First vorticity-velocity-pressure numerical scheme for the Stokes problem

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Abstract

We consider the bidimensional Stokes problem for incompressible fluids and recall the vorticity, velocity and pressure variational formulation, which was previously proposed by one of the authors, and allows very general boundary conditions. We develop a natural implementation of this numerical method and we describe in this paper the numerical results we obtain. Moreover, we prove that the low degree numerical scheme we use is stable for Dirichlet boundary condition on the vorticity. Numerical results are in accordance with the theoretical ones. In the general case of unstructured meshes, a stability problem is present for Dirichlet boundary conditions on the velocity, exactly as in the stream function-vorticity formulation. Finally, we show on some examples that we observe numerical convergence for regular meshes or embedded ones for Dirichlet boundary conditions on the velocity.

Key words: Stokes problem, vorticity-velocity-pressure formulation, stream function-vorticity formulation, mixed finite elements method, inf-sup conditions.
PACS: 65N30

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* Article publié dans *Computer Methods in Applied Mechanics and Engineering*, volume 192, numéros 44-46, p. 4877-4907, novembre 2003. Edition du 03 sept. 2005.

1 Vorticity-velocity-pressure formulation for the Stokes problem

1.1 Statement of the problem

Let Ω be a bounded connected domain of $\mathbb{R}^2$ with an assumed regular boundary $\partial\Omega \equiv \Gamma$. The Stokes problem models the stationary equilibrium of an incompressible viscous fluid when the velocity u is sufficiently small in order to neglect the nonlinear terms (see *e.g.* Landau-Lifchitz [LL71]). From a mathematical point of view, this problem is the first step in order to consider the nonlinear Navier-Stokes equations of incompressible fluids, as proposed for example in Girault-Raviart [GR86]. The Stokes problem can be classically written with primal formulation involving velocity u and pressure p :

$$\begin{cases} -\nu \Delta u + \nabla p = f \text{ in } \Omega \\ \operatorname{div} u \qquad\;\, = 0 \text{ in } \Omega \\ u \qquad\qquad\;\, = 0 \text{ on } \Gamma, \end{cases} \tag{1}$$

where $\nu > 0$ is the kinematic viscosity and f the datum of external forces. For the sake of simplicity, we shall take $\nu = 1$ in all the following.

The HAWAY method (Harlow and Welch MAC scheme [HW65], Arakawa C-grid [Ara66], Yee translated grids for Maxwell equations [Yee66]) was developed on quadrangular and regular meshes to solve the Navier-Stokes or Maxwell equations. Results are so satisfying that the method is used in many industrial softwares (Flow3d [HHS83], Phoenics [PS72] among others). Our idea is to extend this method to unstructured triangular meshes, *ie* obtaining exactly the same degrees of freedom as those in the HAWAY method on triangles (see Figures 1 and 2). The analysis for a finite element method leads to a new formulation involving the three fields : vorticity, velocity and pressure. A similar approach using finite volumes method was analysed by Nicolaïdes [Nic91].

In this paper, we recall the variational formulation previously proposed and studied in [Dub92] and [Dub02]. As in the classical stream function-vorticity formulation, we choose to introduce the vorticity as a new unknown and to work with divergence free velocity. But in our case we prefer not to write the divergence free velocity with the help of a stream function. Indeed, the stream function is not uniquely defined in three-dimensions spaces and even in two-dimensions for flows with sources and sinks (Foias-Temam [FT78]). So,

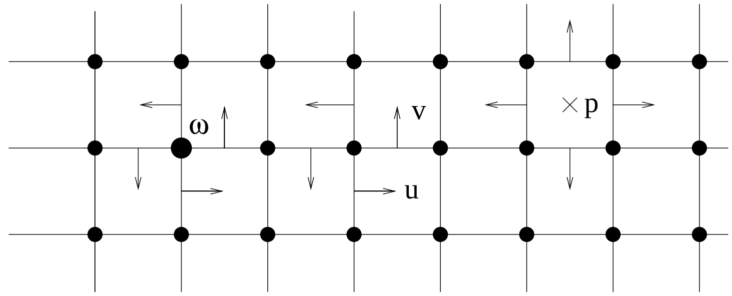

Fig. 1. HAWAY discretization on a cartesian mesh.

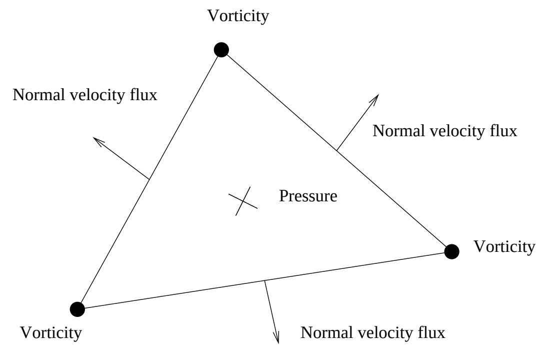

Fig. 2. Degrees of freedom on a triangular mesh.

this new formulation appears as an alternative to the classical one for three-dimensional domains. Moreover, the boundary conditions can be considered in a more general way, as a generalization of previous works of Beghe, Conca, Murat and Pironneau [BCMP87] and Girault [Gir88].

Up to now, numerical aspects have only been studied in dimension two and, in this paper, we restrict ourselves to bidimensional domains. The scope of this work is the following. We present in Section 1 the variational formulation involving the three fields of vorticity, velocity and pressure. In Section 2, we give the numerical discretization and prove a convergence result in a particular case of boundary conditions. Then, in Section 3, we give numerical results and observe that they are in accordance with the above theory. Finally, Section 4 is dedicated to numerical experiments and numerical comparison with the stream function-vorticity formulation analyzed by Glowinski [Glo73], Ciarlet-Raviart [CR74], Glowinski-Pironneau [GP79], Bernardi, Girault and Maday [BGM92] among others.

1.2 Notation and functional spaces

• Let Ω be a given bounded connected domain of $\mathbb{R}^2$ with a regular boundary Γ. We shall consider the following spaces (see for example Adams [Ada75]). We note $L^2(\Omega)$ the space of all (classes of) functions which are square integrable on Ω, equipped with its natural inner product, denoted by $(.,.)$, and the associated norm $\| \, . \, \|_{0,\Omega}$. The subspace of $L^2(\Omega)$ containing square integrable functions whose mean value is zero, is denoted by $L^2_0(\Omega)$.

• The space $H^1(\Omega)$ will be the space of functions $\varphi \in L^2(\Omega)$ for which the first partial derivatives (in the distribution sense) belong to $L^2(\Omega)$:

$$H^1(\Omega) = \left\{ \varphi \in L^2(\Omega) \ / \ \frac{\partial \varphi}{\partial x_i} \in L^2(\Omega) \text{ for } i \ \in \ \{1,2\} \right\} .$$

The usual norm in space $H^1(\Omega)$ is denoted by $\| \, . \, \|_{1,\Omega}$ while the semi-norm is written $| \, . \, |_{1,\Omega}$. In a similar way, we define space $H^2(\Omega)$ as the space of functions of $H^1(\Omega)$ for which the first partial derivatives belong to $H^1(\Omega)$. The associated norms and semi-norms are respectively noted $\| \, . \, \|_{2,\Omega}$ and $| \, . \, |_{2,\Omega}$. We also introduce space $H^1_0(\Omega)$ which is the closure of the space of all indefinitely differentiable functions with compact support in Ω for the norm $\| \, . \, \|_{1,\Omega}$.

• Finally, for all vector field v in $\mathbb{R}^2$, the divergence of v is defined by :

$$\text{div } v \ = \ \frac{\partial v_1}{\partial x_1} + \frac{\partial v_2}{\partial x_2} \quad .$$

Then, the space $H(\text{div}, \Omega)$ is the space of vector fields that belong to $(L^2(\Omega))^2$ with divergence (in the distribution sense) in $L^2(\Omega)$:

$$H(\text{div}, \Omega) = \left\{ v \in (L^2(\Omega))^2 \ / \ \text{div } v \in L^2(\Omega) \right\} \quad , \tag{2}$$

which is a Hilbert space for the norm :

$$\| \, v \, \|_{\text{div},\Omega} \quad = \quad \left(\sum_{j=1}^{2} \| \, v_j \, \|^2_{0,\Omega} + \| \, \text{div } v \, \|^2_{0,\Omega} \right)^{1/2} \quad . \tag{3}$$

1.3 Variational formulation

• Following [Dub92], we propose to write the Stokes problem by means of a vorticity-velocity-pressure formulation. So, we introduce the **vorticity** ω as :

$$\omega = \text{curl } u \quad . \tag{4}$$

Let us recall that, if v is a vector field on Ω, when $\Omega \subset \mathbb{R}^2$, then curl v is the scalar field defined by :

$$\text{curl } v \;=\; \frac{\partial v_1}{\partial x_2} - \frac{\partial v_2}{\partial x_1} \quad . \tag{5}$$

In the following, we shall also use the curl of a scalar fied, say φ, which is the bidimensional field defined by :

$$\text{curl } \varphi \;=\; \left(\frac{\partial \varphi}{\partial x_2}, -\frac{\partial \varphi}{\partial x_1} \right)^t \quad . \tag{6}$$

• Then, we suppose that the boundary Γ of the domain Ω is split into two independent partitions :

$$\Gamma \;=\; \overline{\Gamma_m} \cup \overline{\Gamma_p} \quad \text{with} \quad \Gamma_m \cap \Gamma_p = \emptyset \quad ; \tag{7}$$

$$\Gamma \;=\; \overline{\Gamma_\theta} \cup \overline{\Gamma_t} \quad \text{with} \quad \Gamma_\theta \cap \Gamma_t = \emptyset \quad . \tag{8}$$

We suppose that different types of data are given on each part of the boundary : normal velocity g_0 on Γ_m, pressure Π_0 on Γ_p, vorticity θ_0 on Γ_θ and tangential velocity σ_0 on Γ_t. In all the sequel, f is a field of external forces assumed to belong to $(L^2(\Omega))^2$. Then, the Stokes problem is written :

$$\omega - \text{curl } u = 0 \text{ in } \Omega \tag{9}$$

$$\text{curl } \omega + \nabla p = f \text{ in } \Omega \tag{10}$$

$$\text{div } u = 0 \text{ in } \Omega \quad , \tag{11}$$

with very general boundary conditions :

$$u \bullet n = g_0 \;\text{ on }\; \Gamma_m \tag{12}$$

$$p = \Pi_0 \;\text{ on }\; \Gamma_p \tag{13}$$

$$\omega = \theta_0 \;\text{ on }\; \Gamma_\theta \tag{14}$$

$$u \bullet t = \sigma_0 \;\text{ on }\; \Gamma_t \quad , \tag{15}$$

where $u \bullet n$ and $u \bullet t$ stand respectively for the normal and the tangential components of the velocity, n being the outer normal vector to the boundary Γ and t the tangent vector, chosen such that (n, t) is direct. For the sake of simplicity, in this section, we restrict ourselves to the case of homogeneous boundary conditions for the normal velocity and for the vorticity :

$$\begin{aligned} u \bullet n &= g_0 = 0 \quad \text{on } \Gamma_m \quad , \\ \omega \quad &= \theta_0 = 0 \quad \text{on } \Gamma_\theta \quad . \end{aligned}$$

• In order to include the above boundary conditions, we introduce the following spaces. For velocity, we define the space X by :

$$X = \{ v \in H(\mathrm{div}, \Omega) \ / \ v \bullet n \ = \ 0 \text{ on } \Gamma_m \} \quad , \tag{16}$$

where Γ_m is the part of the boundary where the normal component of the vector field $v \bullet n$ is given.

Remark 1 *This normal component has to be considered in a weak form. More precisely, for any arbitrary decomposition of the boundary of the form :*

$$\Gamma = \overline{\Gamma_1} \cup \overline{\Gamma_2} \quad \text{with} \quad \Gamma_1 \cap \Gamma_2 = \emptyset \quad , \tag{17}$$

the expression $v \bullet n$ is defined in the dual space $\Big(H_{00}^{1/2}(\Gamma_1) \Big)'$ of scalar fields on Γ that are equal to zero on Γ_2 (see e.g. Lions-Magenes [LM68], Amrouche and al [ABDG98] or [Dub02]). Then, writing $v \bullet n \ = \ 0$ on Γ_m means rigorously that the normal trace of v is zero in space $\Big(H_{00}^{1/2}(\Gamma_m) \Big)'$.

For the vorticity, we set :

$$W = \Big\{ \varphi \in H^1(\Omega) \ / \ \gamma\varphi \ = \ 0 \text{ on } \Gamma_\theta \Big\} \quad . \tag{18}$$

Let us remark that the boundary condition is related to the trace of the function φ, that we have noted $\gamma\varphi$. Finally, the space for the pressure is governed by the fact that meas (Γ_p) is zero or not. We set :

$$Y = \begin{cases} L^2(\Omega) & \text{if meas}\,(\Gamma_p) \neq 0 \quad , \\ L_0^2(\Omega) & \text{if meas}\,(\Gamma_p) = 0 \quad . \end{cases} \tag{19}$$

• To obtain the variational formulation, we multiply the first equation (9) by a test function φ in W and we integrate by parts :

$$(\omega, \varphi) \ - \ (\mathrm{curl}\ u, \varphi) \ = \ (\omega, \varphi) \ - \ (\mathrm{curl}\ \varphi, u) \ - \ < u \bullet t \ , \ \gamma\varphi >_\Gamma \quad .$$

In this expression, $< .,. >_\Gamma$ stands for a boundary integral. Then, introducing boundary condition (15), we obtain :

$$(\omega, \varphi) \; - \; (\; \mathrm{curl} \; \varphi, u) \; = \; < \sigma_0, \gamma\varphi >_\Gamma \quad \forall \varphi \in W \quad .$$

Equation (10) is multiplied by a field v in X. As we have :

$$(\nabla p, v) \; = \; - \; (p, \mathrm{div} \; v) \; + \; < p, v \bullet n >_\Gamma \quad ,$$

with the boundary condition (13), we obtain :

$$(\mathrm{curl} \; \omega, v) \; - \; (p, \mathrm{div} \; v) \; = \; (f, v) \; - \; < \Pi_0, v \bullet n >_\Gamma \quad \forall v \in X \quad .$$

Finally equation (11) is multiplied by q in Y and becomes :

$$(\mathrm{div} \; u, q) = 0 \quad \forall q \in Y \quad .$$

Then, the vorticity-velocity-pressure formulation is the following :

$$\begin{cases} \text{Find } (\omega, \, u, \, p) \text{ in } W \times X \times Y \quad \text{such that :} & & \\ (\omega, \varphi) - (\mathrm{curl} \; \varphi, u) & = < \sigma_0, \gamma\varphi >_\Gamma & \forall \; \varphi \in W \\ (\mathrm{curl} \; \omega, v) - (p, \mathrm{div} \; v) & = (f, v) - < \Pi_0, v \bullet n >_\Gamma & \forall \; v \in X \\ (\mathrm{div} \; u, q) & = 0 & \forall \; q \in Y \quad . \end{cases} \tag{20}$$

In this paper, we will not deal with the hypotheses which make the continuous problem (20) well-posed in the general case. A first result was established in [Dub02] and [Sal99], but substantial improvements have been obtained later (see [DSS01]). Among many other technical points, it needs the definition of a new functional space for the vorticity, similar to what Bernardi, Girault and Maday [BGM92] did in two dimensions problems, and Amara, Barucq and Duloué [ABD99] in three dimensions for the stream function-vorticity formulation.

2 Discretization and analysis for Dirichlet vorticity condition

2.1 Numerical discretization

• Let $\mathcal{T}$ be a triangulation of the domain Ω. For the sake of simplicity, we shall assume that Ω is polygonal, in such a way that it is entirely covered by the mesh $\mathcal{T}$. Moreover, we will suppose that the trace of the triangulation on the boundary is such that the boundary edge of any triangle does not overlap different parts of the boundary, Γ_m and Γ_p on the one hand, Γ_θ and Γ_t on the other hand. Then, we denote by $\mathcal{E}_{\mathcal{T}}$ the set of triangles in $\mathcal{T}$.

Definition 2 *Family $\mathcal{U}_\sigma$ of regular meshes.*
We suppose that $\mathcal{T}$ belongs to the set $\mathcal{U}_\sigma$ of triangulations satisfying :

$$\exists\ \sigma > 0\ ,\ \forall\ K \in \mathcal{E}_{\mathcal{T}}\ ,\ \ \frac{h_K}{\rho_K} \leq \sigma$$

where $h_K = \mathrm{diam}\ K$ and ρ_K is the diameter of the circle inscribed in K.

