad \forall u, v \in E_h$
- Energy norm on E_h
  $\|v\|_{e,h}^2 = \sum_{K \in \mathcal{T}_h} ||\nabla v||_K^2$
- Weak Coercivity inequality in V_h^r :
  $a_h^\gamma(v,v) = \|v\|_{e,h}^2 \quad \forall v \in V_h^r$

OBB DG : A priori estimations

- Energy estimate if $u \in H^s$ with $s \geq 2$:
  - Lemma (Rivière, Wheeler, Girault) : for $r \geq 3$ there is an interpolate ζ in V_h^r such that
  $$a_h^0(\zeta - u, v) = 0, \quad \forall v \quad \text{piecewise constant}$$
  - $\|u - u_h^0\|_{e,h} \leq ch^{\mu-1}|u|_{s,\Omega}$ with $\mu = \min(s, r)$

- Inf-Sup Condition : (Larson, Niklasson, 2004)
  - Theorem : in $2d$, if $r \geq 3$ then there is a constant $\alpha > 0$ such that
  $$\inf_{u \in V_h^r} \sup_{v \in V_h^r} \frac{a_h^0(u, v)}{\|u\|_{1,h}\|v\|_{1,h}} > \alpha$$
  - Proof is based on the direct sum $V_h^r = V_c + V_d$ with

$$V_d = \left\{ v \in V_h^r \quad / \quad \sum_{K \in \mathcal{T}_h} (\nabla w, \nabla v)_K - \sum_{e \in \mathcal{E}^I \cup \mathcal{E}_D^B} \langle \{\partial_n w\}, [v] \rangle_e \quad \forall w \in V_h^r \right\}$$
$$V_c = \left\{ v \in V_h^r \quad / \quad \sum_{e \in \mathcal{E}^I \cup \mathcal{E}_D^B} \langle \{\partial_n w\}, [v] \rangle_e = 0 \quad \forall w \in V_d \right\}$$

LDG : link with the mixed FEM

First step : Rewrite the pb in a first order system

$$q = \nabla u \quad \text{and} \quad -\nabla \cdot q = f \quad \text{in} \quad \Omega,$$
$$u = g_D \quad \text{on} \quad \Gamma_D \quad \text{and} \quad q \cdot n = g_N \quad \text{on} \quad \Gamma_N \,.$$

Second step : Find $(q_h, u_h) \in (V_h^r)^d \times V_h^r$ such that

$$\int_\Omega q_h \cdot z_h \, dx = -\int_\Omega u_h \nabla \cdot z_h \, dx + \sum_{e \in \mathcal{E}^I \cup \mathcal{E}^B} \langle \hat{u}_h, [z_h] \cdot n \rangle_e \qquad \forall z_h \in (V_h^r)^d$$

$$\int_\Omega q_h \cdot \nabla v_h \, dx = \int_\Omega f v_h \, dx + \sum_{e \in \mathcal{E}^I \cup \mathcal{E}^B} \langle \hat{q}_h \cdot n, [v_h] \rangle_e \qquad \forall v_h \in V_h^r \,.$$

Third step : choice of the numerical flux ($\alpha_{11} > 0$ and $\alpha_{22} \geq 0$)

$$\hat{u}_h(u_h, q_h) = \{u_h\} + \alpha_{12} \cdot n[u_h] - \alpha_{22} \, n \cdot [q_h]$$
$$\hat{q}_h(q_h, u_h) = \{q_h\} - \alpha_{11} \, n[u_h] - \alpha_{12} \cdot n[q_h]$$

For instance : $\hat{u}_h = u_h^-|_e$ and $\hat{q}_h = q_h^+|_e - h^{-1}[u_h]$.

