The G^α-scheme for Approximation of Fractional Derivatives: Application to the Dynamics of Dissipative Systems

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Abstract. – The Gear scheme is a three-level step algorithm, backward in time and second order accurate for the approximation of classical time derivatives. In this contribution, the formal power of this scheme is proposed to approximate fractional derivative operators in the context of finite difference methods. Some numerical examples are presented and analyzed in order to show the effectiveness of the present Gear scheme at the power α (G^α-scheme) when compared to the classical Grünwald-Letnikov approximation. In particular, for a fractional damped oscillator problem, the combined G^α-Newmark scheme is shown to be second-order accurate.

Keywords: Fractional derivatives, Viscoelasticity, Multi-step scheme, Linear dynamics.

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1 Introduction

The importance of fractional calculus for modeling viscoelastic materials has been recognized by the mechanical scientific community since the article of (Bagley and Torvik, 1983). The numerical approximation of such systems has been intensively studied since the work of (Padovan, 1987). On the other side, the numerical community is interested in the approximation of fractional derivatives. One refers to the pioneering theoretical work of (Lubich, 1986) and the state of the art proposed by (Diethelm et al., 2005). Most applications use the discrete convolution formula proposed by Grünwald-Letnikov (GL-scheme). Another direction could be autonomous systems in the context of diffusive representations (Matignon and Montseny (Eds.), 1998; Yuan and Agrawal, 2002; Trinks and Ruge, 2002).

In this work, we focus on the application of a numerical method based on the Gear scheme for the approximation of fractional derivatives in linear dynamics. Such a scheme is called here G^α-scheme. It should be stressed that preliminary tests of convergence have been performed in a recent work (see (Galucio et al., 2006)). Finally, two examples are presented and analyzed. The first one deals with the study of a harmonic oscillator with fractional damping in order i) to validate the method and ii) to derive an order of convergence. The use of the G^α-scheme is then extended to viscoelastic beams submitted to an applied time-dependent force.

2 The G^α-scheme

Let us introduce the G^α-operator, which is based in the Gear scheme, to approximate fractional derivatives

$$\mathrm{G}^\alpha = \frac{1}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \left[I - \frac{4}{3}\delta^- + \frac{1}{3}(\delta^-)^2\right]^\alpha \tag{1}$$

where Δt, which is supposed to be fixed, is the time step.

Let u be a time dependent function known only by its discretized values u^n at each time t^n, where n is a positive integer. The function u^n is approximated by $u(t^n)$ with $t^n = n\Delta t$. The α-derivative of u at time t^n can be approximated by

$$(\mathrm{G}^\alpha u)^n = \frac{1}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \sum_{j=0}^{\infty} g_{j+1} u^{n-j} \tag{2}$$

where g is a rational number. The calculation of these G^α-coefficients is a hard task due to cumulative numerical errors. In order to overcome such a difficulty, the method employed here consists of calculating these coefficients analytically using Symbolic Matlab

Toolbox. For illustrative purposes, the reader is referred to Table 1, where the first ten G^α-coefficients are presented for three values of α: 1/3, 1/2, and 3/4.

Table 1: First ten coefficients g_{j+1} of the formal power series (2).

j	$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 3/4$
0	1	1	1
1	$-\frac{4}{9}$	$-\frac{2}{3}$	-1
2	$-\frac{7}{81}$	$-\frac{1}{18}$	$\frac{1}{12}$
3	$-\frac{104}{2187}$	$-\frac{1}{27}$	$-\frac{1}{108}$
4	$-\frac{643}{19683}$	$-\frac{17}{648}$	$-\frac{1}{96}$
5	$-\frac{4348}{177147}$	$-\frac{19}{972}$	$-\frac{7}{864}$
6	$-\frac{92809}{4782969}$	$-\frac{59}{3888}$	$-\frac{193}{31104}$
7	$-\frac{683552}{43046721}$	$-\frac{71}{5832}$	$-\frac{151}{31104}$
8	$-\frac{5164958}{387420489}$	$-\frac{2807}{279936}$	$-\frac{5813}{1492992}$
9	$-\frac{358288744}{31381059609}$	$-\frac{10627}{1259712}$	$-\frac{128713}{40310784}$
10	$-\frac{2805807422}{282429536481}$	$-\frac{109159}{15116544}$	$-\frac{430313}{161243136}$

3 The fractional damped oscillator problem

Consider a fractional one-dof system submitted to a constant step load f for $t > 0$ with zero initial conditions. The damping is taken into account by introducing a fractional damping term or a *spring-pot* element in the formulation. The corresponding governing equation as well as the initial conditions are given by

$$\begin{cases} m\ddot{u} + c\tau^\alpha \mathcal{D}^\alpha u + ku = f, \quad t > 0 \\ u(0) = \dot{u}(0) = 0 \end{cases} \tag{3}$$

where m and k are mass and stiffness constants; and $c\tau^\alpha$ is a fractional damping constant with τ the relaxation time and c the classical damping constant.