Moreover, $\mathcal{A}_{\mathcal{T}}$ will be the set of all edges of triangles of $\mathcal{T}$. Finally, $h_{\mathcal{T}}$ is the maximum of the diameters of the triangles of $\mathcal{T}$.

• Now, we shall introduce finite-dimensional spaces, say $W_{\mathcal{T}}$, $X_{\mathcal{T}}$ and $Y_{\mathcal{T}}$ which are respectively contained in W, X and Y.

For the vorticity, we choose piecewise linear continuous functions :

$$P^1_{\mathcal{T}} = \left\{ \varphi \in H^1(\Omega)\ /\ \varphi_{|K} \in \mathbb{P}^1(K)\ ,\ \forall K \in \mathcal{E}_{\mathcal{T}} \right\} \quad . \tag{21}$$

Then, including the boundary conditions, we set the following subspace of W :

$$W_{\mathcal{T}} = \left\{ \varphi \in P^1_{\mathcal{T}}\ /\ \gamma\varphi \ = \ 0 \text{ on } \Gamma_\theta \right\} \quad . \tag{22}$$

The velocity is given by its fluxes through edges of the triangles, by the use of the Raviart-Thomas finite element of lowest degree [RT77] :

$$RT^0_{\mathcal{T}} = \left\{ v \in H(\mathrm{div}, \Omega)\ /\ v_{|K} = \begin{pmatrix} a_{|K} \\ b_{|K} \end{pmatrix} + c_{|K} \begin{pmatrix} x \\ y \end{pmatrix}\ ,\ \forall K \in \mathcal{E}_{\mathcal{T}} \right\} \quad . \tag{23}$$

Now, we can state the discrete space for velocity :

$$X_{\mathcal{T}} = \left\{ v \in RT^0_{\mathcal{T}}\ /\ v \bullet n \ = \ 0 \text{ on } \Gamma_m \right\} \quad . \tag{24}$$

Finally, the pressure is chosen piecewise constant. Setting :

$$P^0_{\mathcal{T}} \ = \ \left\{ q \in L^2(\Omega)\ /\ q_{|K} \in \mathbb{P}^0(K)\ ,\ \forall K \in \mathcal{E}_{\mathcal{T}} \right\} \quad , \tag{25}$$

we define :

$$Y_{\mathcal{T}} \ = \ \left\{ q \in P^0_{\mathcal{T}}\ /\ \int_\Omega q\, \mathrm{d}x \ = \ 0 \ \text{ if } \mathrm{meas}(\Gamma_p) \ = \ 0 \right\} \quad . \tag{26}$$

The discrete problem is then to find $(\omega_{\mathcal{T}}\ ,\ u_{\mathcal{T}}\ ,\ p_{\mathcal{T}})$ in $W_{\mathcal{T}} \times X_{\mathcal{T}} \times Y_{\mathcal{T}}$ such that :

$$\begin{cases} (\omega_{\mathcal{T}}, \varphi) - (\mathrm{curl}\ \varphi, u_{\mathcal{T}}) & = < \sigma_0, \gamma\varphi >_{\Gamma} \quad \forall\ \varphi \in W_{\mathcal{T}} \\ (\mathrm{curl}\ \omega_{\mathcal{T}}, v) - (p_{\mathcal{T}}, \mathrm{div}\ v) & = (f, v) - < \Pi_0, v{\bullet}n >_{\Gamma} \quad \forall\ v \in X_{\mathcal{T}} \\ (\mathrm{div}\ u_{\mathcal{T}}, q) & = 0 \quad \forall\ q \in Y_{\mathcal{T}} \quad . \end{cases} \tag{27}$$

2.2 Interpolation errors

• We introduce the classical Lagrange interpolation operator, denoted by $\Pi^1_{\mathcal{T}}$ and we recall the following well-known result (see *e.g.* [Cia78]) :

Theorem 3 *Interpolation error for vorticity.*
Let us assume that the mesh $\mathcal{T}$ belongs to a regular family of triangulations (see Definition 2). Then, there exists a strictly positive constant C, independent of $h_{\mathcal{T}}$, such that, for all $\omega \in H^2(\Omega)$, we have :

$$\| \ \omega - \Pi^1_{\mathcal{T}} \omega \ \|_{1,\Omega} \ \leq \ C \ h_{\mathcal{T}} \ |\omega|_{2,\Omega} \quad .$$

Remark 4 *There exists a more precise result with the semi-norm $|.|_{1,\Omega}$ instead of the complete norm $\| \ . \ \|_{1,\Omega}$. But it is useless here as, in all the sequel, the complete norm is needed in the estimates.*

• Now, following [RT77], let us recall how the interpolation operator is defined for the velocity.

Definition 5 *Interpolation operator in $H(\mathrm{div}, \Omega)$.*
For all vector field v in $(H^1(\Omega))^2$, the interpolation operator $\Pi^{\mathrm{div}}_{\mathcal{T}}$ is such that :

$$\forall\ a \in \mathcal{A}_{\mathcal{T}} \quad , \quad \int_a \Pi^{\mathrm{div}}_{\mathcal{T}} v{\bullet}n \,\mathrm{d}\gamma \ = \ \int_a v{\bullet}n \,\mathrm{d}\gamma \quad ,$$

where n is the unit normal vector to the edge a.

Let us also notice the following basic property :

Proposition 6 *For all v in $(H^1(\Omega))^2$ and for all q in $Y_{\mathcal{T}}$, we have :*

$$\int_\Omega q \ \mathrm{div}(\Pi^{\mathrm{div}}_{\mathcal{T}} v \ - \ v) \,\mathrm{d}x \ = \ 0 \quad .$$

Proof
It is a direct consequence of the Stokes formula as :

$$\begin{aligned}\int_\Omega q \,\mathrm{div}\, (\Pi_{\mathcal{T}}^{\mathrm{div}} v - v)\,\mathrm{d}x &= \sum_{K\in\mathcal{E}_\mathcal{T}} q_{|K} \int_K \mathrm{div}\,(\Pi_{\mathcal{T}}^{\mathrm{div}} v - v)\,\mathrm{d}x \\ &= \sum_{K\in\mathcal{E}_\mathcal{T}} q_{|K} \int_{\partial K} (\Pi_{\mathcal{T}}^{\mathrm{div}} v - v)\bullet n\,\mathrm{d}\gamma \\ &= \sum_{K\in\mathcal{E}_\mathcal{T}} q_{|K} \sum_{a\in\partial K} \int_a (\Pi_{\mathcal{T}}^{\mathrm{div}} v - v)\bullet n\,\mathrm{d}\gamma \\ &= 0\end{aligned}$$

by definition of the $\Pi_{\mathcal{T}}^{\mathrm{div}}$ interpolation operator. ■

Remark 7 *It is possible to define the interpolation operator for a less regular function* ie *for a function v belonging to $(H^\epsilon(\Omega))^2 \cap H(\mathrm{div}, \Omega)$ [Mat89]. Moreover, for $\epsilon > 1/2$, the interpolation operator is defined as the usual one introduced in definition 5 so the proposition 6 is still valid.*

Then, we recall the associated interpolation error (see [Tho80]) :

Theorem 8 *Interpolation error for velocity.*
Let us assume that the mesh $\mathcal{T}$ belongs to a regular family of triangulations. Then, there exists a strictly positive constant C, independent of $h_\mathcal{T}$, such that, for all v in $(H^1(\Omega))^2$, we have :

$$\| v - \Pi_{\mathcal{T}}^{\mathrm{div}} v \|_{0,\Omega} \;\leq\; C\; h_\mathcal{T} \;\; \| v \|_{1,\Omega} \quad .$$

If, moreover, div v *belongs to $H^1(\Omega)$, we obtain :*

$$\| \mathrm{div}\; v - \mathrm{div}\; \Pi_{\mathcal{T}}^{\mathrm{div}} v \|_{0,\Omega} \;\leq\; C\; h_\mathcal{T} \;\; \| \mathrm{div}\; v \|_{1,\Omega} \quad .$$

Theorem 9 *Finer result for velocity [Mat89].*
Let $\epsilon > 0$ and let Ω be a two-dimensional polygonal region. Let us assume that the mesh $\mathcal{T}$ belongs to a regular family of triangulations. Let v belong to $(H^\epsilon(\Omega))^2 \cap H(\mathrm{div}, \Omega)$. Then, first the interpolation operator $\Pi_{\mathcal{T}}^{\mathrm{div}}$ is well defined and there exists a strictly positive constant C, independent of $h_\mathcal{T}$, such that, for all v in $(H^\epsilon(\Omega))^2 \cap H(\mathrm{div}, \Omega)$, we have :

$$\| v - \Pi_{\mathcal{T}}^{\mathrm{div}} v \|_{0,\Omega} \;\leq\; C(\epsilon,\Omega)\; h_\mathcal{T}^\epsilon \;\; \| v \|_{\epsilon,\Omega} \quad .$$

• Finally, for the pressure, we introduce the L^2-projection operator on space $Y_\mathcal{T}$, denoted by $\Pi_\mathcal{T}^0$, which is defined for all q in $L^2(\Omega)$ by :

$$\int_K (\Pi_\mathcal{T}^0 q - q)\,\mathrm{d}x \;=\; 0 \quad \text{for all} \quad K \;\in\; \mathcal{E}_\mathcal{T} \quad ,$$

and we recall the following result (see *e.g.* [GR86]) :

Theorem 10 *Interpolation error for pressure.*
There exists a strictly positive constant C, independent of $h_{\mathcal{T}}$, such that, for all $q \in H^1(\Omega)$, we have :

$$\| \, q - \Pi^0_{\mathcal{T}} q \, \|_{0,\Omega} \;\leq\; C \; h_{\mathcal{T}} \; |q|_{1,\Omega} \quad .$$

2.3 Discrete inf-sup conditions

• As we work with a three-fields formulation, the analysis of this mixed problem leads to two inf-sup conditions (see [LU68], [Bab71], [Bre74]) : a first classical one between pressure and velocity and a second one between vorticity and velocity. First, we give the discrete inf-sup condition between velocity and pressure. In [RT77], an analogous result is proven without boundary condition on the velocity field. In our case, we have to deal with this aspect.

Proposition 11 *Inf-sup condition on velocity and pressure*
Let us recall the partition of the boundary $\Gamma \;=\; \Gamma_m \,\cup\, \Gamma_p$. Let us assume that Ω is polygonal and bounded, and that the mesh $\mathcal{T}$ belongs to a regular family of triangulations. Then, there exists a strictly positive constant a, independent of $h_{\mathcal{T}}$, such that :

$$\inf_{q_{\mathcal{T}} \in Y_{\mathcal{T}}} \; \sup_{v_{\mathcal{T}} \in X_{\mathcal{T}}} \frac{(q_{\mathcal{T}} \;,\; \mathrm{div}\; v_{\mathcal{T}})}{\| \, v_{\mathcal{T}} \, \|_{\mathrm{div},\Omega} \| \, q_{\mathcal{T}} \, \|_{0,\Omega}} \geq a \quad . \tag{28}$$

Proof
◦ Let $q_{\mathcal{T}}$ be an arbitrary element of $Y_{\mathcal{T}}$, and ψ the solution of the following Neumann problem :

$$\begin{cases} \Delta\psi = q_{\mathcal{T}} \text{ in } \Omega \\[2ex] \dfrac{\partial \psi}{\partial n} = g \quad \text{on } \Gamma, \quad \text{with} \quad g = \begin{cases} 0 & \text{on } \Gamma_m \subset \Gamma. \\ \dfrac{1}{|\Gamma_p|} \displaystyle\int_{\Omega} q_{\mathcal{T}} \, \mathrm{d}x & \text{on } \Gamma_p = \Gamma_m^C. \end{cases} \end{cases}$$

Notice that on the one hand, if Γ_p is empty, $Y_{\mathcal{T}} = \left\{ q \in P^0_{\mathcal{T}} / \displaystyle\int_{\Omega} q \, \mathrm{d}x = 0 \right\}$ and the previous problem is an homogeneous Neumann problem that is well-posed as the compatibility condition is verified. On the other hand, if Γ_p is not empty, $Y_{\mathcal{T}} = \left\{ q \in P^0_{\mathcal{T}} \right\}$, and the previous problem is a well-posed Neumann problem as g is defined to still ensure the compatibility condition :

$$\int_\Omega \Delta\psi \,\mathrm{d}x = \int_\Gamma \frac{\partial\psi}{\partial n}\,\mathrm{d}\gamma = \int_{\Gamma_m} \underbrace{\frac{\partial\psi}{\partial n}}_{=0}\,\mathrm{d}\gamma + \int_{\Gamma_p} \underbrace{\frac{\partial\psi}{\partial n}}_{\frac{1}{|\Gamma_p|}\int_\Omega q_\mathcal{T}\,\mathrm{d}x}\,\mathrm{d}\gamma = \int_\Omega q_\mathcal{T}\,\mathrm{d}x.$$

When Ω is assumed polygonal, the solution ψ of the previous problem is unique in $H^{s+1}(\Omega)$, if q belongs to $H^{s-1}(\Omega)$ and g to $H^{s-1/2}(\Gamma)$ where $3/2 \le s \le 2$ depends on the biggest angle of Ω (see [Gri85], [Mat89] and references herein). Moreover, there exists a constant C strictly positive such that :

$$\| \psi \|_{s+1,\Omega} \;\le\; C\;(\| q_\mathcal{T} \|_{s-1,\Omega} + \| g \|_{s-1/2,\Gamma}).$$

Let us observe that we can choose $s = 1/2 + \eta$ with $0 < \eta < 1/2$ because q is in $L^2(\Omega)$ which is contained in $H^{s-1}(\Omega) = H^{\eta-1/2}(\Omega)$ as $\eta < 1/2$. And, g, which is piecewise constant, belongs to $H^{s-1/2}(\Gamma) = H^\eta(\Gamma)$ with $\eta < 1/2$. So, ψ verifies (C will denote various constants independent of the mesh) :

$$\begin{aligned} \| \psi \|_{3/2+\eta,\Omega} &\le C\;(\| q_\mathcal{T} \|_{\eta-1/2,\Omega} + \| g \|_{\eta,\Gamma}) \\ &\le C\;\; \| q_\mathcal{T} \|_{0,\Omega}, \end{aligned} \tag{29}$$

as first, $\| q_\mathcal{T} \|_{\eta-1/2,\Omega} \;\le\; \| q_\mathcal{T} \|_{0,\Omega}$. Second, we remark that $g = \chi \mathbb{1}_{\Gamma_p}$ where $\chi = \dfrac{1}{|\Gamma_p|}\displaystyle\int_\Omega q\,\mathrm{d}x$ and $\mathbb{1}_{\Gamma_p}$ is the characteristic function of Γ_p. Note that $|\chi| \le C \,\| q_\mathcal{T} \|_{0,\Omega}$ by Cauchy-Schwarz, though :

$$\| g \|_{\eta,\Gamma} = |\chi| \;\| \mathbb{1}_{\Gamma_p} \|_{\eta,\Gamma} \le C \,\| q_\mathcal{T} \|_{0,\Omega}\;.$$

$\circ$ Let us now introduce the vector field : $v \;=\; \nabla\psi$. Then, v belongs to $(H^{1/2+\eta}(\Omega))^2 \cap H(\mathrm{div},\Omega)$ and satisfies the boundary condition as $v \bullet n \;=\; \dfrac{\partial\psi}{\partial n} \;=\; 0$ on Γ_m.

So we can define $\Pi_\mathcal{T}^{\mathrm{div}} v$ for v belonging to $(H^{1/2+\eta}(\Omega))^2 \cap H(\mathrm{div},\Omega)$ (see Remark 7 with $\epsilon = 1/2 + \eta$).

