

# On quantum models for opinion and voting intention polls

François Dubois

*Conservatoire National des Arts et Métiers, Paris, France.*

*francois.dubois@cnam.fr*

10 september 2013 <sup>1</sup>

**Abstract.** In this contribution, we construct a connection between two quantum voting models presented previously. We propose to try to determine the result of a vote from associated given opinion polls. We introduce a density operator relative to the family of all candidates to a particular election. From an hypothesis of proportionality between a family of coefficients which characterize the density matrix and the probabilities of vote for all the candidates, we propose a numerical method for the entire determination of the density operator. This approach is a direct consequence of the Perron-Frobenius theorem for irreducible positive matrices. We apply our algorithm to synthetic data and to operational results issued from the French presidential election of April 2012.

**Key words:** density matrix, Perron-Frobenius theorem, French presidential election.

**AMS classification:** 65F15, 81Q99, 91C99.

## 1) Introduction

Electoral periods are favorable to opinion polls. We keep in mind that opinion polls are intrinsically complex (see *e.g.* Gallup [14]) and give an approximate picture of a possible social reality. They are traditionally of two types: popularity polls for various outstanding political personalities and voting intention polls when a list of candidates is known. We have two different informations and to construct a link between them is not an easy task. In particular, the determination of the voting intentions is a quasi intractable problem! Predictions of votes classically use of so-called “voting functions”. Voting functions have been developed for the prediction of presidential elections in the United States. They are based on correlations between economical parameters, popularity polls and other technical parameters. We refer to Abramowitz [1], Lewis-Beck [22], Campbell [10] and Lafay [20].

---

<sup>1</sup> This contribution was presented at the Seventh International Quantum Interaction Conference, Leicester, 25th - 27th July 2013. Published in the *Lecture Notes in Computer Science*, volume 8369, pages 286-295, H. Atmanspacher, E. Haven, K. Kitto, D. Raine Editeurs, DOI 10.1007/978-3-642-54943-4, Springer, 2014.

We do not detail here the mathematical difficulties associated with the question of voting when the number of candidates is greater than three [2] [6] [9]. They conduct to present-day researches like range voting, independently proposed by Balinski and Laraki [3] [4] and by Rivest and Smith [24] [25]. It is composed by two steps: grading and ranking. In the grading step, all the candidates are evaluated by all the electors. This first step is quite analogous to a popularity investigations and we will merge the two notions in this contribution. The second step of range voting is a majority ranking; it consists of a successive extraction of medians.

In this contribution, we adopt quantum modelling (see *e.g.* Bitbol *et al* [5] for an introduction), in the spirit of authors like Khrennikov and Haven [17] [16], La Mura and Swiatczak [21] and Zorn and Smith [28] concerning voting processes. Moreover, Wang *et al.* [27] present a quantum model for question order effects found with gallup poles. The fact of considering quantum modelling induces a specific vision of probabilities. We refer *e.g.* to the classical treatise on quantum mechanics of Cohen-Tannoudji *et al.* [8], to the so-called contextual objectivity proposed by Grangier [15], or to the elementary introduction proposed by Busemeyer and Trueblood [7] in the context of statistical inference.

This contribution is organized as follows: we recall in Sections 1 and 2 two quantum models for the vote developed previously [11] [12] and a first tentative [13] to connect these two models (Section 4). In Section 5, we develop the main idea of this paper. We construct a link between opinion polls and voting. This idea is tested numerically in Sections 6 and 7 for synthetic data and a “real life” election.

## 2) A fundamental elementary model

In a first tentative [11], we have proposed to introduce an Hilbert space  $V_\Gamma$  formally generated by the candidates  $\gamma_j \in \Gamma$ . In this space, a candidate  $\gamma_j$  is represented by a unitary vector  $|\gamma_j\rangle$  and this family of  $n$  vectors is supposed to be orthogonal. Then an elector  $\ell$  can be decomposed in the space  $V_\Gamma$  of candidates according to

$$(1) \quad |\ell\rangle = \sum_{j=1}^n \theta_j |\gamma_j\rangle .$$

The vector  $|\ell\rangle \in V_\Gamma$  is supposed to be a unitary vector to fix the ideas. According to Born’s rule, the probability for a given elector  $\ell$  to give his voice to the particular candidate  $\gamma_j$  is equal to  $|\theta_j|^2$ . The violence of the quantum measure is clearly visible with this example: the opinions of an elector  $\ell$  never coincide with the program of any candidate. But with a voting system where an elector has to choice only one candidate among  $n$ , his social opinion is *reduced* to the one of a particular candidate.