DGM for 1st order hypberbolic pb

Let consider

$$\nabla \cdot (\beta u) + \sigma u = f \quad \text{in} \quad \Omega \quad \text{and} \quad u = g_D \quad \text{on} \quad \Gamma^- = \{x \in \partial\Omega : \beta \cdot n < 0\}$$

Let us assume $\sigma(x) + \frac{1}{2}\nabla \cdot \beta(x) \geq c_0 > 0 \quad a.e.$

Find $u_h \in V_h^r$ such that $\forall v_h \in V_h^r$

$$\sum_{K \in \mathcal{T}_h} \int_K -u_h(\beta \cdot \nabla v_h) + \sigma u_h v_h \, dx + \sum_{e \in \mathcal{E}, e \notin \Gamma^-} \langle \{\beta u_h\}, [v_h]\, n \rangle_e$$

$$+ \sum_{e \in \mathcal{E}, e \notin \Gamma} \langle |\beta \cdot n|/2 \, [u_h] \, , [v_h] \rangle_e = \int_\Omega f v_h \, dx - \sum_{e \in \Gamma^-} \langle (\beta \cdot n) g, v_h \rangle_e$$

Stability and error estimate for the norm $|||u||| = \left(\|u\|_{0,\Omega}^2 + \sum_{e \in \mathcal{E}} \| |\beta \cdot n|^{1/2} \, [u_h] \, \|_{0,e}^2 \right)^{1/2}$

$$|||u - u_h||| \leq c h^{r - \frac{1}{2}} |u|_{r,\Omega} \quad r \geq 1$$

DGM : motivation and advantage

- Hybrid approach combining FV and FE methods
- Local method :
  - Local Conservation
  - Non conforming mesh
  - h and r refinement, small matrix stencil, local degrees
  - Functions adapted to pb (ex. local zero divergence)
  - Parallelisation and efficient preconditionners
- Discontinuous functions and high order :
  - Easy high order scheme
  - Upwinding and capturing discontinuities of solutions
  - Esay computation of the gradient
- No nodal degrees of freedom :
  - computatuional structure more easy to handle

Drawbacks

- High number of degrees of freedom and same order of CV

Example in 2d
Mesh with T triangles

p	1	2	3	4
CGM	$T/2$	$2T$	$9T/2$	$8T$
DGM	$3T$	$6T$	$10T$	$15T$

- Edge integration : quadrature formula
- Oriented edges : Version Baker
- High order : quadrature formula
- Non conforming mesh : Not so easy to implement
- Penalty terms : NIPG et SIPG
  - unknown γ, depends on the mesh and Poincaré, Inverse inequalities constant
  - $\gamma \longrightarrow \infty$
- DG and NIPG $\Longrightarrow$ non symmetric formulation
- DG weakly stable, instable for P1, sub optimal a priori estimates
- Hyperbolic case : loss of positivity and slope limiting strategy

Literature review 1/2

- 70 : Aubin (DF), Babuška (EF) : non consistant penalty term to impose Dirichlet boundary
- 71 : Nitsche (EF) : consistant penalty of the D.B.C (results on CV)
- 73 : Babuška/Zlamal : Int. penalty terms. : cont. 4th order pb (non consistant)
- 75 : Douglas/Dupont : penalty on the jump of the normal derivative (2nd order pb)
- 77 : Baker : penalty on the jump of normal derivative (4th order pb)
- 78-82 : Wheeler et Arnold : SIPG
- 90' : Baker, Jureidini, Karakashian, Katsaounis : Stokes, N.S.
- 97 : Babuška, Baumann, Oden : ODD DG
- 98-02 : Riviere, Wheeler, Girault : Th. CV of DG and NIPG
- 99 : Castillo/Cockburn/Perugia/Schotzau: LDG
- 99 : Discontinuous Galerkin method congress
- 99 : Arnold/Brezzi/Cockburn/Marini : unified analysis
- 02 : Romkes/Oden/Prudhomme : Stabilized DGM

