Evil twins:
Obstructions to the extension problem of generating families

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Introduction

0.1 Obstructions to generating families constructions

This is based on an idea of Sylvain Courte, who observed, while working with Vivek Shende (about the main theorem of "Generating families are sheaves" [She15]), that there are homotopical obstructions to construct (equivalence classes of) generating families of functions for Legendrian knots, and these cannot be detected by sheaves.

The typical counterexample occurs when constructing a generating family from left to right beginning with some linear function. If the front have crossings, we may have choices when we want to extend the GF (Generating Family) past a crossing in the front. Basically these correspond to the linking of the ascending sphere and the descending sphere of the two critical points involved in the crossing. If we make bad choices - involving non-trivial linking - at some point we are prevented from extending the GF past a further crossing. However we can always extend the associated sheaf.

Formally, these obstructions live in the stable homotopy groups of spheres (would we be forbidden to stabilize our GFs, the obstructions would instead live in the unstable homotopy groups of adequate dimension spheres).

The same obstructions can also be responsible for the existence of non-equivalent GFs, induing the same sheaf and the same augmented DGA. It is easy to build them for the Hopf link with crossings of positive Maslov degree.

In the same vein, he noted that other homotopical obstructions could be responsible for other surprisingly non-equivalent GFs. Briefly put the possible obstructions are:

- Homotopy groups of $S$ the sphere spectrum which are these stable homotopy groups of sphere mentioned earlier, involved in the linking of ascending and descending spheres.

- Homotopy groups of the homotopy fiber $G/o$ where $G = F$ is the spectrum of self-equivalences of spheres, and $O$ is the orthogonal spectrum. Examining this, he can combine a full proof of the uniqueness of the GF for the 1-dimensional unknot based on the work of Jean Cerf namely the uniqueness of births and deaths and a surgery argument equivalent to $\pi_1(G/o) = 0$.

- Homotopy groups of $O/o \times o$, the space of "quadratic forms on $\mathbb{R}^\infty$" which is defined here and where $\pi_1(O/o \times o) \neq 0$ is used to produce the counterexample. It is equivalent to $BO \times \mathbb{Z}$ the classifying space of finite-dimensional vector bundles.
Only when these obstructions vanish, can we prove that sheaves and GFs are the same. Courte and Shende can thus prove this theorem for 1-dimensional links which admit special unobstructed fronts called "ordered".

### 0.2 Application to the problem of GF-compatible surgery

We will sketch here how the last class of obstructions can cause the possible failure in general of the GF-surgery theorem 4.2 of [BST15]. We are in a position to surger a Legendrian manifold \( \Lambda^- \subset J^1(M) \) with a (standard) GF \( f^- \) and get \( \Lambda^+ \) via a conical Legendrian cobordism in \( J^1(M \times \mathbb{R}) \) without Reeb chords (i.e. an embedded cylindrical exact Lagrangian cobordism) \( L : \Lambda^- \to \Lambda^+ \). Can we upgrade this to a cobordism with generating families?

This problem can be cast in three successive questions:

(a) Does there exist a GF (among the standard class of GFs we are considering) for \( L \) ?

(b) If the answer to (a) is no, then the GF-surgery is obstructed in the standard class considered. This surgery is simply not admissible in our category of GF-Legendrians.

But if yes, can we find a GF for \( L \), such that \( F|_{M \times (-\infty)} = f^- \) modulo strict equivalences (i.e. up to fibered diffeomorphisms and stabilisations by a constant quadratic form) ?

(c) If yes to (b), the surgery is unobstructed.

However if the answer is no, we can ask if our fixed GF \( f_- \) is equivalent - in a generalised sense of equivalence - to one of these unobstructed standard GFs ?