The aim of this section is to solve the set of equations (3) with a direct time integration method (Newmark) in conjunction with an approximation for the α-derivative $\mathcal{D}^\alpha u$ (G^α- or GL-scheme). Furthermore, in order to validate such combinations, the approximated solution is compared to an exact solution proposed by (Galucio et al., 2006). Finally, error estimates in L^∞ norm are performed. For a fixed time step $\Delta t = 1/2^m$, this error is computed by

$$e_\infty^m = \max\{|u(j\Delta t) - u^j|\,,\ j = 0, \cdots, 2^m\} \tag{4}$$

where m is a positive integer.

3.1 Algorithm

As mentioned above, the average acceleration algorithm is used to solve Eq. (3). The displacement history arising from the α-derivative approximation (damping term) is shifted to the right-hand side of Eq. (3) (Galucio et al., 2004). Therefore, using (2), the governing equation in its discretized form is written as

$$m\ddot{u}^{n+1} + (k+\kappa)u^{n+1} = f^{n+1} + \phi^{n+1} \tag{5}$$

where the non-classical terms κ and ϕ arise from the approximation of the α-derivative:

$$\kappa = \frac{c\tau^\alpha}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \tag{6a}$$

$$\phi^{n+1} = -\frac{c\tau^\alpha}{\Delta t^\alpha}\left(\frac{3}{2}\right)^\alpha \sum_{k=1}^{N} g_{k+1}u^{n+1-k} \tag{6b}$$

One notes that the stiffness term κ is constant in time, depending only on the time step, which is supposed to be fixed. Concerning the modified loading ϕ, it depends on the displacement history.

3.2 Results

In all calculations performed below, we assume that $m = k = \tau = f = 1$ in a suitable unit system. In Table 2, as well in Figures 1–2, one assumes that $c = 1$. Moreover, three values of α are tested. The final time is chosen to be $T = 15$ for various values of time step.

It should be pointed out that the error estimates in L^∞ norm are obtained using an exact solution based on formal power series as mentioned above.

In Table 2, error estimates in L^∞ norm are presented. One notes that the combined G^α-Newmark scheme keeps the second-order accuracy for any value of α. However, the use of a GL-Newmark algorithm decreases the order of accuracy to 1.

Figures 1–2 show the evolution of the displacement for two values of α as well the error estimates on L^∞ norm, when using a combined G^α- or GL-Newmark scheme.

In Figures 1–2 (a), the exact solution of Eq. (3) and its corresponding numerical approximations (GL and G^α methods) are presented, with a time discretization corresponding to $2^6 = 64$ time steps. One can easily note that the solution obtained by using the

Table 2: Rate of convergence computed with the L^∞ norm for three values of α.

	$\alpha = 1/3$	$\alpha = 1/2$	$\alpha = 3/4$
G^α	1.99	1.96	1.90
GL	1.00	0.99	0.99

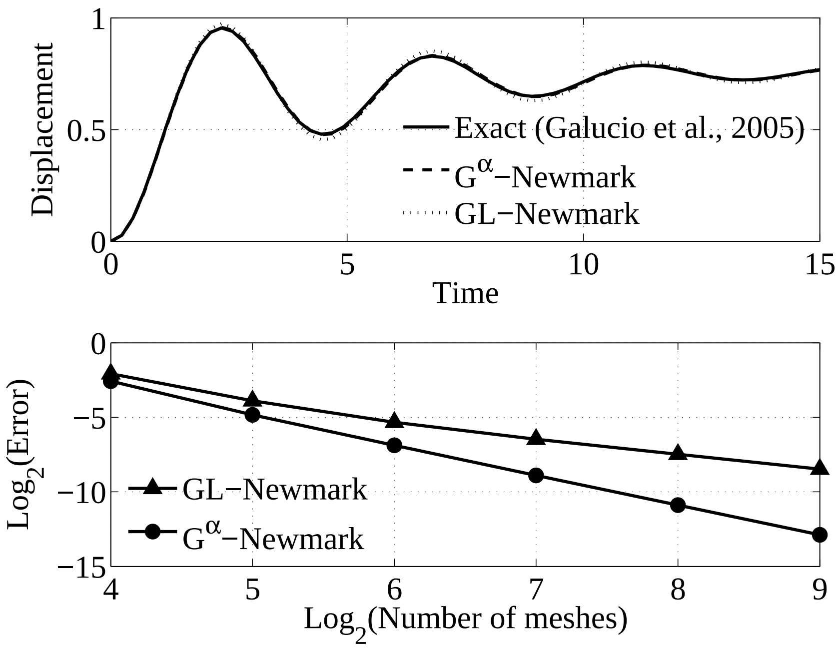

Figure 1: (a) Exact and approximated solutions of (3) for $\alpha = 1/3$ and $\Delta t = T/2^6$; (b) Error estimates in L^∞ norm

G^α-scheme is very close to the exact solution while that one obtained by the GL-method is overestimated.