Moreover, using the interpolation error for velocity (see Theorem 9), we have :

$$\| \Pi_\mathcal{T}^{\mathrm{div}} v \|_{0,\Omega} \;\le\; \| v \|_{0,\Omega} \;+\; \| v - \Pi_\mathcal{T}^{\mathrm{div}} v \|_{0,\Omega} \;\le\; \| v \|_{0,\Omega} \;+ C\, h_\mathcal{T}^{1/2+\eta}\;\| v \|_{1/2+\eta,\Omega} \quad,$$

with $v \;=\; \nabla\psi$. So, using (29), we obtain :

$$\| \Pi_\mathcal{T}^{\mathrm{div}} v \|_{0,\Omega} \;\le\; C\;\; \| q_\mathcal{T} \|_{0,\Omega} \;+\; C\; h_\mathcal{T}^{1/2+\eta}\;\; \| q_\mathcal{T} \|_{0,\Omega},$$

or else, as $h_{\mathcal{T}}$ is bounded :

$$\| \, \Pi_{\mathcal{T}}^{\mathrm{div}} v \, \|_{0,\Omega} \;\leq\; C \;\, \| \, q_{\mathcal{T}} \, \|_{0,\Omega} \, . \tag{30}$$

By definition of the $\Pi_{\mathcal{T}}^{\mathrm{div}}$ interpolation operator (see Definition 5 and Proposition 6), we have :

$$\int_{\Omega} q \, \mathrm{div}(\Pi_{\mathcal{T}}^{\mathrm{div}} v \;-\; v) \, \mathrm{d}x \;=\; 0 \quad , \tag{31}$$

for all q in $Y_{\mathcal{T}}$. As div $\Pi_{\mathcal{T}}^{\mathrm{div}} v$ belongs to $Y_{\mathcal{T}}$, this relation means that div $\Pi_{\mathcal{T}}^{\mathrm{div}} v$ is the L^2-projection of div v on $Y_{\mathcal{T}}$, and we obtain :

$$\| \, \mathrm{div} \; \Pi_{\mathcal{T}}^{\mathrm{div}} v \, \|_{0,\Omega} \;\;\leq\;\; \| \, \mathrm{div} \; v \, \|_{0,\Omega} \;\;=\;\; \| \, \Delta\psi \, \|_{0,\Omega} \;\;=\;\; \| \, q_{\mathcal{T}} \, \|_{0,\Omega} \, .$$

This inequality and (30) obviously lead to :

$$\| \, \Pi_{\mathcal{T}}^{\mathrm{div}} v \, \|_{\mathrm{div},\Omega} \;\;=\;\; \| \, \Pi_{\mathcal{T}}^{\mathrm{div}} \; \nabla\psi \, \|_{\mathrm{div},\Omega} \;\;\leq\;\; C \;\, \| \, q_{\mathcal{T}} \, \|_{0,\Omega} \quad , \tag{32}$$

with C independent of the mesh size.
∘ Finally, from (31), we deduce that

$$(q_{\mathcal{T}}, \mathrm{div} \; \Pi_{\mathcal{T}}^{\mathrm{div}} v) = (q_{\mathcal{T}}, \mathrm{div} \; v) = (q_{\mathcal{T}}, \Delta\psi) = \|q_{\mathcal{T}}\|_{0,\Omega}^2,$$

and we obtain the discrete inf-sup condition for all $q_{\mathcal{T}}$ of $Y_{\mathcal{T}}$ thanks to (32) :

$$\sup_{v_{\mathcal{T}} \in X_{\mathcal{T}}} \frac{(\mathrm{div} \; v_{\mathcal{T}} \;,\; q_{\mathcal{T}})}{\| \, v_{\mathcal{T}} \, \|_{\mathrm{div},\Omega}} \geq \frac{(\mathrm{div} \; \Pi_{\mathcal{T}}^{\mathrm{div}} \; \nabla\psi \;,\; q_{\mathcal{T}})}{\| \, \Pi_{\mathcal{T}}^{\mathrm{div}} \; \nabla\psi \, \|_{\mathrm{div},\Omega}} \geq \frac{1}{C} \, \| \, q_{\mathcal{T}} \, \|_{0,\Omega} \, .$$

■

• Let us now express the link between vorticity and velocity. In a first step, we have to define the discrete kernel of the divergence operator. So we set :

$$V_{\mathcal{T}} \;\;=\;\; \{ v \in X_{\mathcal{T}} \; / \; (\mathrm{div} \; v \;,\; q) \;\;=\;\; 0 \;\;,\;\; \text{for all } q \in Y_{\mathcal{T}} \} \quad . \tag{33}$$

Then, we have the following result :

Proposition 12 *Characterization of space* $V_{\mathcal{T}}$*.*

$$V_{\mathcal{T}} \;\;=\;\; \{ v \in X_{\mathcal{T}} \; / \; \mathrm{div} \; v \;\;=\;\; 0 \;\; \text{in } \Omega \} \quad . \tag{34}$$

Proof
Let v be an arbitrary element of $X_{\mathcal{T}}$. Due to the definition of this space (see (23)), we have :

$$\text{div } v_{|K} \;=\; \partial_1 v_1 \;+\; \partial_2 v_2 \;=\; 2\, c_K \quad ,$$

which is constant on each triangle. So div v belongs to $P^0_{\mathcal{T}}$. Moreover, because of the Stokes formula, if Γ_m is equal to Γ, we have : $v \bullet n \;=\; 0$ on Γ and then div v belongs to $L^2_0(\Omega)$ and then to $Y_{\mathcal{T}}$, which is contained in $L^2_0(\Omega)$ in this particular case. So, in both cases ($\Gamma_p = \emptyset$ or $\Gamma_p \neq \emptyset$), for all v in $V_{\mathcal{T}}$, div v belongs to $Y_{\mathcal{T}}$ and we can take $q \;=\;$ div v in the definition of $V_{\mathcal{T}}$. It leads to : $\|$ div $v \|^2_{0,\Omega} \;=\; 0$ which gives the result. ∎

Then, we can study the link between elements of $V_{\mathcal{T}}$ and $W_{\mathcal{T}}$.

Proposition 13 *Link between velocity and vorticity.*
Let us assume that Ω is simply connected and let Γ' be a part of the boundary Γ whose measure is non zero. For all vector field v of $RT^0_{\mathcal{T}}$, divergence free, such that $v \bullet n \;=\; 0$ on Γ', there exists a scalar field φ in $P^1_{\mathcal{T}}$ such that $\gamma\varphi \;=\; 0$ on Γ' and $v \;=\;$ curl φ in Ω. Conversely, for all scalar field φ in $P^1_{\mathcal{T}}$ such that $\gamma\varphi \;=\; 0$ on Γ', $v \;=\;$ curl φ is a divergence free vector field of $RT^0_{\mathcal{T}}$, such that $v \bullet n \;=\; 0$ on Γ'.

Proof
∘ Let v be a vector field of $RT^0_{\mathcal{T}}$ such that $v \bullet n \;=\; 0$ on Γ'. If v is divergence free in Ω, which is simply connected, there exists a scalar function φ in $H^1(\Omega)$ such that $v \;=\;$ curl φ on Ω and $\gamma\varphi \;=\; 0$ on Γ' (see [GR86]). Moreover, we have div $v \;=\; 0$ on each triangle K. It implies that $v_{|K} \;=\; (a_K, b_K)^T$. From $v \;=\;$ curl φ on Ω, we deduce that $v_{|K} \;=\;$ curl $\varphi_{|K}$ on each triangle :

$$\begin{pmatrix} a_K \\ b_K \end{pmatrix} \;=\; \begin{pmatrix} \partial_2 \varphi_{|K} \\ -\partial_1 \varphi_{|K} \end{pmatrix} \quad .$$

These equations lead to :

$$\varphi_{|K}(x_1, x_2) \;=\; a_K x_2 + f(x_1) \;=\; -\, b_K x_1 + g(x_2) \quad ,$$

which implies : $-a_K x_2 + g(x_2) \;=\; b_K x_1 + f(x_1)$ for all point (x_1, x_2) of K. It means that these two expressions are equal to a constant, say c_K. Finally, we obtain : $\varphi_{|K}(x_1, x_2) \;=\; a_K x_2 - b_K x_1 + c_K$. It is a first degree polynomial function on K. As we have seen that φ belongs to $H^1(\Omega)$, this proves that φ is in $P^1_{\mathcal{T}}$, and achieves the first part of the proof.
∘ Conversely, it suffices to observe that, on the one hand, the curl of an element of $H^1(\Omega)$ is a divergence free vector field of $H(\text{div}, \Omega)$, and, on the other hand, that the curl of a piecewise linear scalar function is a piecewise constant vector

function. So the curl of any scalar field of $P^1_{\mathcal{T}}$ is a divergence free vector field of $RT^0_{\mathcal{T}}$. For the boundary condition, we remark that :

$$\mathrm{curl}\ \varphi{\bullet}n \;=\; \partial_2\varphi\ n_1 \;-\; \partial_1\varphi\ n_2 \;=\; \partial_1\varphi\ t_1 \;+\; \partial_2\varphi\ t_2 \;=\; \frac{\partial\varphi}{\partial t} \quad ,$$

which is the tangential derivative of φ. So, if $\gamma\varphi$ is zero on Γ', we have $v{\bullet}n \;=\; 0$ on Γ' for $v \;=\; \mathrm{curl}\ \varphi$. ■

This property leads naturally to the following result.

Proposition 14 *Inf-sup condition on vorticity and velocity.*
Let us assume that Ω is simply connected. Let us recall the two partitions of the boundary :

$$\Gamma \;=\; \Gamma_m \;\cup\; \Gamma_p \;=\; \Gamma_\theta \;\cup\; \Gamma_t \quad .$$

Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is contained in Γ_m :

$$\Gamma_\theta \subset \Gamma_m .$$

Then, there exists a strictly positive constant b, independent of $h_{\mathcal{T}}$, such that :

$$\inf_{v_{\mathcal{T}}\in V_{\mathcal{T}}}\ \sup_{\varphi_{\mathcal{T}}\in W_{\mathcal{T}}} \frac{(v_{\mathcal{T}}, \mathrm{curl}\ \varphi_{\mathcal{T}})}{\|\ v_{\mathcal{T}}\ \|_{\mathrm{div},\Omega}\|\ \varphi_{\mathcal{T}}\ \|_{1,\Omega}} \geq b \quad . \tag{35}$$

Proof
Let $v_{\mathcal{T}}$ be an arbitrary element of $V_{\mathcal{T}}$. Then, due to Proposition 13, we know that there exists a scalar field φ_0 in $P^1_{\mathcal{T}}$ such that $\gamma\varphi_0 \;=\; 0$ on Γ_m and $v_{\mathcal{T}} \;=\; \mathrm{curl}\ \varphi_0$ on Ω. As we have supposed Γ_θ contained in Γ_m, φ_0 belongs to $W_{\mathcal{T}}$. Then, we have :

$$\sup_{\varphi_{\mathcal{T}}\in W_{\mathcal{T}}} \frac{(v_{\mathcal{T}}, \mathrm{curl}\ \varphi_{\mathcal{T}})}{\|\ \varphi_{\mathcal{T}}\ \|_{1,\Omega}} \;\geq\; \frac{(v_{\mathcal{T}}, \mathrm{curl}\ \varphi_0)}{\|\ \varphi_0\ \|_{1,\Omega}} \;=\; \frac{\|\ v_{\mathcal{T}}\ \|^2_{0,\Omega}}{\|\ \varphi_0\ \|_{1,\Omega}} \quad .$$

Let us observe that, as $v_{\mathcal{T}}$ is divergence free, we have : $\|\ v_{\mathcal{T}}\ \|^2_{0,\Omega} \;=\; \|\ v_{\mathcal{T}}\ \|^2_{\mathrm{div},\Omega}$. Moreover, using the generalized Poincaré inequality, as Γ_m has a strictly positive measure, there exists a strictly positive constant C, independent of $h_{\mathcal{T}}$, such that :

$$\|\ \varphi_0\ \|_{1,\Omega} \;\leq\; C\ \|\ \nabla\varphi_0\ \|_{0,\Omega} \;=\; C\ \|\ \mathrm{curl}\ \varphi_0\ \|_{0,\Omega} \;=\; C\ \|\ v_{\mathcal{T}}\ \|_{0,\Omega} \quad .$$

These results lead to the expected inequality with $b \;=\; 1/C$. ■

• Using the two inf-sup conditions, we can prove that the discrete problem (27) is well-posed.

Proposition 15 *The discrete variational formulation has a unique numerical solution.*
Let us recall the two partitions of the boundary :

$$\Gamma \;=\; \Gamma_m \;\cup\; \Gamma_p \;=\; \Gamma_\theta \;\cup\; \Gamma_t \quad .$$

Then, we assume that Γ_m *has a strictly positive measure and that* Γ_θ *is contained in* Γ_m *:*

$$\Gamma_\theta \subset \Gamma_m .$$

We also assume that Ω *is polygonal, bounded and simply connected and that the mesh* $\mathcal{T}$ *belongs to a regular family of triangulations.*
Then, the discrete problem which consists in finding $(\omega_{\mathcal{T}}\ ,\ u_{\mathcal{T}}\ ,\ p_{\mathcal{T}})$ *in the space* $W_{\mathcal{T}} \times X_{\mathcal{T}} \times Y_{\mathcal{T}}$ *such that :*

$$\begin{cases} (\omega_{\mathcal{T}}, \varphi) - (\mathrm{curl}\ \varphi, u_{\mathcal{T}}) & = < \sigma_0, \gamma\varphi >_\Gamma & \forall\ \varphi \in W_{\mathcal{T}} \\ (\mathrm{curl}\ \omega_{\mathcal{T}}, v) - (p_{\mathcal{T}}, \mathrm{div}\ v) & = (f, v) - < \Pi_0, v \bullet n >_\Gamma & \forall\ v \in X_{\mathcal{T}} \\ (\mathrm{div}\ u_{\mathcal{T}}, q) & = 0 & \forall\ q \in Y_{\mathcal{T}} \quad . \end{cases}$$

has a unique solution.

Proof
First, let us observe that the hypotheses are such that the two inf-sup conditions (28) and (35) are true. Second, as we consider a finite-dimensional square linear system, the only point to prove is that the solution associated with σ_0, f and Π_0 equal to zero, is zero. For this, in the above system, we choose $\varphi = \omega_{\mathcal{T}}$, $v = u_{\mathcal{T}}$ and $q = p_{\mathcal{T}}$, and we add the three equations. We obtain :

$$(\omega_{\mathcal{T}}, \omega_{\mathcal{T}}) \;=\; 0 \quad ,$$

which implies $\omega_{\mathcal{T}} = 0$. Then, the second equation becomes :

$$(p_{\mathcal{T}}, \mathrm{div}\ v) \;=\; 0 \;\;,\;\; \forall\ v \in X_{\mathcal{T}} \quad .$$

Then, using the inf-sup condition (28), we deduce that $p_{\mathcal{T}} = 0$. Finally, the third equation shows that $u_{\mathcal{T}}$ belongs to $V_{\mathcal{T}}$, and the first one becomes :

$$(\mathrm{curl}\ \varphi, u_{\mathcal{T}}) \;=\; 0 \;\;,\;\; \forall\ \varphi \in W_{\mathcal{T}} \quad ,$$

as $\omega_{\mathcal{T}} = 0$. So $u_{\mathcal{T}}$ is zero thanks to the inf-sup condition (35). ∎

• We can now study the stability of the discrete problem. So, let (ω, u, p) be the solution in $W \times X \times Y$ of the continuous problem :

$$
\begin{cases}
(\omega, \varphi) - (\mathrm{curl}\ \varphi, u) & = < \sigma_0, \gamma\varphi >_\Gamma \qquad \forall\ \varphi \in W \\
(\mathrm{curl}\ \omega, v) - (p, \mathrm{div}\ v) & = (f, v) - < \Pi_0, v\bullet n >_\Gamma \ \forall\ v \in X \\
(\mathrm{div}\ u, q) & = 0 \qquad \forall\ q \in Y \quad ,
\end{cases}
$$

and $(\omega_\mathcal{T}, u_\mathcal{T}, p_\mathcal{T})$ in the space $W_\mathcal{T} \times X_\mathcal{T} \times Y_\mathcal{T}$, the solution of the discrete problem :

$$
\begin{cases}
(\omega_\mathcal{T}, \varphi_\mathcal{T}) - (\mathrm{curl}\ \varphi_\mathcal{T}, u_\mathcal{T}) & = < \sigma_0, \gamma\varphi_\mathcal{T} >_\Gamma \qquad \forall\ \varphi_\mathcal{T} \in W_\mathcal{T} \\
(\mathrm{curl}\ \omega_\mathcal{T}, v_\mathcal{T}) - (p_\mathcal{T}, \mathrm{div}\ v_\mathcal{T}) & = (f, v_\mathcal{T}) - < \Pi_0, v_\mathcal{T}\bullet n >_\Gamma \ \forall\ v_\mathcal{T} \in X_\mathcal{T} \\
(\mathrm{div}\ u_\mathcal{T}, q_\mathcal{T}) & = 0 \qquad \forall\ q_\mathcal{T} \in Y_\mathcal{T} \quad .
\end{cases}
$$