For example, in the linear at infinities GF class with strict equivalences, a negative answer to the last question is a non-obvious counter-example to the surgery theorem - this is the case exposed in detail in section 2.2. However we can try to amend it by enlarging our equivalence relation (more precisely by changing the notion of stabilisation), and hoping that we cannot find non-amendable counterexamples. \(^1\)

Assuming we want to do a \( k \)-surgery (i.e. there is a surgery box containing a neighborhood of a \( k \)-sphere of cusps in \( \Lambda^- \), and a horizontal disk bounding it), we will show that (some) obstructions to positive answers are

- in \( \pi_{k+1}(U/o) \) for question (a)
- in \( \pi_k(O/o \times o) \) for question (b)

The first obstruction comes from the fact that the Gauss map of \( \Lambda^+ \) has to be stably trivial, because a GF for \( L \) would induce a GF for \( \Lambda^+ \) (from theorem 1.1.15). More precisely, any disk in \( \Lambda^- \) bounding the surgery sphere, \( D^{k+1} \to \Lambda^- \) with \( \partial D^{k+1} = S^k \), yields a \( k+1 \)-sphere in the Lagrangian Grassmannian. Indeed the cusp edge in the surgery box is assumed to be horizontal, so that each point of \( S^k \) maps to the horizontal Lagrangian plane. Up to constant stabilisation, this defines a class \( \alpha \in \pi_{k+1}(U/o) \) where \( U/o \) is the infinite Lagrangian Grassmannian. After the surgery, if \( \alpha \neq 0 \), the Gauss map of \( L \), \( \gamma(L) \in [L, U/o] \), becomes non-trivial, which prevents the existence of a GF.

The second obstruction shows up because along the \( k \)-sphere in the cusp edge of \( \Lambda^- \), each function in the family is a birth function (i.e generalised Morse function), whose birth-death

\(^1\)Such an amendable case \( f_- \) is what I denote by "evil twin" of a good - i.e. surgerable - equivalent GF \( \tilde{f}_- \).
happens in a field of lines in its tangent domain. This can be straightened out to a fixed direction by constant stabilisation and fibered diffeomorphism (because the infinite dimensional sphere is contractible), so that in an orthogonal subspace, we get Morse singularities, i.e. a $k$-sphere of quadratic forms. Stabilizing more, we get a homotopy class of map $\alpha$ from $S^k$ to $O/O \times O$. Now, would we be allowed to make the surgery, there would be a disk $D^{k+1}$ in the cusp edge of $L$, bounding the original $S^k$. Following the quadratic forms along the disk to a point makes an homotopy from $\alpha$ to a constant map.

0.3 Description of sections

In the first section we roughly sketch a tentative theory of generating families, whereas in the second section we explain in detail our counterexample.

1 Different models of generating families and their equivalences classes

1.1 Definitions of "standard" generating families

1.1.1 Families of functions and fiber derivative

Let $M$ be a smooth manifold of dimension $n$ without boundary, possibly non-compact.

**Definition 1.1.1** (Giroux). A semi-local family of functions on $M$ is a smooth function $f$ defined on a codimension zero submanifold $N$ (possibly with boundary) of a finite rank vector bundle $E$ over $M$:

\[
E \supset N \xrightarrow{f} \mathbb{R} \xrightarrow{p} M
\]

If $N = E$, we have the more usual notion :

**Definition 1.1.2.** A (global) family of functions on $M$ is a smooth function $f$ defined on a finite rank vector bundle over $M$:

\[
E \xrightarrow{f} \mathbb{R} \xrightarrow{p} M
\]

In any case we can define the object that will allow us to generate something from these families. Fix a connection on $E$ so that the horizontal bundle is defined.

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\[ It seems that we are some more general notion of families of functions, where any bundle over $M$ could be used (for Eliashberg-Gromov, even non-locally trivial fibrations are acceptable). In fact we will see on some examples in [1.1.23] that it is plausible that we can assume the fiber $F$ of a bundle $E$ to be a finite dimensional vector space. The idea is to build a Morse-Bott function on $\mathbb{R}^N$ whose critical set is exactly $F$. This is an example of model function as in [1.1.3]. \]
Definition 1.1.3. The fiber derivative $\partial_\eta f$ is the restriction of $df$ on the vertical bundle. Analogously the base derivative $\partial_q f$ is the restriction of $df$ on the horizontal bundle, so that $df = \partial_q f + \partial_\eta f$. And the fiber critical set or critical locus $\Sigma_f$ is simply $\partial_\eta f^{-1}(0)$.

1.1.2 Generating families of functions

Let $\Lambda$ be a immersed Legendrian smooth submanifold (possibly non-compact and with boundary but we will not discuss subtleties at the boundary) of $M$.