Error estimates in L^∞ norms are presented in Figures 1–2 (b). In both situations, the combined G^α-Newmark scheme shows a better accuracy than the GL one. The rates of convergence presented in Table 2 are computed with 7–9 meshes, otherwise the slopes are wrongly estimated.

It should be emphasized that the order of the fractional derivative does not affect the rate of convergence (according to Table 2 and Figures 1–2) when using a Newmark integrator. The influence of α is observed in the mechanical behavior of the fractional damped oscillator by means of a damping factor. In other words, when α decreases, the damping and the time required to achieve the quasistatic time solution increase.

In order to show the influence of an added damping, the results presented below are computed for a fixed value of $\alpha = 1/2$ and different values of the classical damping

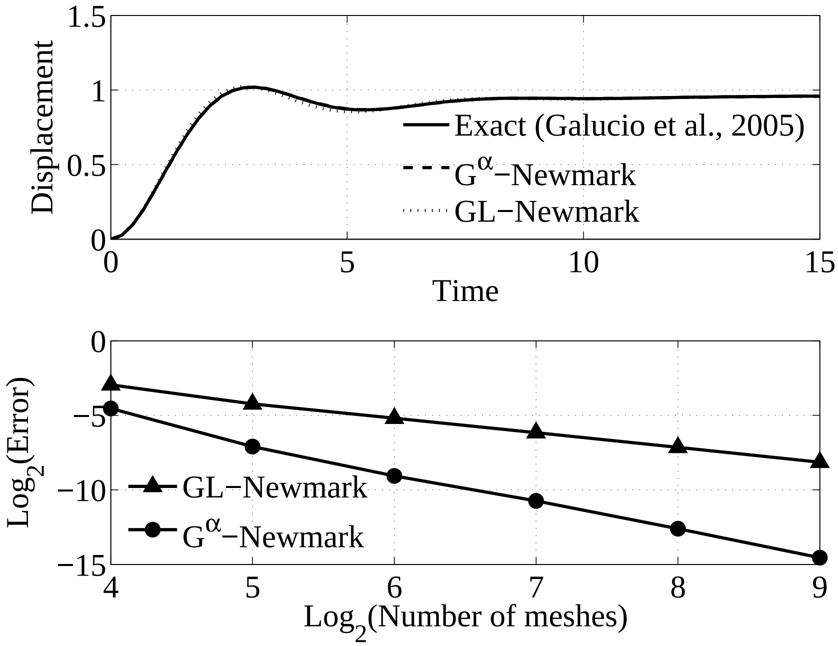

Figure 2: (a) Exact and approximated solutions of (3) for $\alpha = 3/4$ and $\Delta t = T/2^6$; (b) Error estimates in L^∞ norm

Table 3: Rate of convergence computed with the L^∞ norm for three values of c.

	$c = 0.50$	$c = 1.00$	$c = 1.50$
G^α	1.98	1.96	1.93
GL	0.96	0.99	1.00

constant c. According to Table 3 (see also Table 2), the rate of convergence remains the same. As in previous results, using the combined G^α-Newmark algorithm, the order of accuracy is at about 2.

For illustrative purposes, the responses of the oscillator computed with the G^α-Newmark scheme are presented in Fig. 3 (a) for three values of damping: $c = 0.5, 1.0, 1.5$ (see Eq. (3)). These results are obtained for a semi-derivative problem. Comparing to the exact solution, the corresponding error in L^∞ norm is plotted in Fig. 3 (b). One notes that the rate of convergence remains the same for all c.