Literature review 2/2

- 73 : Reed/Hill et Le Saint/Raviart : introduction discontinuous FEM for neutronic transport equation
- 74 : LeSaint/Raviart : mathematical analysis
- 86 : Johnson/Pitkaranka : analysis of the DGM for scalar hyperbolic eq.
- 97-98 : Bassy/Rebay, Cockburn/Shu/Dawson : LDG for N.S. and convection-diffusion
- 99 : Cockburn/Karniadakis/Shu : A review
- 01 : Dolejsi/Feistauer : DGM compressible flow
- 02 : Houston/Schwab/Suli : DGM for advection-diffusion-reaction pbs
- 04 : Brezzi/Marini/Suli : DGM for first-order hyperbolic pbs

- 00/05 : 254 articles with *discontinuous Galerkin* in the title (Zentralblatt)
- 00/05 : 439 articles with *discontinuous Galerkin* in the global index (Zentralblatt)

Approximation results

In $H^1_{g,D}(\Omega) = \{v \in H^1(\Omega) \,|\, v = g_D \text{ on } \Gamma_D\}$ with g_D restriction to Γ_D of a fct in $V_h^r \cap H^1(\Omega)$

For any $v_h \in V_h^r$, there exists $\chi \in V_h^r \cap H^1_{g,D}(\Omega)$ satisfying

$$\sum_{K \in \mathcal{T}_h} \|v_h - \chi\|^2_{i,K} \leq C \left(\sum_{e \in \mathcal{E}^I} h_e^{1-2i} |[v_h]|_e^2 + \sum_{e \in \mathcal{E}^B_D} h_e^{1-2i} |v_h - g_D|_e^2 \right)$$

C is a constant independent on v_h, on χ and on h. $i = 0, 1$.

In $H^1(\Omega)$

For any $v_h \in V_h^r$, there exists $\chi \in V_h^r \cap H^1(\Omega)$ satisfying

$$\sum_{K \in \mathcal{T}_h} \|v_h - \chi\|^2_{i,K} \leq C \sum_{e \in \mathcal{E}^I} h_e^{1-2i} |[v_h]|_e^2$$

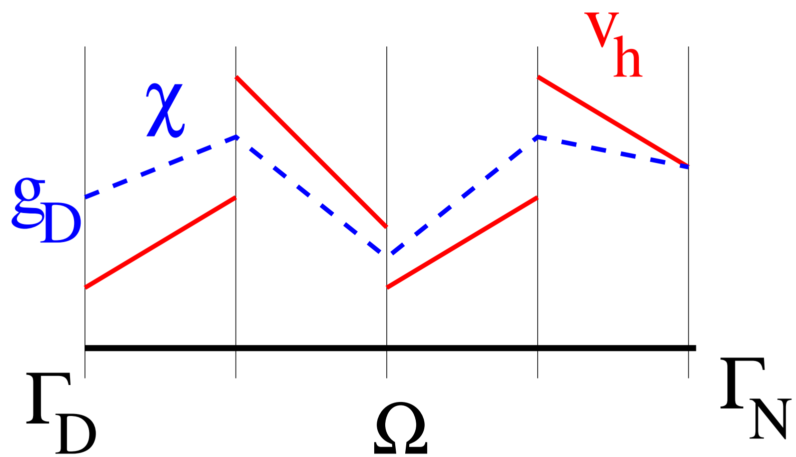
C is a constant independent on v_h, on χ and on h. $i = 0, 1$.

Approximation results : sketch of the proof

Constructive method with the
continuous Lagrange nodal basis : $\Phi^{(\nu)}$

$\nu \in \mathcal{N} = \mathcal{N}^I \cup \mathcal{N}_N^B \cup \mathcal{N}_D^B$ = set of nodes

If supp $\Phi^{(\nu)} = \bigcup_{K \in \omega_\nu} K, \quad \Phi^{(\nu)}|_K = \Phi_K^{(j)}$