Definition 1.1.4. A (semi-local) family of functions $(f, N \subset E \xrightarrow{p} M)$ on $M$ is a generating family (abbreviated as GF) of $\Lambda$ if

- $0$ is a regular value of $\partial_\eta f$ so that $\Sigma_f$ is a submanifold of $N$,
- The map $j^1_q f(e) = (q, f(e), \partial_q f(e))$ for $e \in \Sigma_f$ and $q = p(e) \in M$ defines an immersion of $\Sigma_f$ in $J^1 M$ whose image is exactly $\Lambda$.

Remark 1.1.5. By considering families of closed 1-forms rather than functions, we can also define generating families for non-exact Lagrangians of $T^* M$ - i.e. which cannot be lifted to Legendrian submanifolds of $J^1 M$.

1.1.3 Strict equivalence

Definition 1.1.6. A fibered diffeomorphism between two semi-local generating families of $\Lambda$ say $(f, N \subset E \xrightarrow{p} M)$ and $(f', N' \subset E' \xrightarrow{p'} M)$, is a diffeomorphism between some open neighborhoods of the critical loci : $N \supset Op(\Sigma_f) \cong Op(\Sigma_{f'}) \subset N'$ that makes everything commute, i.e. such that

\[
\begin{align*}
\phi(\Sigma_f) &= \Sigma_{f'} \\
p \circ \phi &= p' \\
f' &= f \circ \phi.
\end{align*}
\]

(1.1.3)

Now the most restrictive kind of stabilisation possible, but necessary so that we consider only stable phenomena, is defined as follows

Definition 1.1.7. Let $(f, N \subset E \xrightarrow{p} M)$ be a semi-local GF. Let $E'$ be another vector bundle over $M$. Fix a Riemannian metric $g$ on $E'$. Consider $N \oplus E' \subset E \oplus E'$, we can extend the function $f$ to $N \oplus E'$ by the formula $f \oplus g$ i.e.

\[
(f \oplus g^2)(e, e') = f(e) + ||e'||^2_{g'}
\]

(1.1.4)

for $e \in N$ and $e' \in E'$ with $p(e) = p'(E') = q \in M$. Clearly $\Sigma_{f \oplus g^2} = \Sigma_f \times 0_{E'}$ because $\partial_\eta (f \oplus g^2) = \partial_\eta f + 2 \ast g(e', \bullet)$.

\[\text{To be more precise, we could consider } i : \Lambda \to J^1 M \text{ as a parametrized immersion, and not just a immersed submanifold, and then ask that there is a diffeomorphism } h : \Lambda \to \Sigma_f \text{ so that } j^1_q f \circ h = i.\]

\[\text{Following the footnote \cite{4}, if we have } h : \Lambda \to \Sigma_f \text{ and } h' : \Lambda \to \Sigma_{f'}, \text{ we also add the condition } \phi \circ h = h'.\]
1.1.4 Tame generating families

**Definition 1.1.8** (Eliashberg-Gromov). A map \( f : X \to Y \) between manifolds is a **fibration at infinity**, or **fibration modulo compact subsets** if there exists one compact \( K \subset Y \) and one compact \( K' \subset f^{-1}(K) \) such that \( f^{-1}(Y \setminus K) \xrightarrow{\sim} Y \setminus K \) and \( f^{-1}(K) \setminus K' \xrightarrow{\sim} K \) are both fibrations.

This property is obviously invariant by action of \( \text{Diff}(X) \times \text{Diff}(Y) \). And it stays true for any compact perturbation of the map \( f \).

If \( Y = \mathbb{R} \) we have the criterion :

**Proposition 1.1.9.** A function \( f : X \to \mathbb{R} \) is a fibration at infinity if and only if, for some complete Riemannian metric on \( X \), there exists a compact \( C \subset X \) such that \( df \) is bounded below on \( X \setminus C \) i.e. there exists \( \varepsilon > 0 \) with \( \|\| df_e \|\| \geq \varepsilon \) for all \( e \in X \setminus C \).