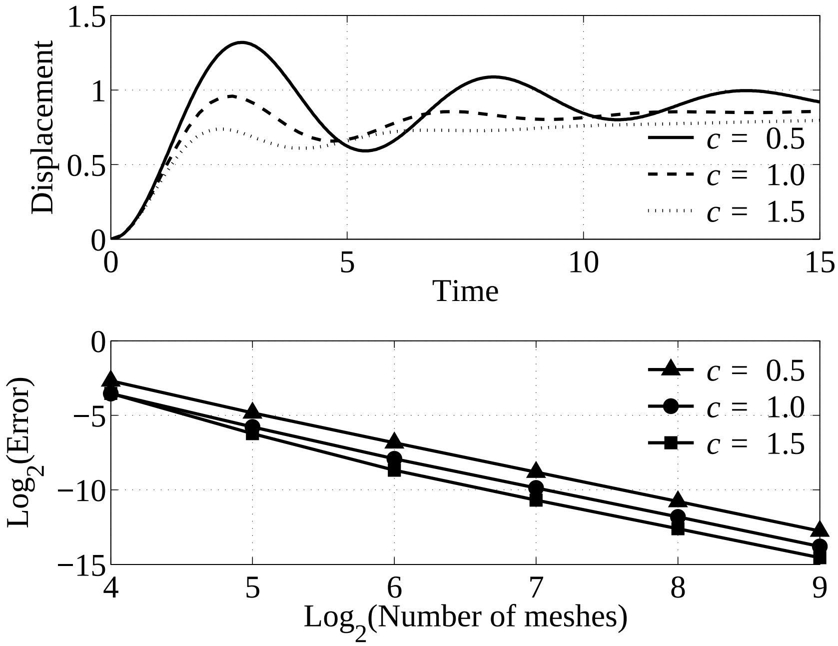

Figure 3: (a) Exact and approximated solutions of (3) for $\alpha = 1/2$ and $\Delta t = T/2^9$; (b) Error estimates in L^∞ norm

4 Extension to viscoelastic beams

4.1 Viscoelastic Constitutive Equations

The one-dimensional fractional Zener model is adopted in this section to describe the behavior of a viscoelastic material (Bagley and Torvik, 1983)

$$\sigma(t) + \tau^\alpha \mathcal{D}^\alpha \sigma(t) = E_o \varepsilon(t) + E_\infty \tau^\alpha \mathcal{D}^\alpha \varepsilon(t) \tag{7}$$

where σ and ε are the stress and the strain, E_o and E_∞ are the relaxed and non-relaxed elastic moduli, and τ is the relaxation time.

In order to facilitate the numerical implementation of this model, let us introduce an internal variable as an "anelastic" strain function:

$$\varepsilon_\alpha = \varepsilon - \sigma / E_\infty \tag{8}$$

This expression when replaced in Eq. (7) results only one fractional derivative operator.

4.2 Algorithm

For the sake of brevity, finite element considerations are not presented in this investigation (for more details, reader is referred to (Galucio et al., 2004)). The governing equation to be solved takes the following form

$$(9) \qquad \begin{cases} \mathbf{M}\ddot{\mathbf{q}}^{n+1} + (\mathbf{K} + \boldsymbol{\mathcal{K}})\mathbf{q}^{n+1} = \mathbf{F}^{n+1} + \boldsymbol{\Phi}^{n+1} \\ \mathbf{q}^0 = \dot{\mathbf{q}}^0 = \mathbf{0} \end{cases}$$

where $\mathbf{M}$ and $\mathbf{K}$ are the mass and stiffness matrices, $\mathbf{q}$ the degree-of-freedom vector and $\mathbf{F}$ a mechanical load. Moreover, $\boldsymbol{\mathcal{K}}$ and $\boldsymbol{\Phi}$ are the terms arising from the viscoelastic behavior of the beam such that

$$(10a) \qquad \boldsymbol{\mathcal{K}} = c_\alpha \frac{E_\infty - E_o}{E_o} \mathbf{K}$$

$$(10b) \qquad \boldsymbol{\Phi}^{n+1} = -c_\alpha \frac{E_\infty}{E_o} \mathbf{K} \sum_{k=1}^{N} g_{k+1} \mathbf{q}_\alpha^{n+1-k}$$

with $c_\alpha = \tau^\alpha / (\tau^\alpha + \Delta t^\alpha)$. We note that, as in the one-dof problem, the added stiffness matrix $\boldsymbol{\mathcal{K}}$ does not depend on time, while the dissipative force $\boldsymbol{\Phi}$ depends on the history of "anelastic" displacements, which are updated at each time step by

$$\mathbf{q}_\alpha^{n+1} = (1 - c_\alpha) \frac{E_\infty - E_o}{E_\infty} \mathbf{q}^{n+1} - c_\alpha \sum_{k=1}^{N} g_{k+1} \mathbf{q}_\alpha^{n+1-k}$$

It is important to observe that the introduction of $\mathbf{q}_\alpha$ in the formulation does not imply an augmentation of the system. It can be considered as an intermediate variable in the time scheme (see (Galucio et al., 2004)).