On conforming mesh if $v_h = \sum_{K \in \mathcal{T}_h} \sum_{j=1}^{m} \alpha_K^{(j)} \Phi_K^{(j)}(x)$ then $\chi = \sum_{\nu \in \mathcal{N}} \beta^{(\nu)} \Phi^\nu$ with

$$\beta^{(\nu)} = \begin{cases} g_D(\nu) & \text{if } \nu \in \mathcal{N}_D^B, \\ \dfrac{1}{|\omega_\nu|} \displaystyle\sum_{x_K^{(j)} = \nu} \alpha_K^{(j)} & \text{if } \nu \in \mathcal{N}^I \cup \mathcal{N}_N^B. \end{cases}$$

Key arguments : $\|\phi_K^{(j)}\|_{i,K}^2 \le c h_K^{d-2i}$ and $\displaystyle\sum_{j=1}^{N} \left| \alpha_j - \frac{1}{N} \sum_{i=1}^{N} \alpha_i \right|^2 \le C \sum_{j=1}^{N-1} |\alpha_{j+1} - \alpha_j|^2$

SIPG : Residual-type a posteriori estimate

$$\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 \leq C\Big\{ \sum_{K\in\mathcal{T}_h} h_K^2 \|f+\Delta u_h^\gamma\|_K^2 + \sum_{e\in\mathcal{E}^I} h_e \big|[\partial_n u_h^\gamma]\big|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}^I} h_e^{-1} |[u_h^\gamma]|_e^2$$
$$+ \sum_{e\in\mathcal{E}_N^B} h_e |g_N - \partial_n u_h^\gamma|_e^2 + \gamma^2 \sum_{e\in\mathcal{E}_D^B} h_e^{-1} |g_D - u_h^\gamma|_e^2 \Big\}$$

The proof is based on a Verfürth-type technique :

Estimation of the residual $a_h^\gamma(e,\eta) = (f,\eta) - a_h^\gamma(u_h^\gamma,\eta)$ with $\eta = v - v_h,\ v \in E_h,\ v_h \in V_h^r$.

Since the weak formulation is not coercive on E_h, take $\eta = e - v_h$ with v_h the best approximating constant of e, then

$$\sum_{K\in\mathcal{T}_h} \|\nabla e\|_K^2 = \sum_{K\in\mathcal{T}_h} (f+\Delta u_h^\gamma,\eta) + \sum_{e\in\mathcal{E}^I} \left(\langle\{\partial_n e\},[\eta]\rangle_e + \langle\{\eta\},[\partial_n e]\rangle_e\right) + \sum_{e\in\mathcal{E}^B} \langle \partial_n e, \eta\rangle_e$$

a) $a_h^\gamma(e, u_h^\gamma + v_h - \chi) = 0, \quad \forall \chi \in V_h^r \cap H^1_{g,D}(\Omega)$

b) terms with η : $\|e - v_h\|_K \leq c h_K \|\nabla e\|$

c) terms with $u_h^\gamma - \chi$: approximation results

SIPG : Optimality of the estimate

Suppose that f is a piecewise polynomial on $\mathcal{T}_h$

(i) For each $K \in \mathcal{T}_h, \quad h_K^2 \|f + \Delta u_h^\gamma\|_K^2 \leq c\, \|\nabla e\|_K^2$

(ii) For $e = K^+ \cap K^- \in \mathcal{E}^I, \quad h_e \big| [\partial_n u_h^\gamma] \big|_e^2 \leq c\Big(\|\nabla e\|_{K+}^2 + \|\nabla e\|_{K-}^2 \Big)$

(iii) For $e = K^+ \cap \partial\Omega \in \mathcal{E}_N^B, \quad h_e \big| g_N - \partial_n u_h^\gamma \big|_e^2 \leq c \|\nabla e\|_{K+}^2$