### 3) A quantum model for range voting

Our second model [12] is adapted to the grading step of range voting [3, 24]. We consider a grid  $G$  of  $m$  types of opinions as one of the two following ones. We have  $m = 5$  for the first grid (2) and  $m = 3$  for the second one (3):

$$(2) \quad ++ \ \succ \ + \ \succ \ 0 \ \succ \ - \ \succ \ --$$

$$(3) \quad + \ \succ \ 0 \ \succ \ -.$$

These ordered grids are typically used for popularity polls. We assume also that a ranking grid like (2) or (3) is a basic tool to represent a social state of the opinion. We introduce a specific grading space  $W_G$  of political appreciations associated with a grading family  $G$ . The space  $W_G$  is formally generated by the  $m$  orthogonal vectors  $|\zeta_i\rangle$  relative to the opinions. Then we suppose that the candidates  $\gamma_j$  are now decomposed by each elector on the basis  $|\zeta_i\rangle$  for  $1 \leq i \leq m$ :

$$(4) \quad |\gamma_j\rangle = \sum_{i=1}^m \alpha_j^i |\zeta_i\rangle, \quad \gamma_j \in \Gamma, \quad 1 \leq j \leq n.$$

Moreover the vector  $|\gamma_j\rangle$  in (4) is supposed to be by a unitary:

$$(5) \quad \sum_{i=1}^m |\alpha_j^i|^2 = 1, \quad \gamma_j \in \Gamma, \quad 1 \leq j \leq n.$$

With this notation, the probability for a given elector to appreciate a candidate  $\gamma_j$  with an opinion  $\zeta_i$  is simply a consequence of the Born rule. The mean statistical expectation of a given opinion  $\zeta_i$  for a candidate  $\gamma_j$  is equal to  $|\alpha_j^i|^2$  on one hand and is given by the popularity polls  $S_{ji}$  on the other hand. Consequently,

$$|\alpha_j^i|^2 = S_{ji}, \quad \gamma_j \in \Gamma, \quad \zeta_i \in G, \quad 1 \leq j \leq n, \quad 1 \leq i \leq m.$$

### 4) A first link between the two previous models

In [13], we have proposed a first link between the two previous models. We simplify the approach (1) and suppose that there exists some equivalent candidate  $|\xi\rangle \in V_\Gamma$  such that the voting intention for each particular candidate  $\gamma_j \in \Gamma$  is equal to  $|\langle \xi, \gamma_j \rangle|^2$ . We interpret the relation (4) in the following way: for each candidate  $\gamma_j \in \Gamma$ , there exists a political decomposition  $A|\gamma_j\rangle \in W_G$  in terms of the grid  $G$ . By linearity, we construct in this way a linear operator  $A : V_\Gamma \rightarrow W_G$  between two different Hilbert spaces. Preliminary results have been presented, in the context of the 2012 French presidential election.

## 5) From opinion polls to the prediction of the vote

In the space  $W_G$  of political appreciations described in Section 3 of this contribution, the opinion polls allow through the relation (4) to determine some knowledge about each candidate  $\gamma_j \in \Gamma$  in the space  $W_G$ . We suppose that each candidate is represented by a unitary vector and the relation (5) still holds. The question is now to evaluate the probability for an arbitrary elector to vote for the various candidates.

We denote by  $\Pi_j \equiv |\gamma_j\rangle\langle\gamma_j|$  the orthogonal projector onto the direction of the state  $|\gamma_j\rangle$ . Then we introduce a density matrix  $\rho$  associated to a statistical representation of the voting population:

$$(6) \quad \rho = \sum_{j=1}^n \alpha_j \Pi_j \equiv \sum_{j=1}^n \alpha_j |\gamma_j\rangle\langle\gamma_j|.$$