As discrete spaces $W_\mathcal{T}$, $X_\mathcal{T}$ and $Y_\mathcal{T}$ are respectively contained in the continuous ones W, X and Y, we can take $\varphi = \varphi_\mathcal{T}$, $v = v_\mathcal{T}$ and $q = q_\mathcal{T}$ in the continuous problem. Then, subtracting each corresponding equation in the two systems, we obtain :

$$
\begin{cases}
(\omega - \omega_\mathcal{T}, \varphi_\mathcal{T}) - (u - u_\mathcal{T}, \mathrm{curl}\ \varphi_\mathcal{T}) & = 0 \qquad \forall\ \varphi_\mathcal{T} \in W_\mathcal{T} \\
(\mathrm{curl}\ (\omega - \omega_\mathcal{T}), v_\mathcal{T}) - (p - p_\mathcal{T}, \mathrm{div}\ v_\mathcal{T}) & = 0 \qquad \forall\ v_\mathcal{T} \in X_\mathcal{T} \\
(\mathrm{div}\ (u - u_\mathcal{T}), q_\mathcal{T}) & = 0 \qquad \forall\ q_\mathcal{T} \in Y_\mathcal{T} \quad .
\end{cases}
$$

Let us now introduce the interpolants on the mesh $\mathcal{T}$ of each field. Then, we assume that the solution is smooth enough in order that these interpolants be well-defined. For the vorticity field, we denote by $\Pi^1_\mathcal{T}$ the classical Lagrange interpolation operator. For the velocity field, the interpolation operator in $H(\mathrm{div}, \Omega)$ is $\Pi^{\mathrm{div}}_\mathcal{T}$ (see Definition 5). Finally, the pressure field is interpolated using the L^2-projection operator on space $Y_\mathcal{T}$, say $\Pi^0_\mathcal{T}$. Then, we have for each equation :

∘ First equation. For all $\varphi_\mathcal{T}$ in $W_\mathcal{T}$:

$$
(\omega_\mathcal{T} - \Pi^1_\mathcal{T}\omega, \varphi_\mathcal{T}) - (u_\mathcal{T} - \Pi^{\mathrm{div}}_\mathcal{T} u, \mathrm{curl}\ \varphi_\mathcal{T}) = (\omega - \Pi^1_\mathcal{T}\omega, \varphi_\mathcal{T}) - (u - \Pi^{\mathrm{div}}_\mathcal{T} u, \mathrm{curl}\ \varphi_\mathcal{T})
$$

∘ Second equation. For all $v_\mathcal{T}$ in $X_\mathcal{T}$:

$$
(\mathrm{curl}\ (\omega_\mathcal{T} - \Pi^1_\mathcal{T}\omega), v_\mathcal{T}) - (p_\mathcal{T} - \Pi^0_\mathcal{T} p, \mathrm{div}\ v_\mathcal{T}) = (\mathrm{curl}\ (\omega - \Pi^1_\mathcal{T}\omega), v_\mathcal{T}) - (p - \Pi^0_\mathcal{T} p, \mathrm{div}\ v_\mathcal{T})
$$

∘ Third equation. For all $q_{\mathcal{T}}$ in $Y_{\mathcal{T}}$:

$$(\text{div } (u_{\mathcal{T}} - \Pi_{\mathcal{T}}^{\text{div}} u), q_{\mathcal{T}}) = (\text{div } (u - \Pi_{\mathcal{T}}^{\text{div}} u), q_{\mathcal{T}})$$

Let us remark that this last equation becomes :

$$(\text{div } (u_{\mathcal{T}} - \Pi_{\mathcal{T}}^{\text{div}} u), q_{\mathcal{T}}) \;=\; 0 \quad ,$$

for all $q_{\mathcal{T}}$ in $Y_{\mathcal{T}}$ because of Proposition 6 (assuming that u belongs to $(H^1(\Omega))^2$). Finally, the following auxiliary problem appears :

Find $(\theta_{\mathcal{T}}, w_{\mathcal{T}}, r_{\mathcal{T}})$ in $W_{\mathcal{T}} \times X_{\mathcal{T}} \times Y_{\mathcal{T}}$ such that :

$$\begin{cases} (\theta_{\mathcal{T}}, \varphi_{\mathcal{T}}) - (w_{\mathcal{T}}, \text{curl } \varphi_{\mathcal{T}}) & = (f, \varphi_{\mathcal{T}}) + (g, \text{curl } \varphi_{\mathcal{T}}) \;\; \forall\, \varphi_{\mathcal{T}} \in W_{\mathcal{T}} \\ (\text{curl } \theta_{\mathcal{T}}, v_{\mathcal{T}}) - (r_{\mathcal{T}}, \text{div } v_{\mathcal{T}}) & = (k, v_{\mathcal{T}}) + (l, \text{div } v_{\mathcal{T}}) \quad \forall\, v_{\mathcal{T}} \in X_{\mathcal{T}} \\ (\text{div } w_{\mathcal{T}}, q_{\mathcal{T}}) & = 0 \qquad\qquad\qquad\qquad\quad\; \forall\, q_{\mathcal{T}} \in Y_{\mathcal{T}} \end{cases} \tag{36}$$

where we have set :

- ∘ $f \;=\; \omega - \Pi_{\mathcal{T}}^1 \omega$, which belongs to $L^2(\Omega)$;
- ∘ $g \;=\; -\,u + \Pi_{\mathcal{T}}^{\text{div}} u$, which belongs to $(L^2(\Omega))^2$;
- ∘ $k \;=\;$ curl $(\omega - \Pi_{\mathcal{T}}^1 \omega)$, which is in $(L^2(\Omega))^2$;
- ∘ $l \;=\; -\,p + \Pi_{\mathcal{T}}^0 p$, which is in $L^2(\Omega)$.

Now, we can prove a stability result, in a very particular case.

Proposition 16 *Stability of the discrete variational formulation.*
Let us recall the two partitions of the boundary :

$$\Gamma \;=\; \Gamma_m \;\cup\; \Gamma_p \;=\; \Gamma_\theta \;\cup\; \Gamma_t \quad .$$

Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is equal to Γ_m :

$$\Gamma_\theta = \Gamma_m.$$

Moreover, we also assume that Ω is polygonal, bounded and simply connected and that the mesh $\mathcal{T}$ belongs to a regular family of triangulations.
Then, the problem (36) is well-posed and there exists a strictly positive constant C, independent of the mesh, such that :

$$\| \,\theta_{\mathcal{T}}\, \|_{1,\Omega} + \| \,w_{\mathcal{T}}\, \|_{\text{div},\Omega} + \| \,r_{\mathcal{T}}\, \|_{0,\Omega} \leq C \Big(\| \,f\, \|_{0,\Omega} + \| \,g\, \|_{0,\Omega} + \| \,k\, \|_{0,\Omega} + \| \,l\, \|_{0,\Omega} \Big)$$

Proof
We observe that the hypotheses are such that the two inf-sup conditions (28)

and (35) are true. As a matter of fact, when Γ_θ is equal to Γ_m, the former is contained in the latter. Then, exactly as in Proposition 15, the problem (36) is well-posed. Moreover, we remark that the third equation of (36) shows that $w_\mathcal{T}$ is divergence free (see Proposition 12). Then, we have :

$$\| w_\mathcal{T} \|_X \;=\; \| w_\mathcal{T} \|_{\text{div},\Omega} \;=\; \| w_\mathcal{T} \|_{0,\Omega} \,.$$

Finally, we recall that, in two dimensions, we have :

$$\| \theta_\mathcal{T} \|_W^2 \;=\; \| \theta_\mathcal{T} \|_{1,\Omega}^2 \;=\; \| \theta_\mathcal{T} \|_{0,\Omega}^2 \;+\; \| \text{curl } \theta_\mathcal{T} \|_{0,\Omega}^2 \quad .$$

So, the proof of the inequality is given in five steps, in which C will denote various constants, independent of the mesh.
∘ **First step**. We take $\varphi_\mathcal{T} = \theta_\mathcal{T}$, $v_\mathcal{T} = w_\mathcal{T}$ and $q_\mathcal{T} = r_\mathcal{T}$ in (36). As $w_\mathcal{T}$ is divergence free, after adding the three equations, we obtain :

$$\begin{aligned} \| \theta_\mathcal{T} \|_{0,\Omega}^2 &= (f, \theta_\mathcal{T}) + (g, \text{curl } \theta_\mathcal{T}) + (k, w_\mathcal{T}) \\ &\le \| f \|_{0,\Omega} \| \theta_\mathcal{T} \|_{0,\Omega} + \| g \|_{0,\Omega} \| \text{curl } \theta_\mathcal{T} \|_{0,\Omega} + \| k \|_{0,\Omega} \| w_\mathcal{T} \|_{0,\Omega} \end{aligned}$$

Then, using the classical inequality : $\alpha\beta \le \frac{1}{2}(\alpha^2 + \beta^2)$, we deduce :

$$\| \theta_\mathcal{T} \|_{0,\Omega}^2 \;\le\; \| f \|_{0,\Omega}^2 \;+\; 2\; \| g \|_{0,\Omega} \| \text{curl } \theta_\mathcal{T} \|_{0,\Omega} \;+\; 2\; \| k \|_{0,\Omega} \| w_\mathcal{T} \|_{0,\Omega} \tag{37}$$

∘ **Second step**. We use the inf-sup condition (28) in the second equation of (36) and obtain :

$$a \| r_\mathcal{T} \|_{0,\Omega} \le \sup_{v \in X_\mathcal{T}} \frac{(\text{div } v, r_\mathcal{T})}{\| v \|_{\text{div},\Omega}} \le \sup_{v \in X_\mathcal{T}} \frac{(\text{curl } \theta_\mathcal{T}, v) - (l, \text{div } v) - (k, v)}{\| v \|_{\text{div},\Omega}} \quad .$$

Using the fact that the norm in X is the norm in $H(\text{div}, \Omega)$, we finally have :

$$a \;\; \| r_\mathcal{T} \|_{0,\Omega} \;\le\; \| \text{curl } \theta_\mathcal{T} \|_{0,\Omega} \;+\; \| l \|_{0,\Omega} \;+\; \| k \|_{0,\Omega} \quad . \tag{38}$$

∘ **Third step**. We apply the inf-sup condition (35) to $w_\mathcal{T}$, which is divergence free, in the first equation of (36). We deduce :

$$b \| w_\mathcal{T} \|_{\text{div},\Omega} \le \sup_{\varphi \in W_\mathcal{T}} \frac{(w_\mathcal{T}, \text{curl } \varphi)}{\| \varphi \|_{1,\Omega}} \le \sup_{\varphi \in W_\mathcal{T}} \frac{(\theta_\mathcal{T}, \varphi) - (f, \varphi) - (g, \text{curl } \varphi)}{\| \varphi \|_{1,\Omega}} .$$

And we obtain :

$$b \;\; \| w_\mathcal{T} \|_{\text{div},\Omega} \;\le\; \| \theta_\mathcal{T} \|_{0,\Omega} \;+\; \| f \|_{0,\Omega} \;+\; \| g \|_{0,\Omega} \quad . \tag{39}$$

∘ **Fourth step**. As we have supposed that Γ_m is equal to Γ_θ, we can take $v_{\mathcal{T}} \;=\; \mathrm{curl}\ \theta_{\mathcal{T}}$ in the second equation of (36) as $v_{\mathcal{T}}$ belongs to $X_{\mathcal{T}}$ (see Proposition 13). Observe that only this point is at fault for general boundary conditions. Then, we have :

$$(\mathrm{curl}\ \theta_{\mathcal{T}}, \mathrm{curl}\ \theta_{\mathcal{T}}) - (r_{\mathcal{T}}, \mathrm{div}\ (\mathrm{curl}\ \theta_{\mathcal{T}})) \;=\; (k, \mathrm{curl}\ \theta_{\mathcal{T}}) + (l, \mathrm{div}\ (\mathrm{curl}\ \theta_{\mathcal{T}})) \quad .$$

As div curl $\equiv$ 0, it remains :

$$\| \mathrm{curl}\ \theta_{\mathcal{T}} \|_{0,\Omega}^2 \;=\; (k, \mathrm{curl}\ \theta_{\mathcal{T}}) \;\leq\; \| k \|_{0,\Omega} \| \mathrm{curl}\ \theta_{\mathcal{T}} \|_{0,\Omega} \quad ,$$

or else :

$$\| \mathrm{curl}\ \theta_{\mathcal{T}} \|_{0,\Omega} \;\leq\; \| k \|_{0,\Omega} \quad . \tag{40}$$

∘ **Fifth step**. Inequalities (38) and (40) lead to :

$$\| r_{\mathcal{T}} \|_{0,\Omega} \;\leq\; \frac{1}{a} \Big(\| l \|_{0,\Omega} \;+\; 2\ \| k \|_{0,\Omega} \Big) \quad . \tag{41}$$

Then, inequalities (37) and (40) give :

$$\| \theta_{\mathcal{T}} \|_{0,\Omega}^2 \;\leq\; \| f \|_{0,\Omega}^2 \;+\; 2\ \| g \|_{0,\Omega} \| k \|_{0,\Omega} \;+\; 2\ \| k \|_{0,\Omega} \| w_{\mathcal{T}} \|_{0,\Omega} \quad ,$$

or else, using again : $\alpha\beta \;\leq\; \dfrac{1}{2}(\alpha^2 \;+\; \beta^2)$, we obtain :

$$\| \theta_{\mathcal{T}} \|_{0,\Omega}^2 \;\leq\; \| f \|_{0,\Omega}^2 \;+\; \| g \|_{0,\Omega}^2 \;+\; \| k \|_{0,\Omega}^2 \;+\; 2\ \| k \|_{0,\Omega} \| w_{\mathcal{T}} \|_{0,\Omega} \quad .$$

Finally, introducing (39) in the above inequality, we have :

$$\begin{aligned}
\| \theta_{\mathcal{T}} \|_{0,\Omega}^2 \leq\; & \| f \|_{0,\Omega}^2 \;+\; \| g \|_{0,\Omega}^2 \;+\; \| k \|_{0,\Omega}^2 \\
& +\; \frac{2}{b} \| k \|_{0,\Omega} \Big(\| \theta_{\mathcal{T}} \|_{0,\Omega} \;+\; \| f \|_{0,\Omega} \;+\; \| g \|_{0,\Omega} \Big) \\
& \leq\; C\ \Big(\| f \|_{0,\Omega}^2 \;+\; \| g \|_{0,\Omega}^2 \;+\; \| k \|_{0,\Omega}^2 \Big) \;+\; \frac{2}{b} \| k \|_{0,\Omega} \| \theta_{\mathcal{T}} \|_{0,\Omega}
\end{aligned}$$

where C is a constant equal to $1 \;+\; \dfrac{2}{b}$. Now, we use the classical inequality : $2\alpha\beta \;\leq\; \dfrac{\alpha^2}{\varepsilon} \;+\; \varepsilon\, \beta^2$, true for all strictly positive real number ε, to obtain :

$$\begin{aligned}
\| \theta_{\mathcal{T}} \|_{0,\Omega}^2 \leq\; & C\ \Big(\| f \|_{0,\Omega}^2 \;+\; \| g \|_{0,\Omega}^2 \;+\; \| k \|_{0,\Omega}^2 \Big) \\
& +\; \frac{1}{b\ \varepsilon} \| k \|_{0,\Omega}^2 \;+\; \frac{\varepsilon}{b} \| \theta_{\mathcal{T}} \|_{0,\Omega}^2 \quad .
\end{aligned}$$

Taking ε equal to $\dfrac{b}{2}$, we finally obtain :

$$\| \theta_{\mathcal{T}} \|_{0,\Omega}^2 \ \leq \ C \ \left(\| f \|_{0,\Omega}^2 \ + \ \| g \|_{0,\Omega}^2 \ + \ \| k \|_{0,\Omega}^2 \right) \quad . \tag{42}$$

So, the two inequalities (40) and (42) lead to :

$$\| \theta_{\mathcal{T}} \|_{1,\Omega}^2 \ \leq \ C \ \left(\| f \|_{0,\Omega}^2 \ + \ \| g \|_{0,\Omega}^2 \ + \ \| k \|_{0,\Omega}^2 \right) \quad ,$$

and then :

$$\| \theta_{\mathcal{T}} \|_{1,\Omega} \ \leq \ C \ \left(\| f \|_{0,\Omega} \ + \ \| g \|_{0,\Omega} \ + \ \| k \|_{0,\Omega} \right) \quad , \tag{43}$$

Finally, introducing (43) in (39) gives :

$$\| w_{\mathcal{T}} \|_{\mathrm{div},\Omega} \ \leq \ C \ \left(\| f \|_{0,\Omega} \ + \ \| g \|_{0,\Omega} \ + \ \| k \|_{0,\Omega} \right) \quad . \tag{44}$$

The final inequality, given in the proposition, is a direct consequence of (41), (43) and (44). ■

2.5 Convergence result

In this subsection, we consider the discrete case which corresponds to the well-posed continuous one analysed in [Dub02]. The following theorem on the discrete problem is thus in accordance with the previous continuous study.