In particular \( f \) verifies the Palais-Smale condition, and we have that the critical points of \( f \) all lies in this compact subset \( C \) of \( X \). Moreover the union of gradient lines emanating from \( C \) which don’t escape to (or come from) infinity stays in a compact subset \( S \). This is where all the Morse homology computation takes place (at least after perturbing a bit \( f \) to make it Morse, or Morse-Bott), so that we are computing the singular homology of \( S \) (shifted by the Thom isomorphism with its descending normal bundle in \( X \)).

From this proposition it follows that any homogeneous polynomial \( f : \mathbb{R}^N \to \mathbb{R} \) with only critical point at zero is a fibration at infinity. Indeed near zero \( df \) must not vanish, so there exist some small circle and \( \varepsilon > 0 \) with \( \|\| df_e \|\| \geq \varepsilon \) for all \( e \) on this circle. Or if \( f \) is \( k \)-homogeneous \( (k > 0) \) then \( df \) is \( k-1 \)-homogeneous, so that \( \|\| df_{\lambda e} \|\| \geq \lambda^{k-1} \varepsilon \) for all \( \lambda > 0 \). As \( \leq \) describes all \( \mathbb{R}^n \) but the small ball around zero (a compact subset), \( f \) satisfies the previous criterion. Moreover if we modify \( f \) with some lower-degree polynomial terms \( g \), this is still a fibration at infinity, as \( \|\| (d(f + g)) \|\| \geq \|\| (d(f)) \|\| - \|\| (d(g)) \|\| \) and \( g \) is bounded by a \( k-1 \)-homogeneous polynomial \( h \) so that

\[
(1.1.5) \quad \|\| (d(f + g))_{\lambda e} \|\| \geq \|\| (d(f))_{\lambda e} \|\| - \|\| (d(h))_{\lambda e} \|\| = \lambda^{k-2} (\|\| df_e \|\| - \|\| (d(h))_{\lambda e} \|\|)
\]

and this is greater than \( \varepsilon/2 \) for \( \lambda \) big enough.

Still in the case \( Y = \mathbb{R} \), we have the

**Proposition 1.1.10** (Addition property). Let \( f_1 \) and \( f_2 \) be functions on \( X_1 \) and \( X_2 \). If they are fibrations at infinity, then so is \( f_1 + f_2 : X_1 \times X_2 \to \mathbb{R} \).

Now let’s proceed to the following definition, which can only make sense for global generating families :

**Definition 1.1.11** (Courte). A global generating family \( (f,E \xrightarrow{\sim} M) \) is **tame at infinity** (or simply **tame**) if \( p \times f : E \to M \times \mathbb{R} \) is a fibration at infinity. So it means there exists one compact \( K \subset M \times \mathbb{R} \) and one compact \( K' \subset (p \times f)^{-1}(K) \) such that these are fibrations:

- \( (p \times f)^{-1}(M \times \mathbb{R} \setminus K) \to M \times \mathbb{R} \setminus K \),
- \( (p \times f)^{-1}(K) \setminus K' \to K \).
The compact subset in \( J^0\mathbb{R} = M \times \mathbb{R} \) must at least cover the front of \( \Lambda \) as a GF can never be a fibration above its fiber-critical values. This implies that if \( M \) is non-compact, this definition is only suitable for compact Legendrians. For non-compact ones we shall resort to this definition (equivalent to the first if \( M \) is compact).

**Definition 1.1.12.** A global generating family \((f, E \xrightarrow{p} M)\) is **slicewise tame** if each \( q \in M \) is a fibration at infinity. Precisely there must exist a subset \( K \subset M \times \mathbb{R} \) whose slices in each \( q \times \mathbb{R} \) are compact and one subset \( K' \subset (p \times f)^{-1}(K) \) whose slices in each \( E_q \) are compact, such that the usual maps are fibrations:

- \((p \times f)^{-1}(M \times \mathbb{R} \setminus K) \to M \times \mathbb{R} \setminus K,
- (p \times f)^{-1}(K) \setminus K' \to K.

This property is necessary if we want to extract Morse-like invariants from a GF - precisely we have the lemma 3.10 of [Tra01] for tame GFs (see also the three lemmas (2.2, 2.3 and 2.4) of [ST13]).

