A STABILITY PROPERTY FOR A MONO-DIMENSIONAL THREE VELOCITIES SCHEME WITH RELATIVE VELOCITY

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In this contribution, we investigate a mono-dimensional 3 velocities linear lattice Boltzmann scheme with relative velocity [**Dubois-Fevrier-Graille-2015**]. Denoting Δx the spatial step, Δt the time step, and $\lambda = \Delta x/\Delta t$ the scheme velocity, this scheme can be described in the framework of D. d'Humières [**d'Humieres-1992**]: (1) the 3 velocities $c_- = -1$, $c_0 = 0$, and $c_+ = 1$; (2) the 3 associated distributions f_-, f_0, and f_+; (3) the 3 moments ρ, $q(u)$, and $\varepsilon(u)$ given by

$$\rho = \sum_{k\in\{-,0,+\}} f_k, \quad q(u) = \lambda \sum_{k\in\{-,0,+\}} (c_k - u) f_\alpha, \quad \varepsilon(u) = 3\lambda^2 \sum_{k\in\{-,0,+\}} (c_k - u)^2 f_k - 2\lambda^2 \sum_{k\in\{-,0,+\}} f_k,$$

where u is a given scalar representing the relative velocity; (4) the equilibrium value of the 3 moments $\rho^{\text{eq}} = \rho$, $q^{\text{eq}}(u) = \lambda(V - u)\rho$, $\varepsilon^{\text{eq}}(u) = \lambda^2(3u^2 - 6uV + \alpha)\rho$, where V and α are given scalars; (5) the 2 relaxation parameters s and s' such that the relaxation phase reads $q^\star(u) = (1-s)q(u) + sq^{\text{eq}}(u)$, $\varepsilon^\star(u) = (1-s')\varepsilon(u) + s'\varepsilon^{\text{eq}}(u)$. The equilibrium values are chosen such that the equilibrium distributions do not depend on the relative velocity u. Indeed, we have:

$$f_k^{\text{eq}} = \tfrac{1}{6}\rho\big(2 + 3c_k V + (3c_k - 2)\alpha\big), \quad k \in \{-,0,+\}.$$

Note that this scheme can be used to simulate a scalar transport equation with constant velocity λV:

$$\partial_t \rho + \lambda V \partial_x \rho = 0.$$

The relaxation phase can be written in a vectorial form $F^\star = R(u)F$, with $R(u) = M^{-1}T(-u)J(u)T(u)M$, $F = (f_- \ f_0 \ f_+)^T$, where the matrices are given by

$$M = \begin{pmatrix} 1 & 1 & 1 \\ -\lambda & 0 & \lambda \\ \lambda^2 & -2\lambda^2 & \lambda^2 \end{pmatrix}, \quad T(u) = \begin{pmatrix} 1 & 0 & 0 \\ -\lambda u & 1 & 0 \\ 3\lambda^2 u^2 & -6\lambda u & 1 \end{pmatrix},$$

$$J(u) = \begin{pmatrix} 1 & 0 & 0 \\ s\lambda(V-u) & 1-s & 0 \\ s'\lambda^2(3u^2 - 6uV + \alpha) & 0 & 1-s' \end{pmatrix}.$$

The velocity V being fixed, we propose to give a full description of the sets

$$\Omega^{V,u} = \big\{(s, s', \alpha) \in \mathbb{R}^4 \text{ such that } R(u) \text{ is a non-negative matrix}\big\}, \quad u \in \mathbb{R}.$$

Indeed, $R(u)$ being a non-negative matrix imposes that all the distributions f_α remain non-negative if they are at the initial time. These sets are first described by a set of 9 inequalities that can be joined into just one. Numerical illustrations are then given to visualize it in the characteristic cases including SRT, MRT, and relative velocity scheme.

References

[Dubois-Fevrier-Graille-2015] F. Dubois, T. Février and B. Graille, Lattice Boltzmann Schemes with Relative Velocities, Commun. Comput. Phys., volume 17, issue 4, pages 1088–1112, 2015.

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