Proof : Verfürth-type technique

(iv) For γ large enough,

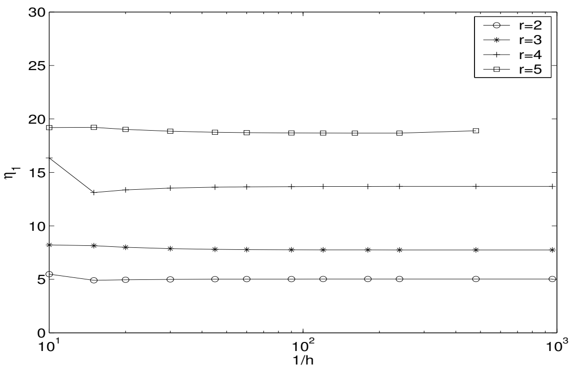
$$\gamma^2 \sum_{e \in \mathcal{E}^I} h_e^{-1} \big| [u_h^\gamma] \big|_e^2 + \gamma^2 \sum_{e \in \mathcal{E}_D^B} h_e^{-1} \big| [g_D - u_h^\gamma] \big|_e^2 \leq c \sum_{K \in \mathcal{T}_h} \|\nabla e\|_K^2$$

Proof : Use the continuous Galerkin approximation $u_h^G \in V_h^r \cap H^1_{g,D}(\Omega)$ solution of

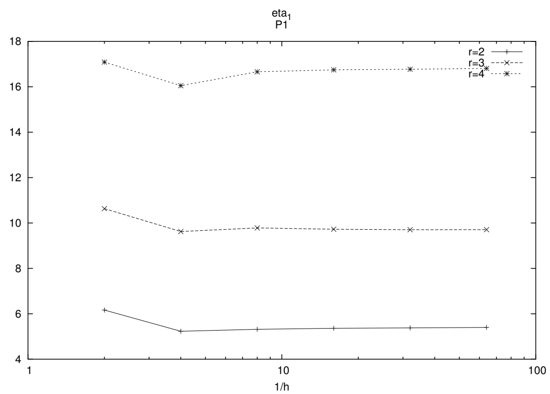
SIPG : Effectiveness of estimators indices

$$\eta_1^2 = \left(\sum_{K \in \mathcal{T}_h} h_K^2 \| f + \Delta u_h^\gamma \|_K^2 + \sum_{e \in \mathcal{E}^I} h_e |[\partial_n u_h^\gamma]|_e^2 + \gamma^2 \sum_{e \in \mathcal{E}^I \cup \mathcal{E}^B} h_e^{-1} |[u_h^\gamma]|_e^2 \right) / \left(\sum_{K \in \mathcal{T}_h} \| \nabla e \|_K^2 \right)$$

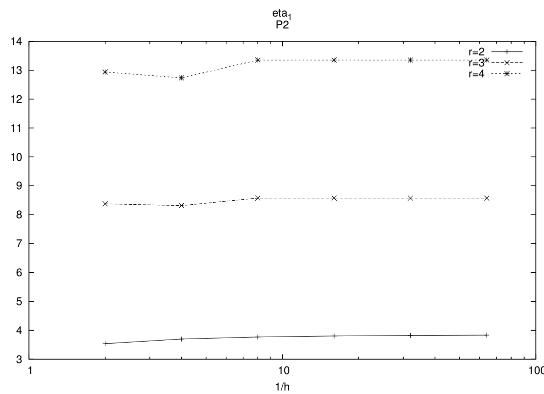
1 d	2 d	2 d
$\Omega = [0,1]$	$\Omega = [0,1]^2$	L-shaped dom
$\Gamma_D = \partial\Omega$	$\Gamma_D = \partial\Omega$	$\Gamma_D = \partial\Omega$
$u(x) = e^{-100(x-\frac{1}{2})^2}$	$u(x,y) = \sin \pi x \sin \pi y$	$u(x,y) = r^{2/3} \sin \frac{2\theta}{3}$
η_1 versus $1/h$	η_1 versus $1/h$	η_1 versus $1/h$



$r = 2, 3, 4, 5$



$r = 2, 3, 4$



$r = 2, 3, 4$

Adaptive mesh strategy (equidistribution)