It is classical that  $\text{tr}\rho = \sum_{j=1}^n \alpha_j$  and if  $\alpha_j \geq 0$  for each index  $j$ , the auto-adjoint operator  $\rho$  is non-negative:

$$\langle\rho\zeta, \zeta\rangle \geq 0, \quad \forall\zeta \in W_G.$$

It is then natural to search the coefficients  $\alpha_j$  such that

$$(7) \quad \begin{cases} \alpha_j \geq 0, & 1 \leq j \leq n \\ \sum_{j=1}^n \alpha_j = 1. \end{cases}$$

In these conditions, the esperance of election  $\langle\gamma_j\rangle$  of the candidate  $\gamma_j$  is given through the relation

$$(8) \quad \langle\gamma_j\rangle = \text{tr}(\rho\Pi_j), \quad 1 \leq j \leq n.$$

We have the following calculus:

$$\begin{aligned} \text{tr}(\rho\Pi_j) &= \sum_{k=1}^n \langle\zeta_k, \rho\Pi_j\zeta_k\rangle = \sum_{k=1}^n \sum_{\ell=1}^n \langle\zeta_k, \alpha_\ell\gamma_\ell\rangle \langle\gamma_\ell, \gamma_j\rangle \langle\gamma_j, \zeta_k\rangle \\ &= \sum_{\ell=1}^n \alpha_\ell \langle\gamma_\ell, \gamma_j\rangle \sum_{k=1}^n \langle\zeta_k, \gamma_\ell\rangle \langle\gamma_j, \zeta_k\rangle \\ &= \sum_{\ell=1}^n \alpha_\ell \langle\gamma_\ell, \gamma_j\rangle \sum_{k=1}^n \langle\gamma_j, \zeta_k\rangle \langle\zeta_k, \gamma_\ell\rangle \\ &= \sum_{\ell=1}^n \alpha_\ell \langle\gamma_\ell, \gamma_j\rangle \langle\gamma_j, \gamma_\ell\rangle = \sum_{\ell=1}^n \alpha_\ell |\langle\gamma_\ell, \gamma_j\rangle|^2. \end{aligned}$$

We introduce the matrix  $A$  composed by the squares of the scalar products of the vectors of candidates:

$$(9) \quad A_{j\ell} = |\langle\gamma_j, \gamma_\ell\rangle|^2, \quad 1 \leq j, \ell \leq n.$$

Then the previous calculus establishes that

$$(10) \quad \langle\gamma_j\rangle = \sum_{\ell=1}^n A_{j\ell} \alpha_\ell.$$

It is interesting to imagine a link between the esperance of election  $\langle \gamma_j \rangle$  of the candidate  $\gamma_j$  and the coefficient  $\alpha_j$  of the density matrix introduced in (6). In general they differ. In the following, we focus our attention to the particular case where these two quantities are proportional, *id est*

$$(11) \quad \exists \lambda \in \mathbb{C}, \quad \forall j = 1, 2, \dots, n, \quad \langle \gamma_j \rangle = \lambda \alpha_j.$$

Because both  $\langle \gamma_j \rangle$  and  $\alpha_j$  are positive, the coefficient  $\lambda$  must be a positive number. Moreover, due to (10), the relation (11) express that the non-null vector  $\alpha \in \mathbb{R}^n$  composed by the coefficients  $\alpha_j$  is an eigenvector of the matrix  $A$ . Then, due to the hypothesis (7), we have  $\alpha_j \geq 0$  and this eigenvector has non-negative components. If we suppose that the matrix  $A$  is irreducible (see *e.g.* in the book of Meyer [23] or Serre [26]), the Perron-Frobenius theorem states that there exists a **unique** eigenvalue (equal to the spectral radius of the matrix  $A$ ) such that the corresponding eigenvector has all non-negative components. Moreover, all the components of this eigenvector are strictly positive. In other words, if the matrix  $A$  defined in (9) is irreducible and if the hypothesis of proportionality (11) is satisfied, the coefficients  $\alpha_j$  of the density matrix are, due to the second relation of (7), completely defined. In the following, we propose to determine the coefficients  $\alpha_j$  of the density matrix (6) and satisfying the conditions (7) as proportional to the positive eigenvector of the matrix  $A$  defined by (9).