Theorem 17 *Convergence of the discrete variational formulation.*
∘ *Let us recall the two partitions of the boundary :*

$$\Gamma \ = \ \Gamma_m \ \cup \ \Gamma_p \ = \ \Gamma_\theta \ \cup \ \Gamma_t \quad .$$

Then, we assume that Γ_m has a strictly positive measure and that Γ_θ is equal to Γ_m :

$$\Gamma_\theta = \Gamma_m.$$

Moreover, we also assume that Ω is polygonal, bounded and simply connected and that the mesh $\mathcal{T}$ belongs to a regular family of triangulations.
∘ *Let (ω, u, p) be the solution in $W \times X \times Y$ of the continuous problem (20) and $(\omega_{\mathcal{T}}, u_{\mathcal{T}}, p_{\mathcal{T}})$ in space $W_{\mathcal{T}} \times X_{\mathcal{T}} \times Y_{\mathcal{T}}$, the solution of the discrete problem (27). We suppose that the solution is such that : $\omega \ \in \ H^2(\Omega)$, $u \ \in \ (H^1(\Omega))^2$, with $\mathrm{div}\ u \ \in \ H^1(\Omega)$, and $p \ \in \ H^1(\Omega)$. Then, there exists a strictly positive*

constant C, independent of the mesh, such that :

$$\| \omega - \omega_{\mathcal{T}} \|_{1,\Omega} + \| u - u_{\mathcal{T}} \|_{\mathrm{div},\Omega} + \| p - p_{\mathcal{T}} \|_{0,\Omega}$$

$$\leq C \, h_{\mathcal{T}} \, \Big(| \omega |_{2,\Omega} + \| u \|_{1,\Omega} + \| \mathrm{div} \, u \|_{1,\Omega} + | p |_{1,\Omega} \Big) \quad .$$

Proof
First, let us recall the basic inequalities :

$$\begin{aligned}
&\| \omega - \omega_{\mathcal{T}} \|_{1,\Omega} \leq \| \omega - \Pi^1_{\mathcal{T}} \omega \|_{1,\Omega} + \| \Pi^1_{\mathcal{T}} \omega - \omega_{\mathcal{T}} \|_{1,\Omega} \quad , \\
&\| u - u_{\mathcal{T}} \|_{\mathrm{div},\Omega} \leq \| u - \Pi^{\mathrm{div}}_{\mathcal{T}} u \|_{\mathrm{div},\Omega} + \| \Pi^{\mathrm{div}}_{\mathcal{T}} u - u_{\mathcal{T}} \|_{\mathrm{div},\Omega} \quad , \\
&\| p - p_{\mathcal{T}} \|_{0,\Omega} \leq \| p - \Pi^0_{\mathcal{T}} p \|_{0,\Omega} + \| \Pi^0_{\mathcal{T}} p - p_{\mathcal{T}} \|_{0,\Omega} \quad .
\end{aligned} \tag{45}$$

In these relations, the first terms are well-known : they are the classical interpolation errors. And the second terms are precisely the solutions of the auxiliary problem (36) where we have :

$$\theta_{\mathcal{T}} = \omega_{\mathcal{T}} - \Pi^1_{\mathcal{T}} \omega \quad , \quad w_{\mathcal{T}} = u_{\mathcal{T}} - \Pi^{\mathrm{div}}_{\mathcal{T}} u \quad , \quad r_{\mathcal{T}} = p_{\mathcal{T}} - \Pi^0_{\mathcal{T}} p \quad .$$

Then, Proposition 16 ensures that there exists a strictly positive constant C, independent of the mesh, such that :

$$\| \omega_{\mathcal{T}} - \Pi^1_{\mathcal{T}} \omega \|_{1,\Omega} + \| u_{\mathcal{T}} - \Pi^{\mathrm{div}}_{\mathcal{T}} u \|_{\mathrm{div},\Omega} + \| p_{\mathcal{T}} - \Pi^0_{\mathcal{T}} p \|_{0,\Omega}$$

$$\leq C \Big(\| f \|_{0,\Omega} + \| g \|_{0,\Omega} + \| k \|_{0,\Omega} + \| l \|_{0,\Omega} \Big) \quad ,$$

where we have set : $f = \omega - \Pi^1_{\mathcal{T}} \omega$, $g = -u + \Pi^{\mathrm{div}}_{\mathcal{T}} u$, $k = \mathrm{curl} \, (\omega - \Pi^1_{\mathcal{T}} \omega)$ and $l = -p + \Pi^0_{\mathcal{T}} p$. Then, the above inequality and (45) lead to :

$$\| \omega - \omega_{\mathcal{T}} \|_{1,\Omega} + \| u - u_{\mathcal{T}} \|_{\mathrm{div},\Omega} + \| p - p_{\mathcal{T}} \|_{0,\Omega}$$

$$\leq C(\| \omega - \Pi^1_{\mathcal{T}} \omega \|_{1,\Omega} + \| u - \Pi^{\mathrm{div}}_{\mathcal{T}} u \|_{\mathrm{div},\Omega} + \| p - \Pi^0_{\mathcal{T}} p \|_{0,\Omega}) \quad ,$$

where C is another constant independent of the mesh size. Finally, using the interpolation errors recalled in Theorems 3, 8 and 10, we obtain the announced result. ∎

• To conclude this subsection, let us remark that, in the above theorem, the regularity assumptions on the exact solution are classical. The main drawback lies in the equality : $\Gamma_\theta = \Gamma_m$ which is clearly restrictive for general applications. The reason of this hypothesis is that, to conclude, we must be able to set $v_{\mathcal{T}} = \text{curl } \theta_{\mathcal{T}}$ (see the fourth step in the proof of Proposition 16). One of the ways studied to improve this result was to build a velocity field, belonging to $X_{\mathcal{T}}$, which realizes, in a weaker sense, the equality $v_{\mathcal{T}} = \text{curl } \theta_{\mathcal{T}}$. Nevertheless, the error bounds were not improved, even if the numerical results are much better (see [Sal99] for results of this methodology).

3 Numerical results for Dirichlet vorticity condition

3.1 Bercovier-Engelman test case

Numerical experiments have been performed first on a unit square with an analytical solution proposed by Bercovier and Engelman [BE79]. The boundary conditions are formulated as follows :

$$\omega = 256(y^2(y-1)^2(6x^2-6x+1) + x^2(x-1)^2(6y^2-6y+1)) \text{ on } \Gamma,$$

$$u.n = 0 \text{ on } \Gamma.$$

So Γ_θ and Γ_m are equal. Figure 7 shows that the scheme is stable on a triangular mesh as announced in Theorem 17, and convergence is as expected : order 1 for the *curl* of the vorticity and for the velocity and more than 1 for the pressure. This last result is better than expected. The order 2 for the vorticity in L^2-norm is a classical consequence of the Aubin-Nitsche lemma as the domain Ω is convex (see *e.g.* Ciarlet [Cia78]). All the numerical results can be found on the following web page [Sal02].

3.2 Ruas test case

Then, we have worked on a circle with an analytical solution proposed by Ruas [Rua97]. The boundary conditions are formulated as follows : $u.n = 0$ on Γ and $\omega = 32 - 16x^2 - 16y^2$ on Γ. The pressure isovalues are presented on Figure 11. The analytical pressure is a constant equal to zero on the domain and the computed extrema vary from -0.02 to 0.02. Here again, the convergence is in accordance with the Theorem 17 (see Figure 12). And we also observe a kind of super-convergence on the pressure.

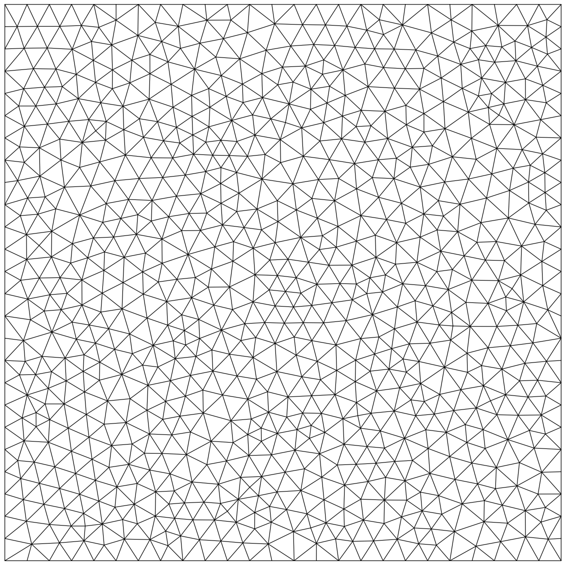

Fig. 3. An unstructured mesh.

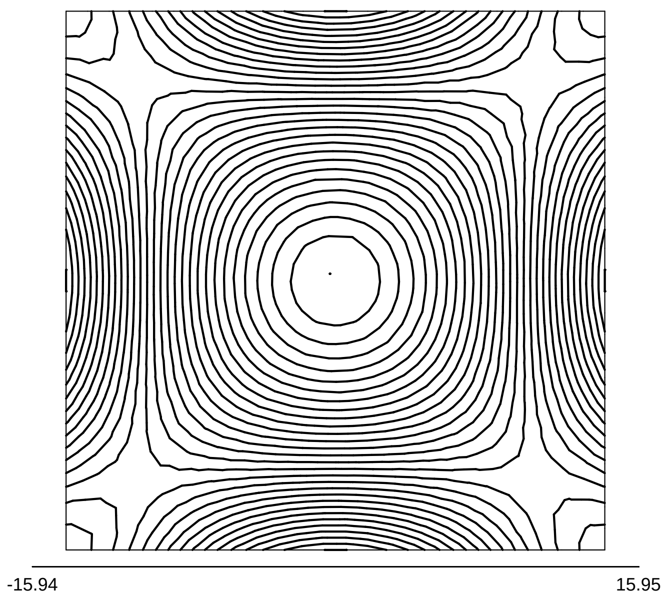

Fig. 4. Numerical vorticity isovalues - Test proposed by Bercovier-Engelman on an unstructured mesh - Case $\Gamma_m = \Gamma_\theta$ - Expected extrema : -16 in the center, +16 on the middle of the boundary - Computed extrema : -15.9 to 15.9.

4 Numerical experiments for Dirichlet velocity boundary conditions

Note that the results obtained in the previous section suppose that the vorticity is known on the part of the boundary where the normal velocity is also

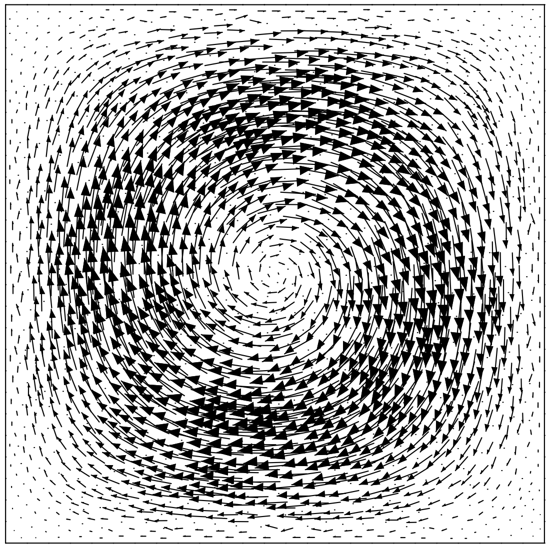

Fig. 5. Velocity vectors - Test proposed by Bercovier-Engelman on an unstructured mesh - Case $\Gamma_m = \Gamma_\theta$.

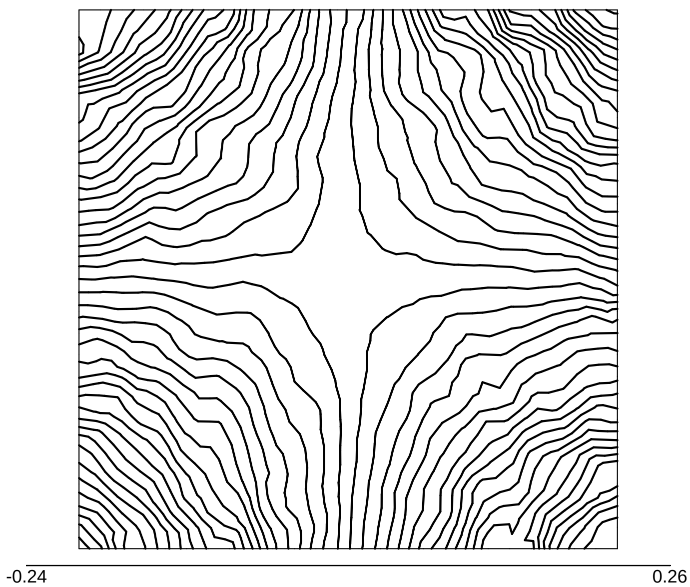

Fig. 6. Numerical pressure isovalues - Test proposed by Bercovier-Engelman on an unstructured mesh - Case $\Gamma_m = \Gamma_\theta$ - Expected extrema : -0.25 to 0.25 on the boundary - Computed extrema : -0.24 to 0.26.

known. Now, we study the numerical behaviour of the scheme with general boundary conditions.

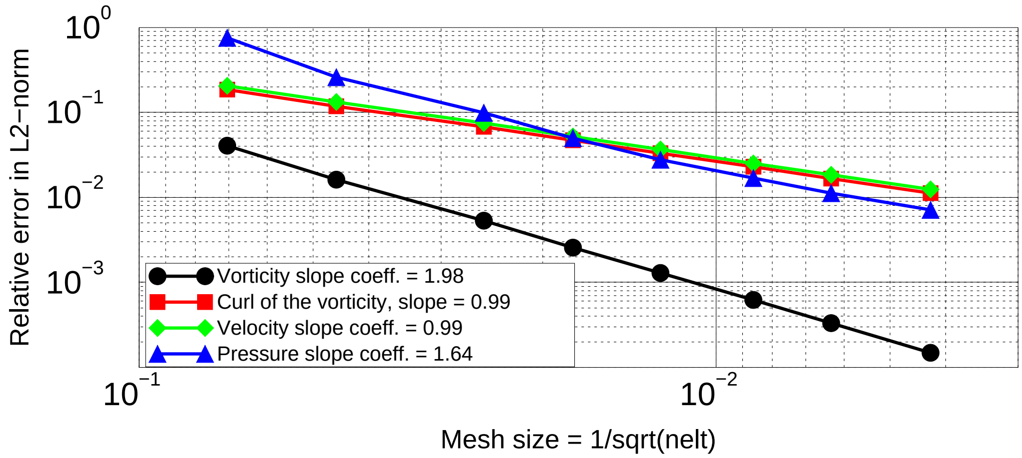

Fig. 7. Convergence order on the Bercovier-Engelman test case - Unstructured mesh - Case $\Gamma_m = \Gamma_\theta$.

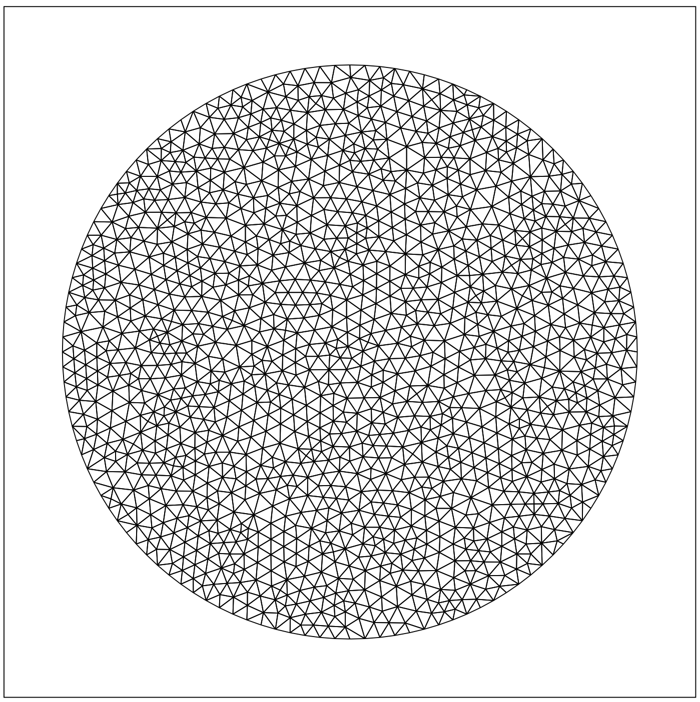

Fig. 8. A reference mesh.