- *(i)* Compute the local estimate : $\eta_K^k(u_k^\gamma)$
- *(ii)* Compute the total error estimate $\eta^k(u_k^\gamma)^2 = \sum_{K\in\mathcal{T}_k} \eta_K^k(u_k^\gamma)^2$,
- *(iii)* Mark the elements $\hat{\mathcal{T}}_k$ to be refined such that for a given parameter θ ($\theta = 0.5$),

$$\left(\sum_{K\in\hat{\mathcal{T}}_k} \eta_K^k(u_k^\gamma)^2 \right)^{1/2} \geq \theta\eta^k(u_k^\gamma) \,,$$

- *(iv)* Refine the mesh and obtain $\mathcal{T}_{k+1}$ by dividing each $K \in \hat{\mathcal{T}}_k$,
- *(v)* Compute the SIPG solution on $\mathcal{T}_{k+1}$,
- *(vi)* $k \leftarrow k+1$ and go to step *(i)*.

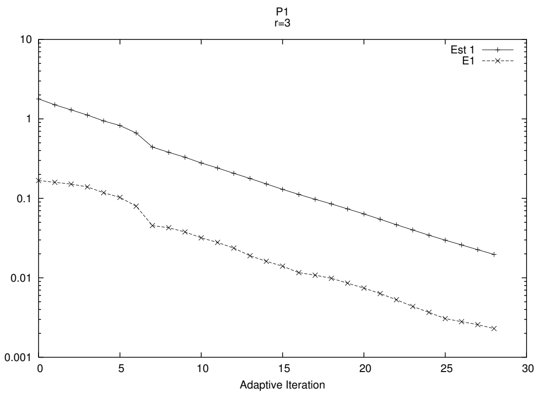
STOP if $\eta^k(u_k^\gamma) \leq tol$.

W. Dörfler $\Rightarrow$ convergence results for continuous Galerkin method

SIPG : numerical results : 1/2

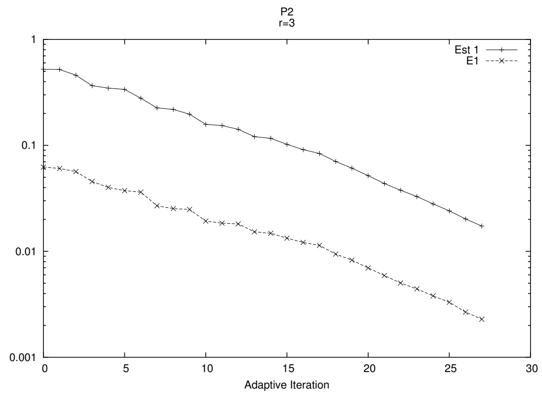
$\Omega = [0,1]^2$ and $\Gamma_D = \partial\Omega$