The above model is not completely satisfactory for the following reason. The underlying order associated to the grading family  $G$  has not been taken into account. To fix the ideas, we suppose that each grade  $\nu_i$  is associated to a number  $\sigma_i$  such that

$$(12) \quad \sigma_1 > \sigma_2 > \dots > \sigma_m.$$

We introduce a ‘‘popularity operator’’  $P_j$  associated to the  $j$ th candidate  $\gamma_j$ :

$$(13) \quad P_j \equiv \sum_{i=1}^m \sigma_i |\langle \gamma_j, \zeta_i \rangle|^2 |\zeta_i\rangle\langle \zeta_i|.$$

We can determine without difficulty the mean value of the operator  $P_j$  for the density configuration  $\rho$  defined in (6):

$$\begin{aligned} \text{tr}(\rho P_j) &= \sum_{k=1}^m \langle \zeta_k, \rho P_j \zeta_k \rangle \\ &= \sum_{k=1}^m \sum_{\ell=1}^n \langle \zeta_k, \alpha_\ell \gamma_\ell \rangle \langle \gamma_\ell, \sum_{i=1}^m \sigma_i |\langle \gamma_j, \zeta_i \rangle|^2 |\zeta_i\rangle\langle \zeta_i, \zeta_k \rangle \\ &= \sum_{i=1}^m \sum_{\ell=1}^n \sigma_i \alpha_\ell \langle \zeta_i, \gamma_\ell \rangle \langle \gamma_\ell, \zeta_i \rangle |\langle \gamma_j, \zeta_i \rangle|^2 \\ &= \sum_{i=1}^m \sum_{\ell=1}^n \sigma_i \alpha_\ell |\langle \gamma_\ell, \zeta_i \rangle|^2 |\langle \gamma_j, \zeta_i \rangle|^2. \end{aligned}$$

In other words, if we set

$$(14) \quad B_{j\ell} \equiv \sum_{i=1}^m \sigma_i |\langle \gamma_j, \zeta_i \rangle|^2 |\langle \gamma_\ell, \zeta_i \rangle|^2,$$

we have:

$$(15) \quad \langle P_j \rangle \equiv \text{tr}(\rho P_j) = \sum_{\ell=1}^n B_{j\ell} \alpha_\ell.$$

We use a positive parameter  $t$  and search the coefficients  $\alpha_j$  in such a way that the mean value of the candidate  $\gamma_j$  with some “upwinding” associated to its popularity is proportional to the above coefficients. In other words, due to (10) and (15), the mean value  $\langle \gamma_j \rangle + t \langle P_j \rangle$  takes the algebraic form

$$(16) \quad \langle \gamma_j \rangle + t \langle P_j \rangle = \sum_{\ell=1}^n (A_{j\ell} + t B_{j\ell}) \alpha_\ell.$$

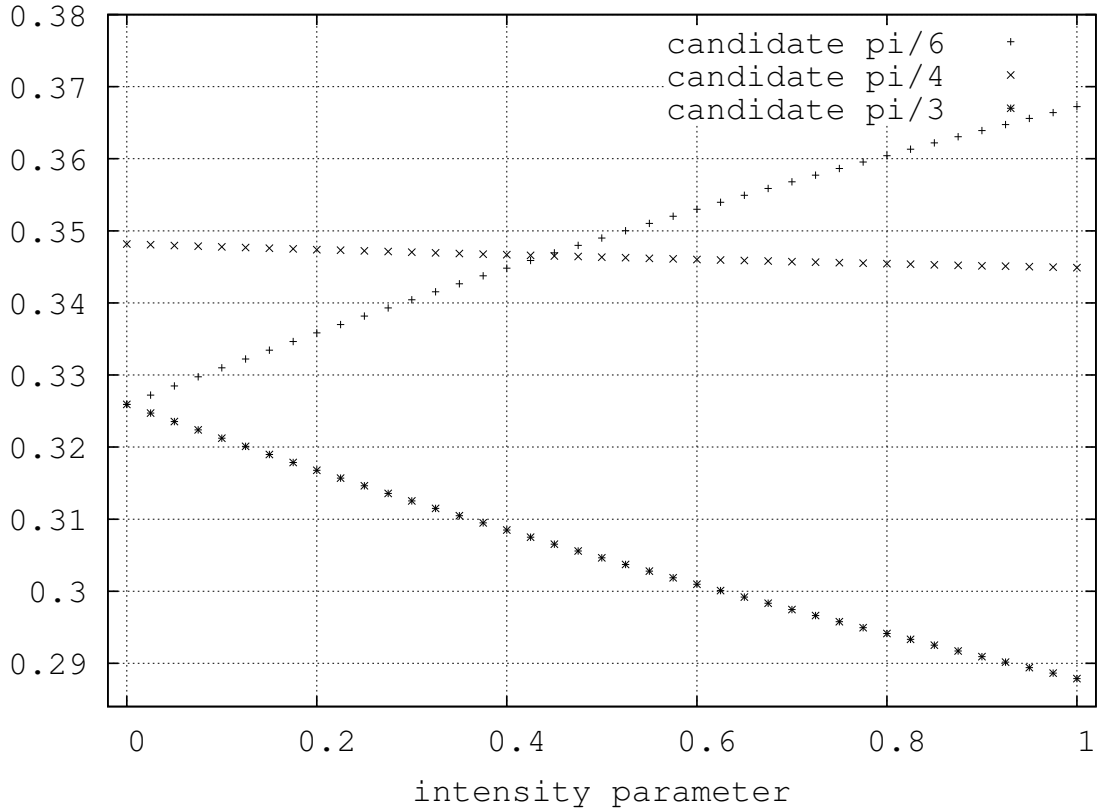
Under the condition that all the coefficients  $A_{j\ell} + t B_{j\ell}$  are positive, *id est* that the parameter  $t$  is small enough, we compute the coefficients  $\alpha_\ell$  with the help of the Perron-Frobenius theorem as presented previously.