4.1 Numerical results on regular meshes

Let us come back to the test case as it was originally proposed by Bercovier and Engelman [BE79] *ie* with homogeneous velocity conditions on the whole boundary : $u = 0$ on Γ. As $\Gamma_m = \Gamma_t = \Gamma$, we know that there exists a unique solution (see Proposition 15) but the stability of the scheme is not established

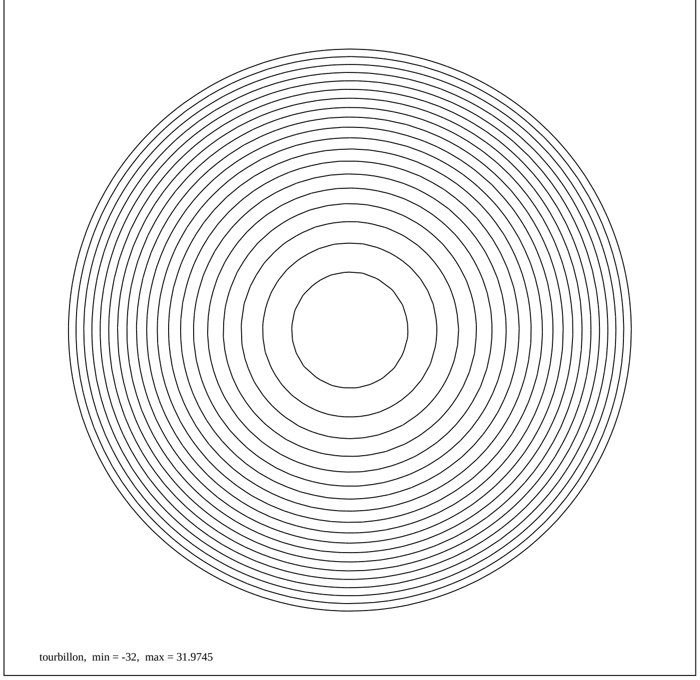

Fig. 9. Numerical vorticity isovalues - Test proposed by Ruas on a reference mesh - Case $\Gamma_m = \Gamma_\theta$ - Expected extrema : -32 on the whole boundary to +32 in the center - Computed extrema : -32 to 31.9.

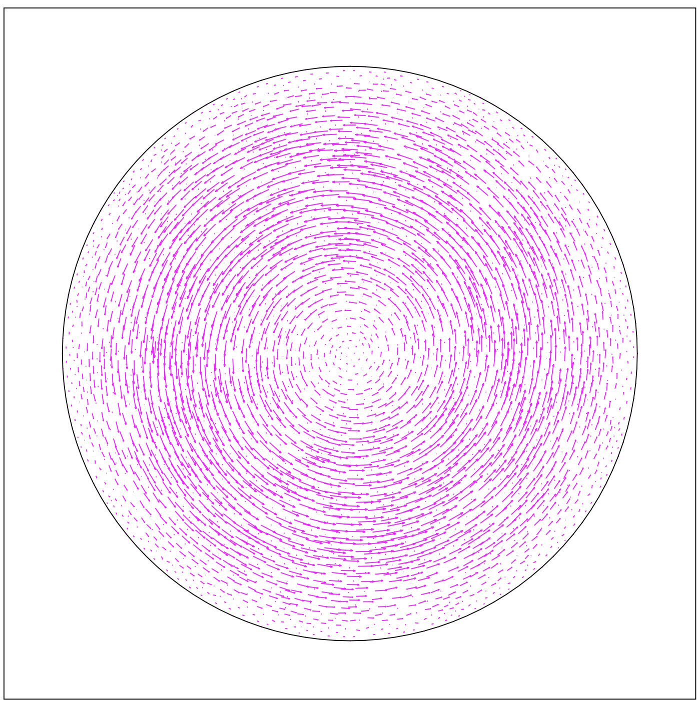

Fig. 10. Velocity vectors - Test proposed by Ruas on a reference mesh - Case $\Gamma_m = \Gamma_\theta$.

in this case (see Theorem 17). Nevertheless, we can exhibit convergence on regular meshes (here criss-cross ones, see Figure 13). Results are very satisfying, see Figure 17. Even the curl of the vorticity, which is not theoretically bounded by the way (see fourth step into the proof of Proposition 16), con-

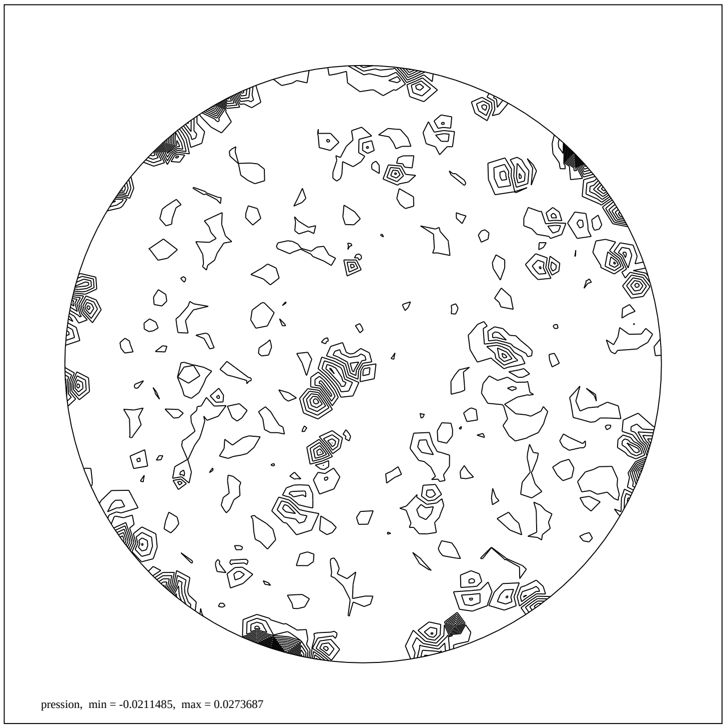

Fig. 11. Numerical pressure isovalues - Test proposed by Ruas on a reference mesh - Case $\Gamma_m = \Gamma_\theta$ - Expected pressure : 0 - Computed extrema : -0.02 to 0.02.

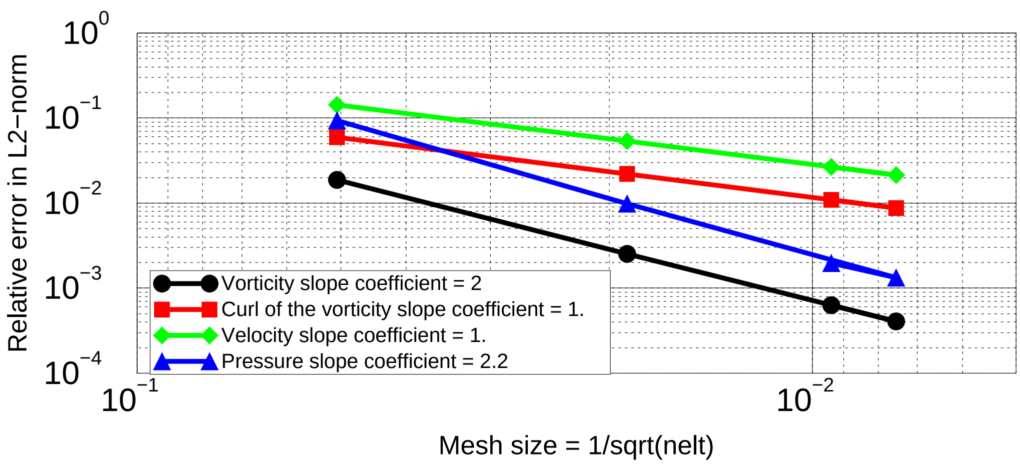

Fig. 12. Convergence order on the Ruas test case - Unstructured meshes - Case $\Gamma_m = \Gamma_\theta$.

verges with an order 1. The convergence on the pressure is still better than expected. It can probably be attributed to super-convergence on quadrilateral meshes (see [GR86]).

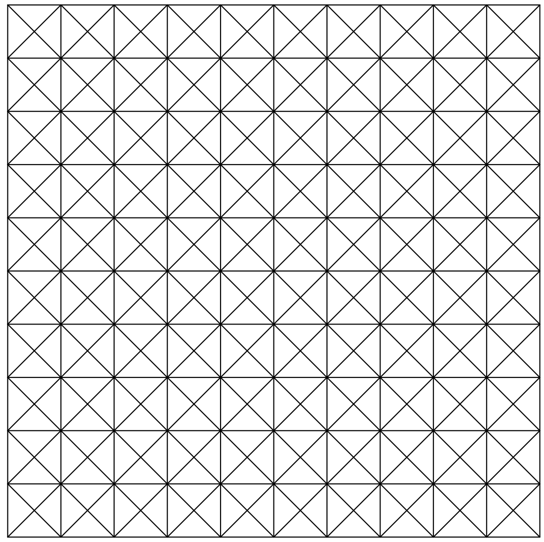

Fig. 13. Criss-cross structured mesh.

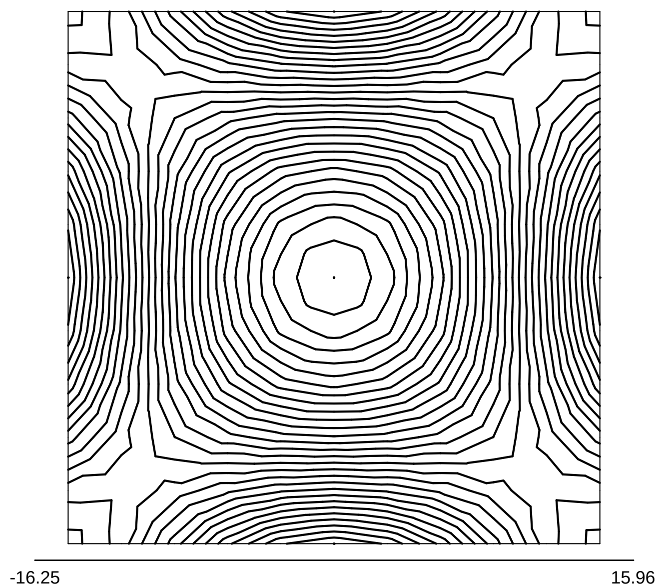

Fig. 14. Numerical vorticity isovalues - Test proposed by Bercovier-Engelman on a structured mesh - Expected extrema : -16 in the center, +16 on the middle of the boundary - Computed extrema : -16.2 to 15.9.

4.2 Numerical results on unstructured meshes

In this subsection, numerical experiments have been performed, first, on the case originally proposed by Bercovier and Engelman [BE79] and, second on the circle proposed by Ruas [Rua97]. In both cases, boundary conditions are such that $\Gamma_m = \Gamma_t = \Gamma$. The results on unstructured triangular meshes (see for instance Figures 3 and 8) are not satisfying for the vorticity and the

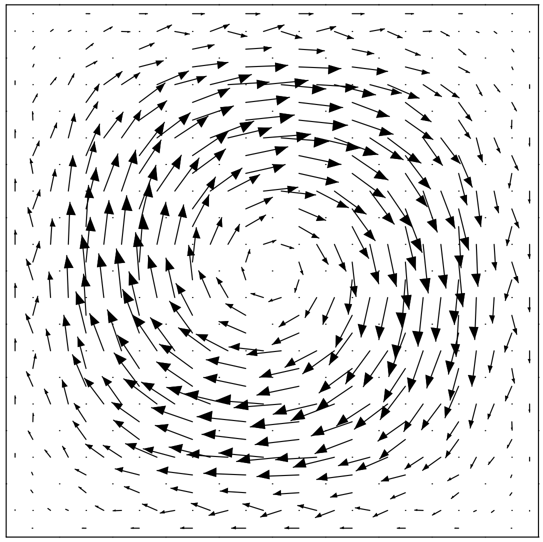

Fig. 15. Velocity vectors - Test proposed by Bercovier-Engelman on a structured mesh.

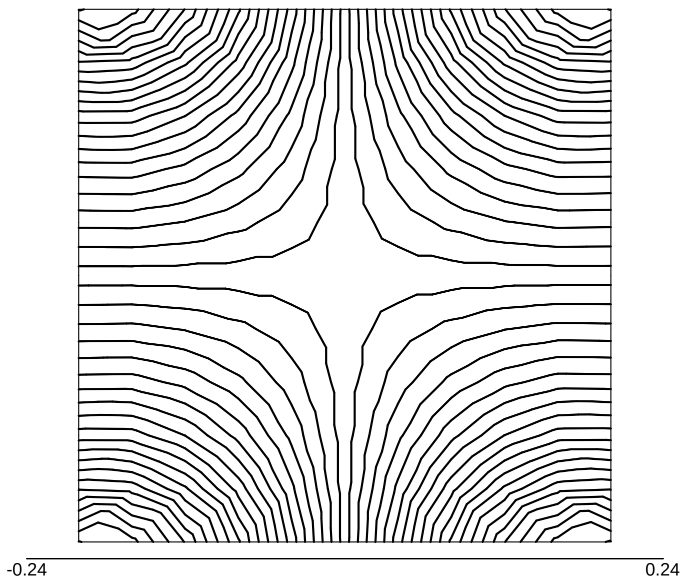

Fig. 16. Numerical pressure isovalues - Test proposed by Bercovier-Engelman on a structured mesh - Expected extrema : -0.25 to 0.25 on the boundary - Computed extrema : -0.24 to 0.24.

pressure fields : they both explode near the boundary (see Figures 18 or 19 for the vorticity, maximum is 27.8 instead of 16 in the Bercovier-Engelman test case). Moreover, error on the pressure remains at a too important level : more than 200% in relative error for the quadratic norm (see Figures 22 and 26). For instance, pressure varies between -7.67 and 6.44 instead of -0.25 and 0.25 in the Bercovier-Engelman case and between -17.56 to 12.83 instead of the constant value zero in the Ruas test. We observe that the rate of convergence

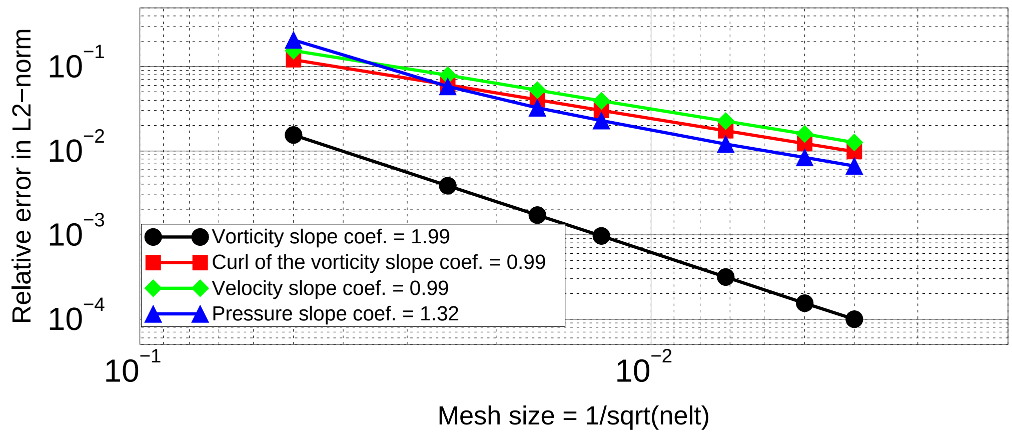

Fig. 17. Convergence order - Structured meshes - Test proposed by Bercovier-Engelman.

is approximatively $\mathcal{O}(\sqrt{h_{\mathcal{T}}})$ for the vorticity and for the pressure and $\mathcal{O}(h_{\mathcal{T}})$ for the velocity (see Figures 22 and 26).