est and err versus k



L-shaped domain and $\Gamma_D = \partial\Omega$

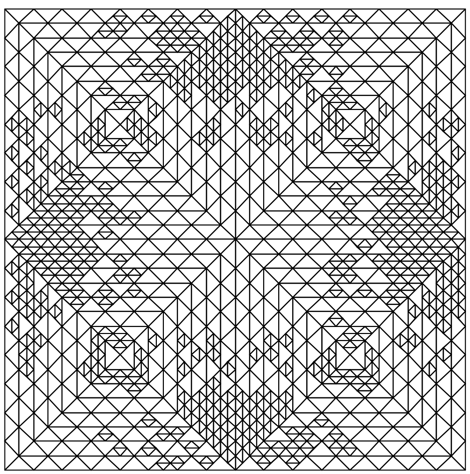
est and err versus k



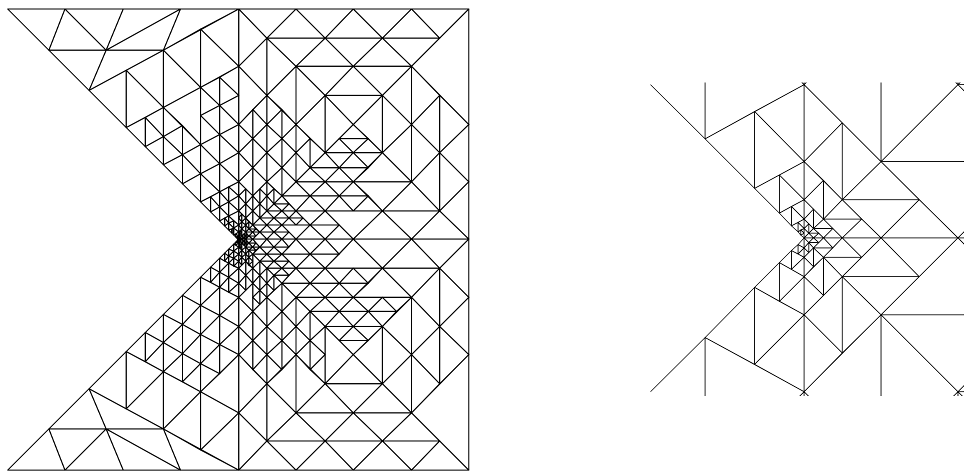
$\epsilon = 0.02$

$r = 3$ (quadratic fcts)

Final mesh



Final mesh and zoom around the corner



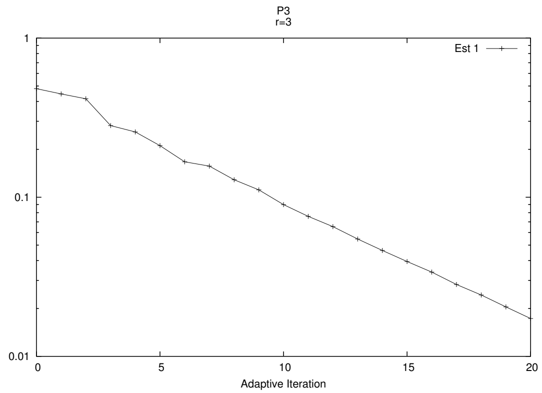
SIPG : numerical results : 2/2

A test case with an incompatibility of the Neumann and Dirichlet data

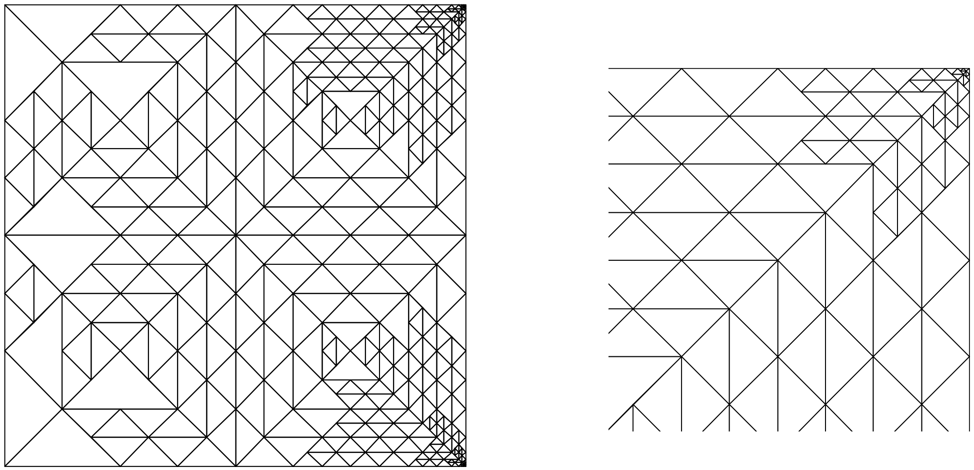
$-\Delta u = 0$ in $\Omega = [0,1]^2$

$u = 0$ on Γ_D and $\nabla u \cdot n = -1$ on Γ_N = the line segment joining (1,0) to (1,1)

est versus k



Final mesh and zoom around the point (1,1)



$\epsilon = 0.02$

$r = 3$ (quadratic fcts)

Implementation of the adaptive code done by Mike Saum, UTK