## 6) A first numerical test case

Our first model uses synthetic data. We suppose that we have three candidates ( $n = 3$ ) and two ( $m = 2$ ) levels of “political” appreciation. We suppose that

$$(17) \quad \begin{cases} |\gamma_1\rangle = \cos\left(\frac{\pi}{6}\right) |\zeta_1\rangle + \sin\left(\frac{\pi}{6}\right) |\zeta_2\rangle \\ |\gamma_2\rangle = \cos\left(\frac{\pi}{4}\right) |\zeta_1\rangle + \sin\left(\frac{\pi}{4}\right) |\zeta_2\rangle \\ |\gamma_3\rangle = \cos\left(\frac{\pi}{3}\right) |\zeta_1\rangle + \sin\left(\frac{\pi}{3}\right) |\zeta_2\rangle . \end{cases}$$

With the choice  $\sigma_1 = 1$  and  $\sigma_2 = 0$  in a way suggested at the relation (12), we can simulate numerically the process presented in the Section 5. The results are presented in Figure 1. When the variable  $t$  is increasing, the first candidate has a better score, due to his best results in the grading evaluation (17).



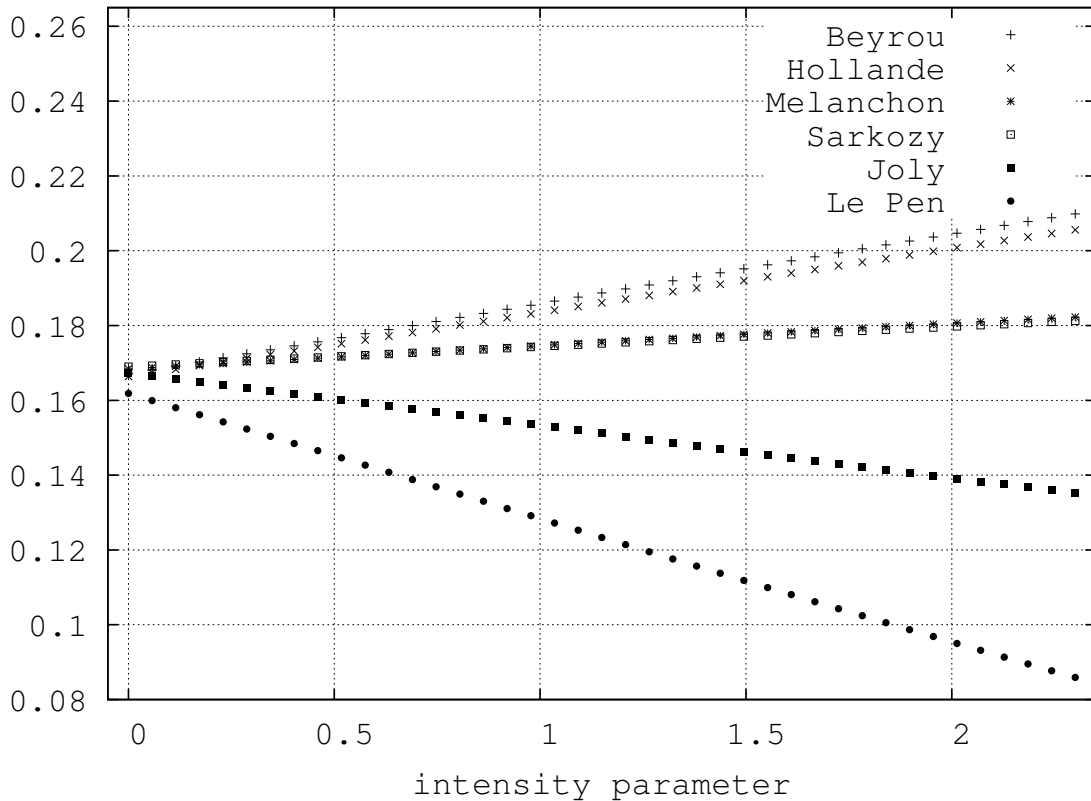
**Figure 1.** Result of the vote obtained by a quantum model from the opinion poll, with synthetic data proposed in (17).