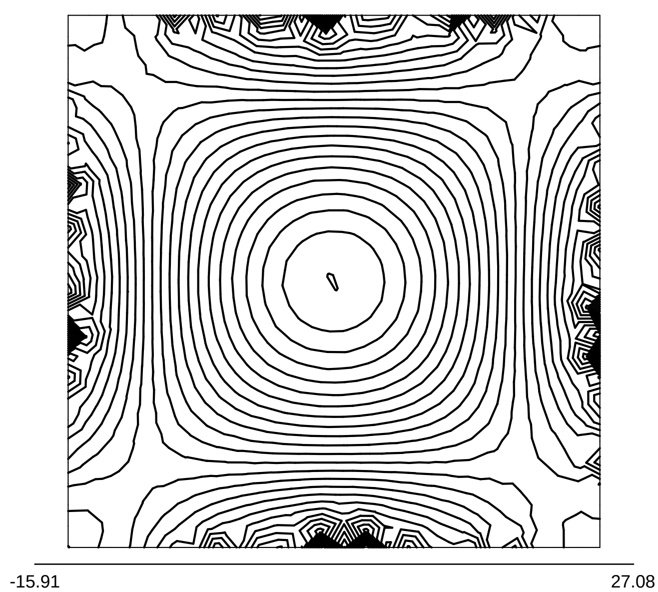

Fig. 18. Numerical vorticity isovalues - Test proposed by Bercovier-Engelman on an unstructured mesh - Expected extrema : -16 in the center, +16 on the middle of the boundary - Computed extrema : -15.9 to 27.

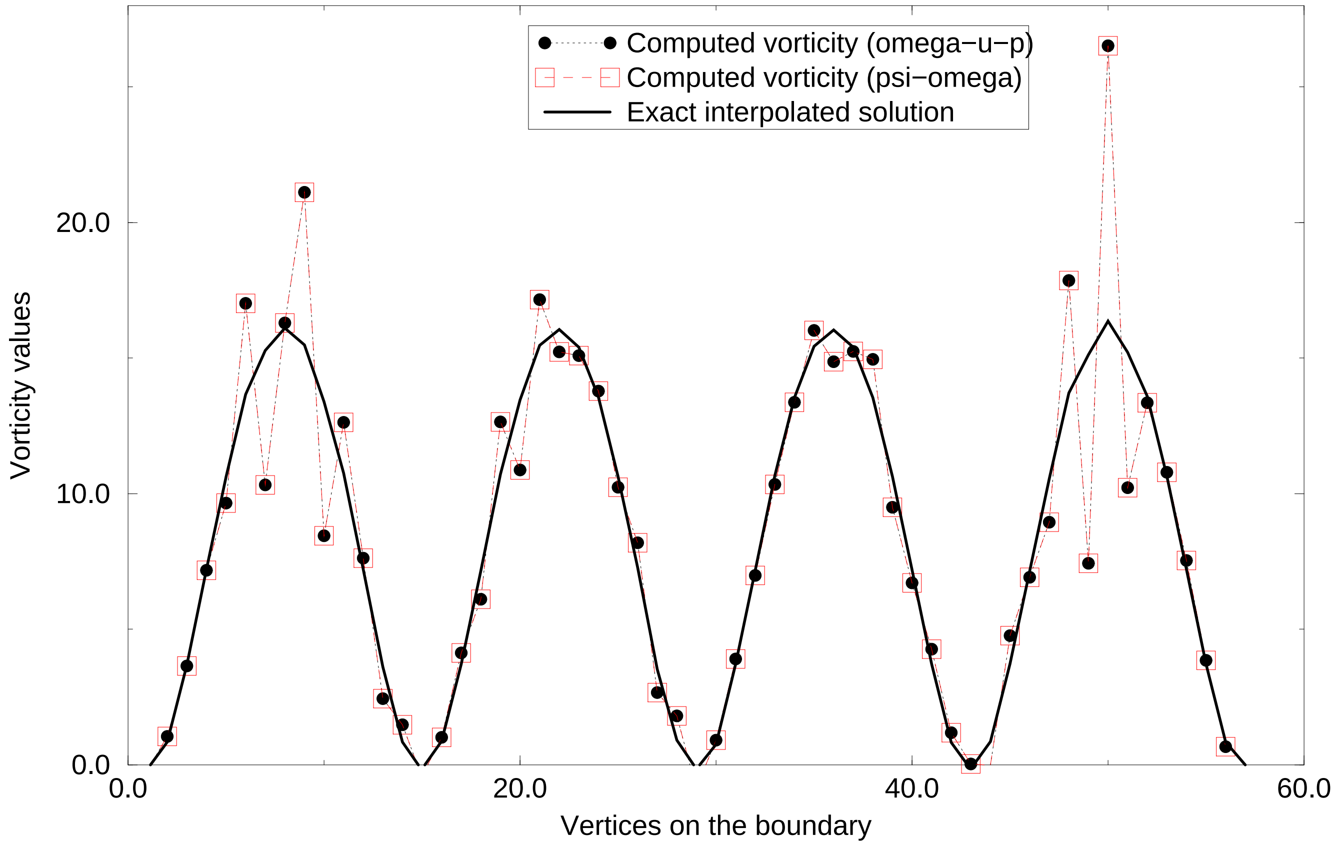

Fig. 19. Value of the vorticity along the boundary - Test proposed by Bercovier and Engelman on an unstructured mesh.

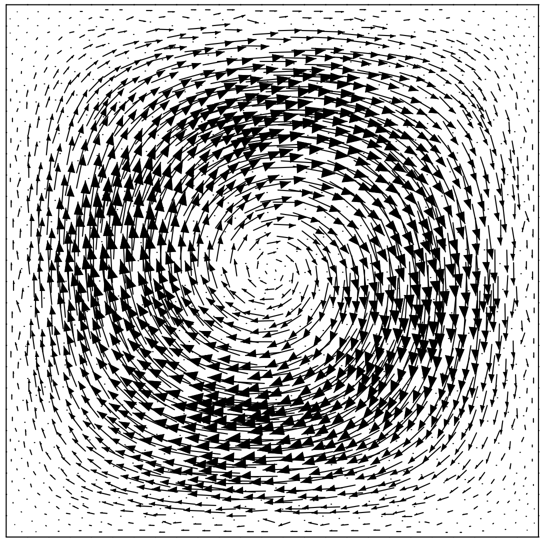

Fig. 20. Velocity vectors - Test proposed by Bercovier and Engelman on an unstructured mesh.

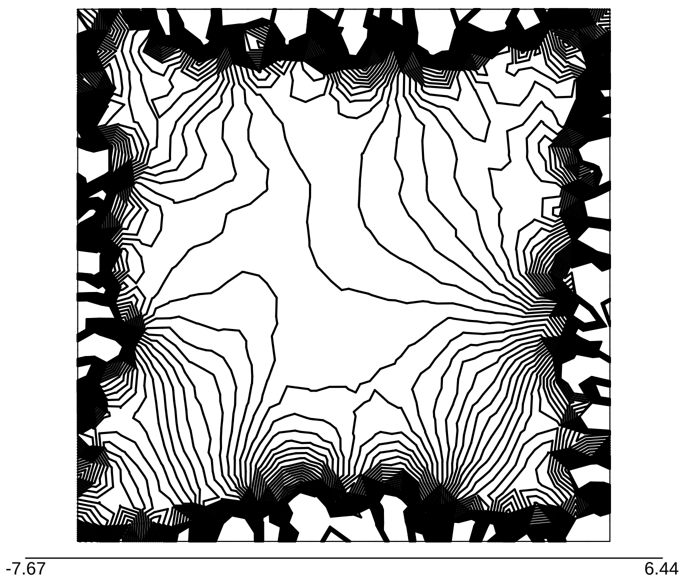

Fig. 21. Numerical pressure isovalues - Test proposed by Bercovier and Engelman on an unstructured mesh- Expected extrema : -0.25 to 0.25 on the boundary - Computed extrema : -7.6 to 6.4 !

4.3 Numerical results on embedded meshes

Some other convergence results with the Bercovier-Engelman test case are numerically obtained on embedded meshes, *ie* meshes obtained from a given one by dividing each triangle in four homothetic ones (see Figures 27 and 28), even if the problem is not stable as we are not in the case analysed in the

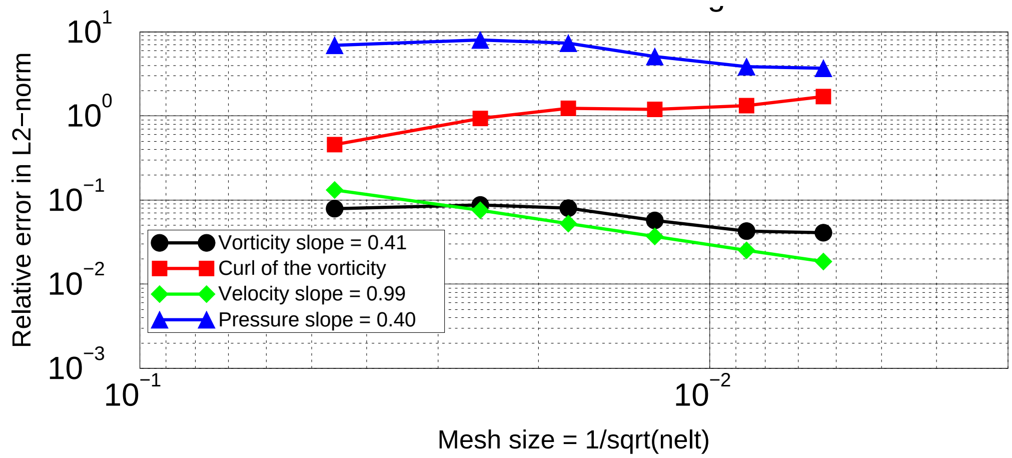

Fig. 22. Convergence order - Unstructured meshes - Test proposed by Bercovier-Engelman.

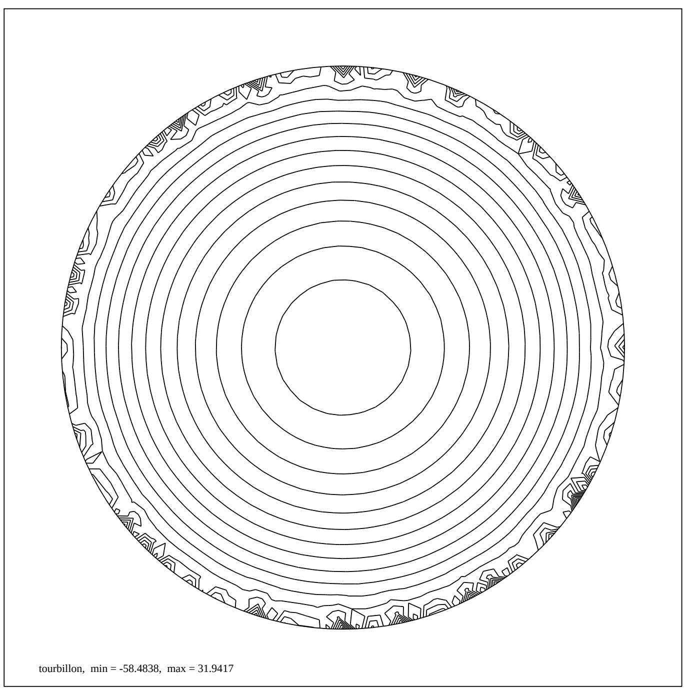

Fig. 23. Numerical vorticity isovalues - Test proposed by Ruas on an unstructured mesh - Expected extrema : -32 on the whole boundary to +32 in the center - Computed extrema : -58.4 to 31.9.

theorem 17 (see Figure 31). In this case, we observe a rate of convergence of order 1 on the pressure and on the vorticity. We also remark that the error on the curl of the vorticity seems to be bounded. Nevertheless, extrema of both fields seem to increase with the number of refinements.

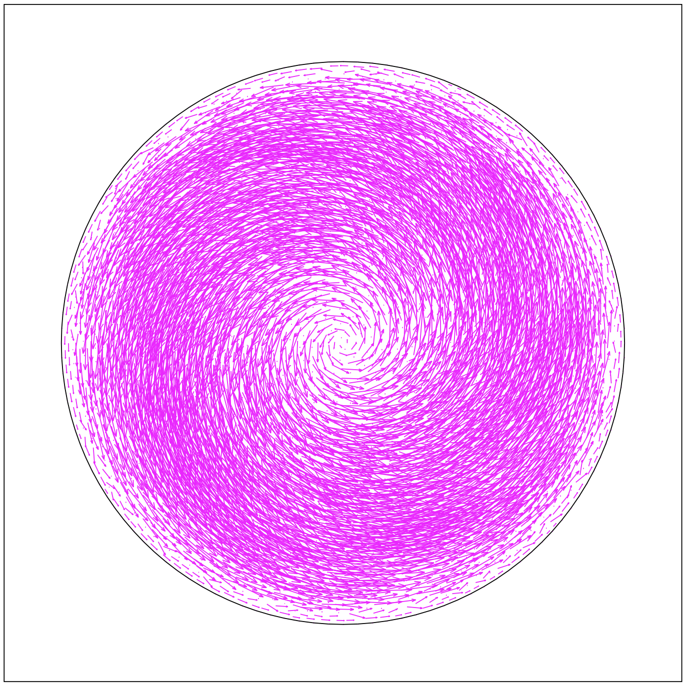

Fig. 24. Velocity vectors - Test proposed by Ruas on an unstructured mesh.

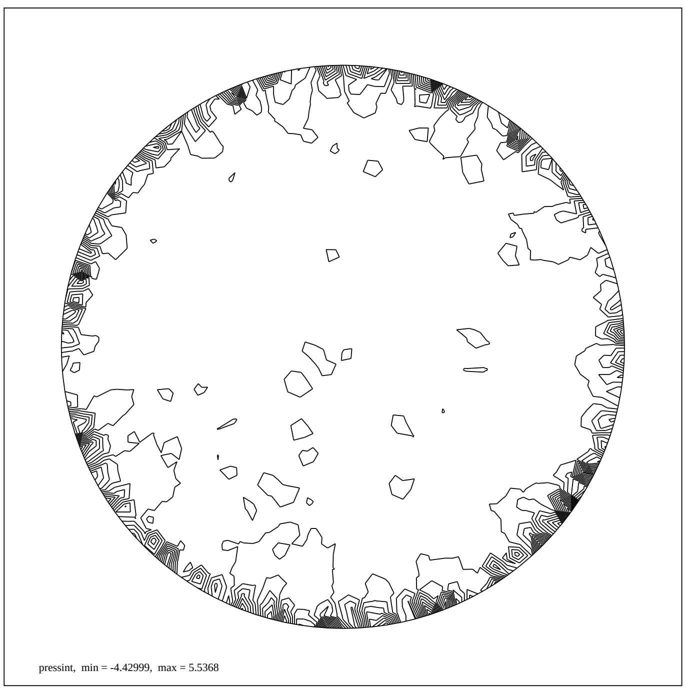

Fig. 25. Numerical pressure isovalues - Test proposed by Ruas on an unstructured mesh - Expected pressure : 0 - Computed extrema : -17.5 to 12.8 !

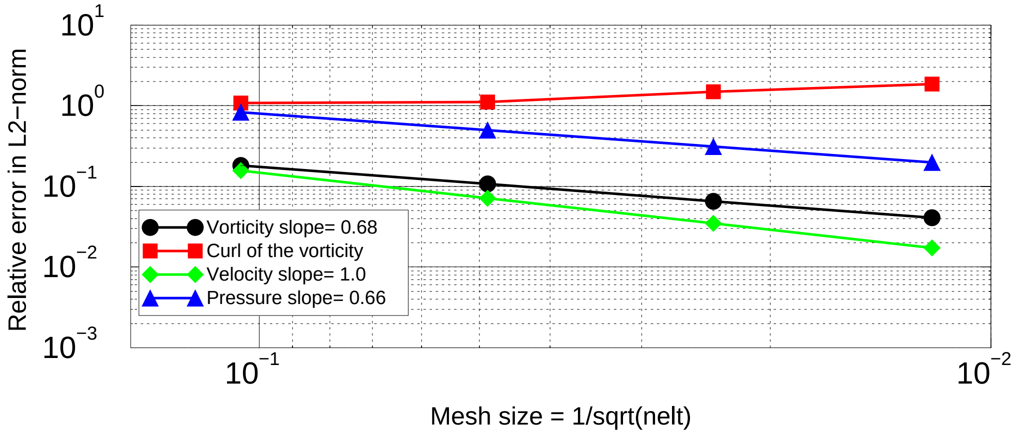

Fig. 26. Convergence order - Unstructured meshes - Test proposed by Ruas.