As in the previous example, the average acceleration algorithm is used to solve Eq. (9).

4.3 Results

Consider a viscoelastic cantilever beam of length $L = 150$ mm, width $b = 25$ mm and thickness $h = 5$ mm, discretized with 5 finite elements. The mechanical characteristics of the fictitious viscoelastic material are: mass density $\rho = 1000$ kg/m^3, Poisson's ratio $\nu = 0.5$, relaxed elastic modulus $E_o = 1$ MPa, non-relaxed elastic modulus $E_\infty = 50$ MPa, relaxation time $\tau = 1$ ms and order of the fractional derivative $\alpha = 0.5$. The beam is subjected to a transversal load at its free end such that

$$F(t) = \begin{cases} F_o t / t_1 \quad , \quad 0 \leq t \leq t_1 \\ F_o \quad , \quad t \geq t_1 \end{cases}$$

where $F_o = 0.01$ N, $t_1 = 50$ ms and $T = 1$ s. Additionally, the time step is $\Delta t = 2$ ms and the whole time history of "anelastic" displacements is used in the calculations.

In Fig. 4, transient responses of the damped viscoelastic beam are presented. The evolution of the tip displacement and the phase-space diagram are plotted in Fig. 4(a) and (b), respectively. As expected, we observe that the oscillations of the viscoelastic beam are damped.

It should be pointed out that these preliminary results show the versatility of the G^α-scheme since its implementation is easy and without additional costs when compared to the GL-one.

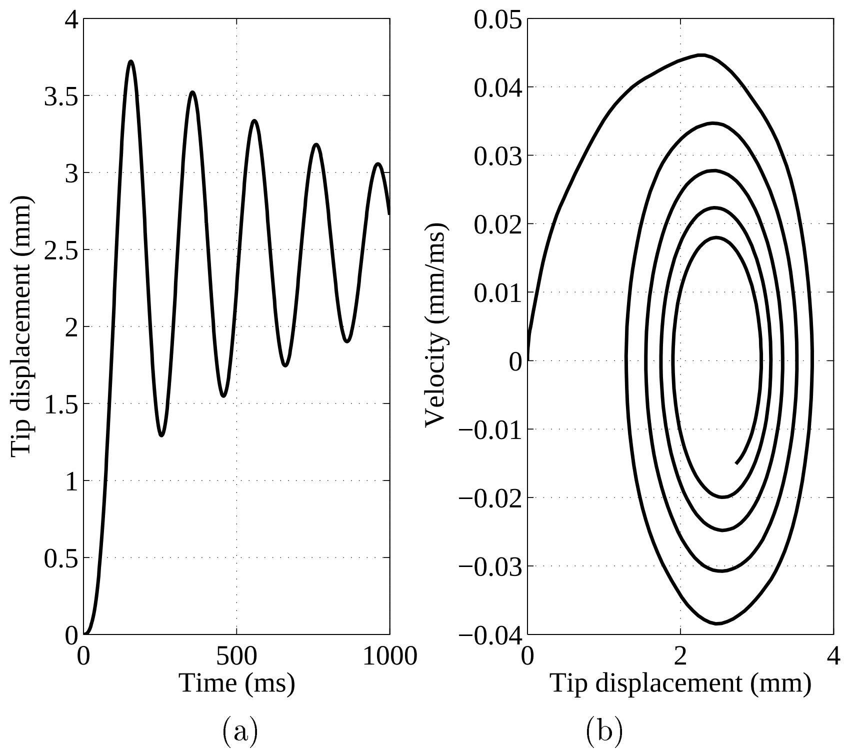

Figure 4: Damped responses of the viscoelastic beam: (a) Tip displacement versus time; (b) Phase-space diagram

5 Conclusion

A numerical method based on the Gear scheme to approximate fractional derivatives is used here to model damping in linear dynamics. This G^α-scheme is written in terms of a formal power series, where the coefficients have to be calculated. The numerical evaluation of G^α-coefficients is delicate due to a bad conditioning of the recurrence formula. However, with the help of formal calculus, cumulative numerical errors are avoided.

Two examples are presented and analyzed. In both cases, the average-acceleration algorithm is used to integrate the governing equation. The first example concerns a single degree-of-freedom oscillator with a fractional damping. In order to validate the presented approach, numerical results are compared to an exact solution for the single dof problem submitted to a constant load (Galucio et al., 2006). The combined algorithm G$^\alpha$-Newmark is a promising tool for dynamic problems since a two-order accuracy is obtained. The second example deals with the finite element implementation of a viscoelastic beam submitted to a mechanical load.

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