## 7) Test of the method with real data

We have also used data coming from the “first tour” of the French presidential election of April 2012. Popularity data [18] and result of voting intentions [19] are displayed in Table 3. The names of the principal candidates to the French presidential election are proposed in alphabetic order with the following abbreviations: “Ba” for François Bayrou, “Ho” for François Hollande, “Jo” for Eva Joly, “LP” for Marine Le Pen, “Mé” for Jean-Luc Mélançon and “Sa” for Nicolas Sarkozy. In Table 3, we have also reported the result of the election of 22 April 2012.

|    | +   | 0   | −   | voting | result |
|----|-----|-----|-----|--------|--------|
| Ba | .56 | .07 | .37 | .095   | .091   |
| Ho | .57 | .03 | .40 | .285   | .286   |
| Jo | .35 | .10 | .55 | .015   | .023   |
| LP | .26 | .05 | .69 | .15    | .179   |
| Mé | .47 | .10 | .43 | .145   | .111   |
| Sa | .49 | .05 | .46 | .29    | .272   |

**Table 1.** Popularity, sounding polls and result, april 2012 [18, 19].



**Figure 2.** Result of the vote obtained by a quantum model from the opinion poll. Data issued from the April 2012 French presidential election.

This test case corresponds to  $n = 6$  and  $m = 3$ . The numerical data relative to the relation (12) are chosen such that  $\sigma_1 = 1$ ,  $\sigma_2 = 0$ , and  $\sigma_3 = -1$ . Then the above Perron-Frobenius methodology is available up to  $t = 2.2$ . The numerical result are presented in Figure 2. It reflects some big tendencies of the real election. But the correlation between the popularity and the result is not always satisfied, as shown clearly by comparison between our simulation in Figure 2 and the result of the election shown in the last column of Table 1.

## 8) Conclusion

In this contribution, we have used a given quantum model for range voting in the context of opinion polls. From these data, we have proposed a quantum methodology for predicting the vote. We introduce a density operator associated to the candidates. The mathematical key point is the determination of a positive eigenvector for a real matrix with non-negative coefficients. Our results are encouraging, even if the confrontation to real life data shows explicitly that other parameters have to be taken into account.

## Acknowledgments

The author thanks the referees for helpful comments on the first edition (April 2013) of this contribution. Some of them have been incorporated into the present edition of the



article.