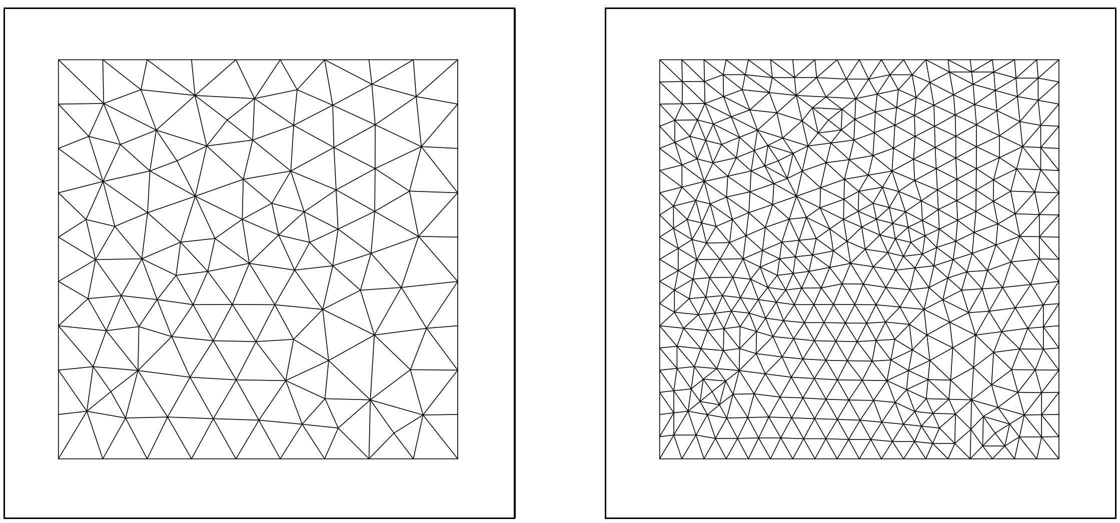

Fig. 27. An unstructured mesh and the same mesh refined once.

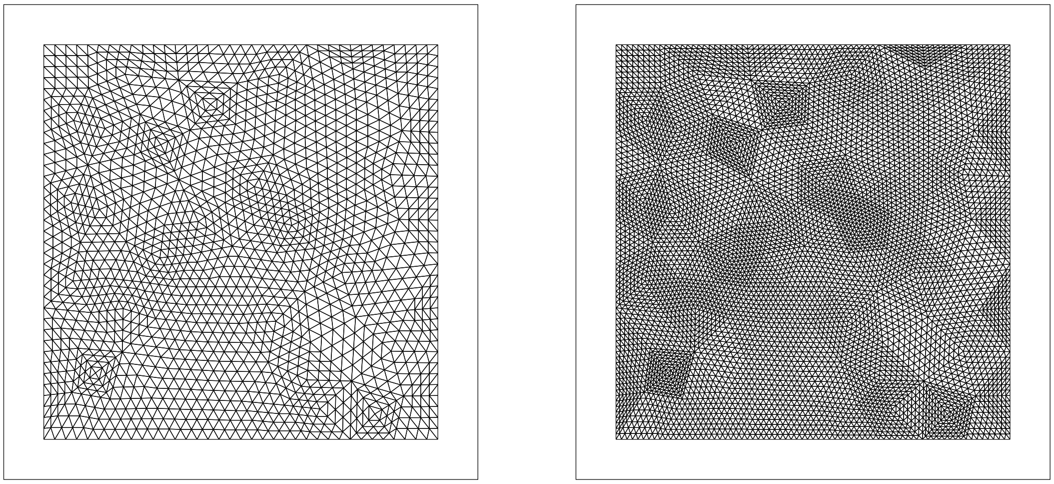

Fig. 28. Same mesh refined twice and third.

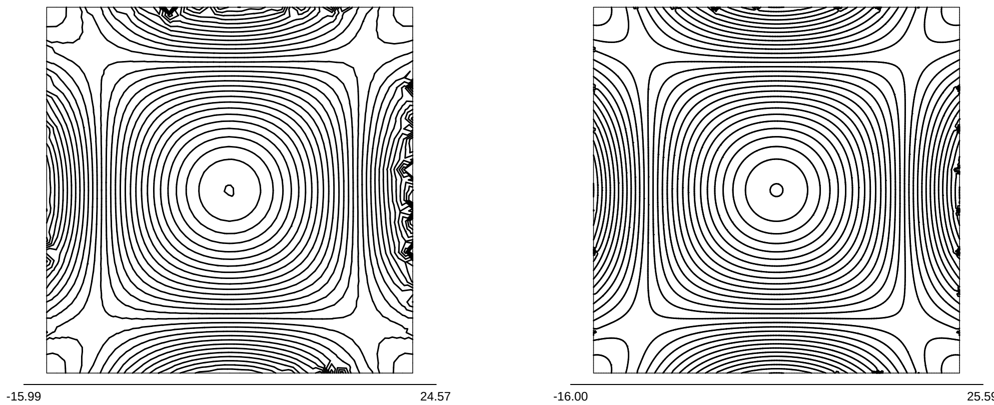

Fig. 29. Numerical vorticity isovalues on the once (left) and four times (right) refined mesh - Test proposed by Bercovier-Engelman - Expected extrema : -16 in the center, +16 on the middle of the boundary - Computed extrema : -15.9 to 24.5 (left), -16 to 25.5 (right).

4.4 Link with the stream function-vorticity formulation

• In all this section, we suppose that Ω is simply connected and that the velocity u is identically zero on the whole boundary Γ. With the notation introduced in (12-15), the above boundary condition corresponds to the following ones in the vorticity-velocity-pressure formulation :

$$\begin{aligned} \Gamma_m &= \Gamma \qquad g_0 \equiv 0 \quad , \\ \Gamma_t &= \Gamma \qquad \sigma_0 \equiv 0 \quad . \end{aligned}$$

The unknown velocity field u belongs to the space X introduced in relation (16) and satisfies also the incompressibility relation (11). Then, Ω being simply

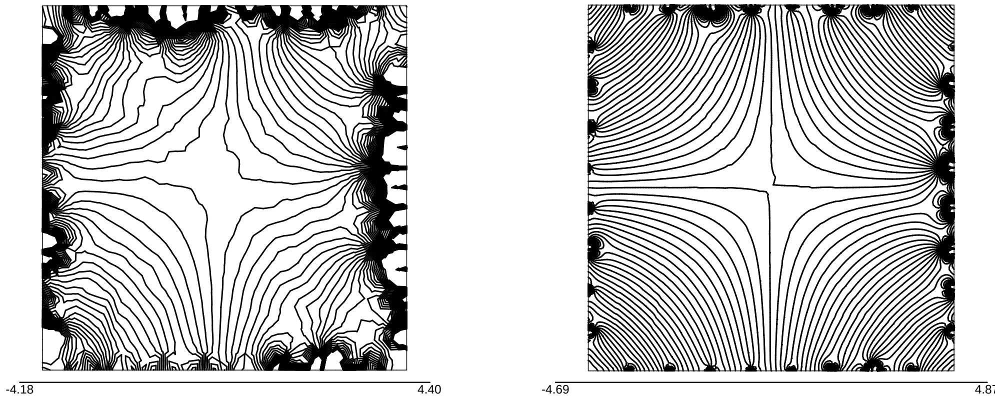

Fig. 30. Numerical pressure isovalues on the once (left) and four times (right) refined mesh - Test proposed by Bercovier-Engelman - Expected extrema : -0.25 to 0.25 on the boundary - Computed extrema : -4.1 to 4.4 (left), -4.6 to 4.8 (right).

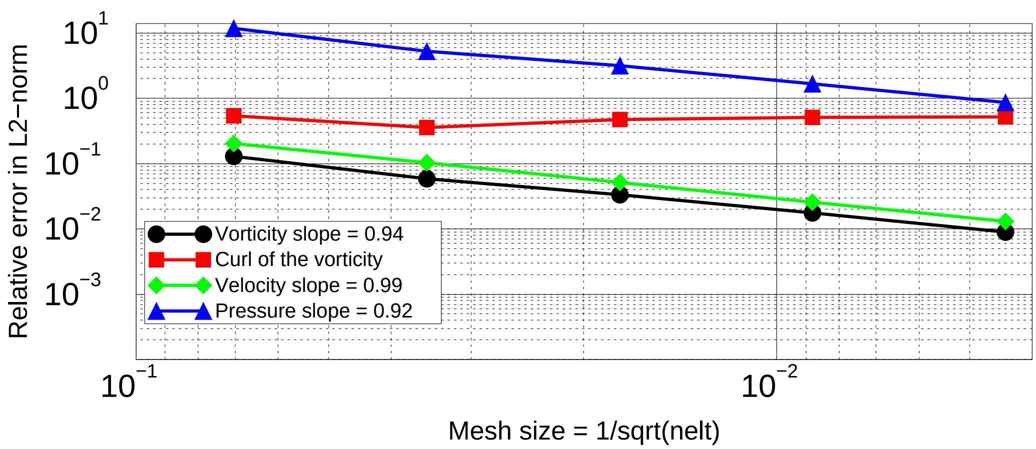

Fig. 31. Convergence order - Embedded meshes - Test proposed by Bercovier-Engelman.

connected, there exists a stream function ψ that belongs to space $H_0^1(\Omega)$ in such a way that u is the curl of ψ (see *e.g.* Girault and Raviart [GR86]) :

$$u = \text{curl } \psi \quad . \tag{46}$$

Then equations (9) and (10) can be written :

$$\omega + \Delta\psi = 0 \quad \text{in } \Omega \quad , \tag{47}$$

$$-\Delta\omega = \text{curl } f \quad \text{in } \Omega \quad . \tag{48}$$

With representation (46), the boundary conditions for the stream function are :

$$\psi \;=\; 0 \;\text{ and }\; \frac{\partial \psi}{\partial n} \;=\; 0 \quad \text{on } \Gamma \quad . \tag{49}$$

These equations are nothing else than those of the Stokes problem in stream function-vorticity formulation which was well studied ([GP79], [GR86]).

• The usual variational form of (47)-(48) can be obtained by multiplying the equation (47) by a function φ in $H^1(\Omega)$, and the equation (48) by a function ξ in $H_0^1(\Omega)$. Then, we obtain :

$$\begin{cases} (\omega, \varphi) - (\text{curl } \psi, \text{curl } \varphi) = 0 & \forall \varphi \in H^1(\Omega) \\ (\text{curl } \omega, \text{curl } \xi) \qquad\qquad\; = (\text{f}, \text{curl } \xi) & \forall \xi \in H_0^1(\Omega) \quad . \end{cases} \tag{50}$$

Here again, we will not discuss the well-posedness of this problem. By the way, it is not well-posed when the vorticity belongs to $H^1(\Omega)$. This is the reason why we said at the end of Section 1 that the vorticity had to be searched in an other space (for more details, we refer to [BGM92]). We have just proved that the vorticity-velocity-pressure problem is formally equivalent to the stream function-vorticity problem when we restrict to bidimensional case and particular boundary conditions. For a more precise study of the link between these two formulations, we refer to [DSS01].

• Let us also observe that, in both discrete schemes, the vorticity is a piecewise continuous polynomial of degree one and the velocity is constant on each triangle. Indeed, in the vorticity-velocity-pressure formulation, the velocity is an exactly divergence free vector of the Raviart-Thomas element thus is constant on each triangle. And in the stream function-vorticity formulation, the stream function being a piecewise polynomial of degree one, its curl is also constant per triangle. On Figure 19, we remark that both numerical methods (vorticity-velocity-pressure and stream function-vorticity codes) give the **same** result for the vorticity. This comparison between the two methods is also illustrated by the **same** convergence rates on the quadratic norm of the vorticity and the velocity obtained by the two schemes (see Figures 32 and 33). Indeed, we observe a convergence of only $\mathcal{O}(\sqrt{h_{\mathcal{T}}})$ for the vorticity (see Figures 22 and 26), as expected in a convex domain by [GR86] and [Sch78], which is the case considered here. Moreover, the quadratic norm of the curl of the vorticity has a divergent behaviour (see Figures 22 and 26), which is well known (see *eg* [Gir96]). Only the velocity is correct and converges in quadratic norm in $\mathcal{O}(h_{\mathcal{T}}^{1-\epsilon}),\ \epsilon > 0$ in both cases.

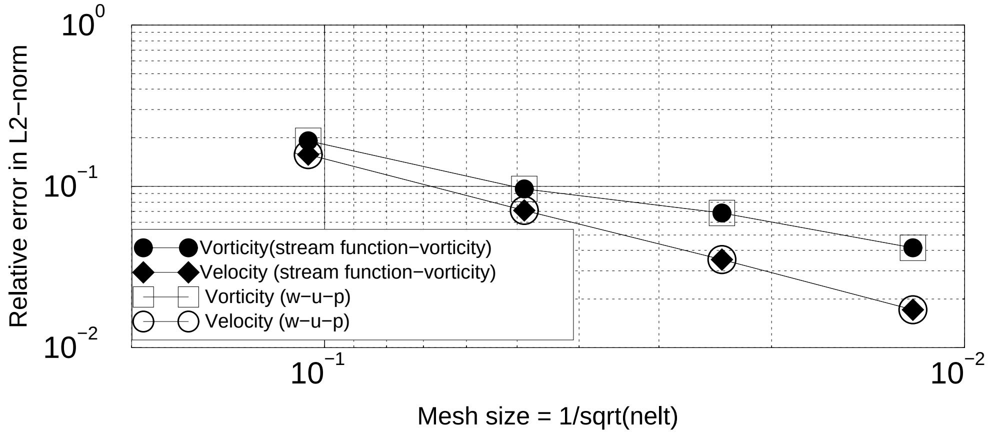

Fig. 32. Comparison between convergence orders - Unstructured meshes - Test proposed by Ruas.

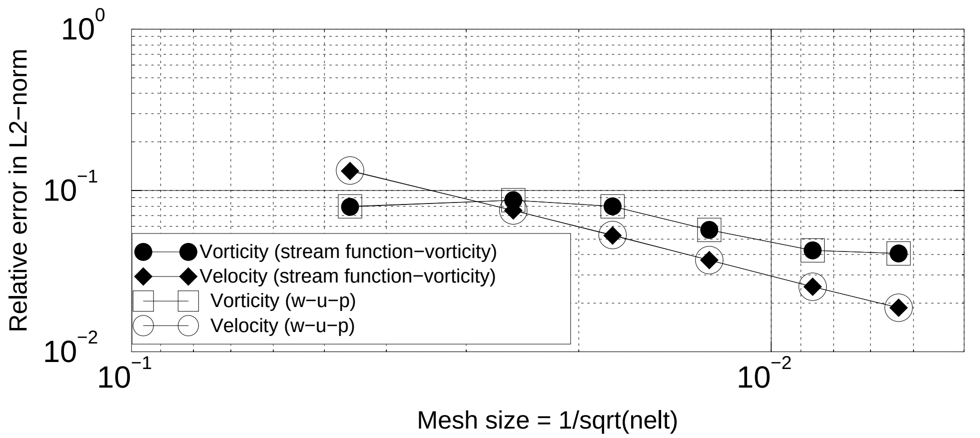

Fig. 33. Comparison between convergence orders - Unstructured meshes - Test proposed by Bercovier and Engelman.

5 Conclusion

The vorticity-velocity-pressure variational formulation of the bidimensional Stokes problem for incompressible fluids was introduced in [Dub92] with the vorticity chosen in space $H(\mathrm{curl}, \Omega) (= H^1(\Omega)$ in bidimensional domains). In this paper, the well-posedness of this problem is theoretically proven for a particular case of Dirichlet vorticity boundary condition. We have here introduced a numerical discretization of the vorticity-velocity-pressure variational formulation and proven theoretically and numerically that our numerical scheme is stable, with an optimal rate of convergence, in this particular case of boundary condition.

However, our numerical experiments show that this scheme, in the general case of boundary conditions, gives correct results on structured meshes, improvable ones on unstructured meshes, and converges on embedded meshes. To our opinion, this formulation is not sufficiently stable in the general case of Dirichlet velocity boundary conditions, exactly as the stream function-vorticity formulation.

For the stream function-vorticity formulation, the problem is solved in a paper of the authors [DSS02] thanks to "discrete harmonic functions". The first extension of the vorticity-velocity-pressure formulation, which is achieved, is to define a good functional frame for our formulation, as Bernardi, Girault and Maday [BGM92] did for the stream function-vorticity one [DSS01]. The second one is to build a numerical scheme fitted to this functional frame as we did in the case of the stream function-vorticity formulation with the help of harmonic functions (see [ASS02] and [ASS01]). The corresponding bi-dimensional numerical results for the vorticity-velocity-pressure formulation are in progress.

Acknowledgements The authors would like to thank the referee for his very interesting remarks and Christine Bernardi for helpful discussions.

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