## References

- [1] A. Abramowitz. “An Improved Model for Predicting the Outcomes of Presidential Elections”, *PS: Political Science and Politics*, vol. 21: p. 843-847, 1988.
- [2] K.J. Arrow. *Social Choice and Individual Values*, J. Wiley and Sons, New York, 1951.
- [3] M. Balinski, R. Laraki. “A theory of measuring, electing and ranking”, *Proceeding of the National Academy of Sciences of the USA*, 22 May 2007, vol. 104, number 21, p. 8720-8725, doi:10.1073/pnas.0702634104, 2007.
- [4] M. Balinski, R. Laraki. “Le Jugement Majoritaire : l’Expérience d’Orsay”, *Commentaire*, vol. 30, number 118, p. 413-420, summer 2007.
- [5] M. Bitbol (Ed). *Théorie quantique et sciences humaines*, CNRS Editions, Paris, 2009.
- [6] J.C. de Borda. “Mémoire sur les élections au scrutin”, *Histoire de l’Académie Royale des Sciences*, Paris, 1781.
- [7] J.R. Busemeyer, J. Trueblood. “Comparison of Quantum and Bayesian Inference Models”, *Lecture Notes in Computer Science*, Proceedings of the Third Quantum Interaction International Symposium, Saarbrücken, vol. 5494 (P. Bruza *et al.* Editors), p. 29-43, Springer, 2009.
- [8] C. Cohen-Tannoudji, B. Diu, F. Laloë. *Mécanique quantique*, Hermann, Paris, 1977.
- [9] N. de Condorcet. *Essai sur l’application de l’analyse à la probabilité des décisions rendues à la pluralité des voix*, Imprimerie Royale, Paris, 1785.
- [10] J.E. Campbell. “Forecasting the Presidential Vote in the States”, *American Journal of Political Science*, vol. 36, p. 386-407, 1992.
- [11] F. Dubois. “On the measure process between different scales”, *Res-Systemica*, vol. 7, 7th European Congress of System Science, Lisboa, December 2008.
- [12] F. Dubois. “On Voting process and Quantum Mechanics”, *Lecture Notes in Computer Science*, Proceedings of the Third Quantum Interaction International Symposium, Saarbrücken, vol. 5494 (P. Bruza *et al.* Editors), p. 200-210, Springer, march 2009.
- [13] F. Dubois. “A quantum approach for determining a state of the opinion”, workshop on “Quantum Decision Theory”, symposium “Foundations and Applications of Utility, Risk and Decision Theory”, Atlanta, Georgia, USA, 30 june - 03 july 2012, unpublished.
- [14] G.H. Gallup. *Guide to Public Opinion Polls*, Princeton university press, 1944.

- [15] P. Grangier. “Contextual objectivity: a realistic interpretation of quantum mechanics”, *European Journal of Physics*, vol. 23, p. 331-337, doi:10.1088/0143-0807/23/3/312, 2002.
- [16] E. Haven, A. Khrennikov. *Quantum Social Science*, Cambridge University Press, 2013.
- [17] A.Y. Khrennikov, E. Haven. “The importance of probability interference in social science: rationale and experiment”, arXiv:0709.2802, september 2007.
- [18] IPSOS. “Le baromètre de l’action politique”, for *Le Point*, 09 april 2012.
- [19] IPSOS-Logica. “Baromètres d’intentions de vote pour l’élection présidentielle”, vague 16, 10 april 2012.
- [20] J.D. Lafay. “L’analyse économique de la politique : raisons d’être, vrais problèmes et fausses critiques”, *Revue Française de Sociologie*, vol. 38, p. 229-244, 1997.
- [21] P. La Mura, L. Swiatczak. “Markovian Entanglement Networks”, Leizig Graduate School of Management, 2007.
- [22] M.S. Lewis-Beck. “French national elections: Political economic forecasts”, *European Journal of Political Economy*, vol. 7, p. 487-496, 1991.
- [23] C. Meyer. *Matrix analysis and applied linear algebra*, SIAM, ISBN 0-89871-454-0, 2000.
- [24] W.D. Smith. “Range Voting”, december 2000.
- [25] R.L. Rivest, W.D. Smith. “Three Voting Protocols: ThreeBallot, VAV, and Twin”, Proceedings of the Electronic Voting Technology’07, Boston, MA, August 6, 2007.
- [26] D. Serre. *Matrices: Theory and Applications*, Springer, 2002.
- [27] Z. Wang, J.R. Busemeyer, H. Atmanspacher, E.M. Pothos. The potential to use quantum theory to build models of cognition. *Topics in Cognitive Science*, 2013, in press.
- [28] C. Zorn, C.E. Smith. “Pseudo-classical Nonseparability and Mass Politics in Two-Party Systems”, *Lecture Notes in Computer Science*, Proceedings of the 5th Quantum Interaction International Symposium, Aberdeen, vol. 7052 (Dawei Song *et al.* Editors), p. 83-94, Springer, june 2011.