**Proposition 1.1.13** (Critical non-crossing). Suppose \((f, E \xrightarrow{p} M)\) is a slicewise tame GF of the Legendrian \( \Lambda \) in \( J^1M \).

Let \( \alpha, \beta : M \supset \mathbb{D}^k \to \mathbb{R} \) be continuous sections over a disc of the bundle \( J^0M = M \times \mathbb{R} \to M \) which avoids the front of \( \Lambda \) (i.e. \( \alpha(q), \beta(q) \) are non-critical values of \( f_q \)) and which never cross each other : \( \alpha(q) < \beta(q) \) for all \( q \in \mathbb{D}^k \). then

\[(1.1.6) \quad H_*(f_{q \leq \beta(q)}, f_{q \leq \alpha(q)}) \simeq H_*(f'_{q' \leq \beta(q')}, f'_{q' \leq \alpha(q')})\]

for all \( q, q' \in \mathbb{D}^n \subset M \).

Where if \( g : X \to \mathbb{R} \) and \( a \in \mathbb{R} \), we have denoted by \( g \leq a = \{ x \in X, g(x) \leq a \} \) the sublevel set of \( g \).

**Proof.** It suffices to show this for \( k = 1 \) and a small interval \([0, 1]\).

**Lemma 1.1.14.** Let \( \alpha, \beta : [0, 1] \to \mathbb{R} \) be continuous functions, and \( t \mapsto (f_t, E \xrightarrow{p} M) \) a continuous 1-parameter deformation of \( f \) for \( t \in [0, 1] \). Assume for all \( t \), \( \alpha(t) < \beta(t) \) and these are regular values of \( f_t \), then the pairs \((f_0 \leq \beta(0), f_0 \leq \alpha(0))\) and \((f_1 \leq \beta(1), f_1 \leq \alpha(1))\) are homotopy equivalent.

Moreover, without this property, the existence of a GF is a homotopical condition (hence flexible). Recall that the rotation class of a Legendrian submanifold \( \Lambda \xrightarrow{i} J^1M \) is its immersion class up to regular homotopy (among immersed Legendrian). It is classified (by Gromov-Lee theorem) by the homotopy class \([\Lambda, M]\) of \( p \circ i \), and the homotopy class of \( U\)-trivialisation of its stable tangent bundle, where is the infinite unitary group. For a manifold to admit a Legendrian immersion in \( J^1M \) with homotopy class \( g \in [\Lambda, M] \), it is enough to admit lift of the stable normal bundle \( \nu_M \) of \( g(\Lambda) \) to the Lagrangian Grassmannian :

\[(1.1.7) \quad \begin{array}{ccc}
\nu_M & \xrightarrow{U/O} & \\nu_M \\
\Lambda & \xrightarrow{\nu_M} & BO
\end{array}\]

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And a choice of such a lift is an affine space directed by \([L,U]\) (this comes from the fact that the vertical map is a homotopy fibration whose homotopy fiber is \(U\)). If \(\Lambda \overset{\sigma}{\rightarrow} J^1 M\) is Legendrian immersion, this lift comes from the tangent map \(Ti\): We embed \(M \in RR^N\) so that we have \(\Lambda \overset{\sigma}{\rightarrow} J^1 \mathbb{R}^N\) isotropic immersion. Then \(Ti\) sends tangent spaces of \(\Lambda\) to isotropic subspaces of \(J^1 \mathbb{R}^N\). Complete these with the Legendrian lift of the normal bundle of \(X \in RR^N\) restricted to \(i(\Lambda)\) and we get Lagrangian subspaces of the canonical contact hyperplane field of \(J^1 \mathbb{R}^N\) which are points in the Lagrangian Grassmannian of dimension \(N\) : \(U(N)/O(N)\) (for details on all this see \[Aud87\]). When \(N\) big enough, this is the stable Gauss map of \(\Lambda \overset{\sigma}{\rightarrow} J^1 M\).  

**Theorem 1.1.15** (Giroux 1991). A compact Legendrian \(\Lambda\) in \(J^1 M\) admits a semi-local generating family if and only if its Gauss map is stably trivial.

**Remark 1.1.16.** (a) The original theorem is written for a priori non-exact Lagrangians and so for generating forms.  

(b) The "if" part of the theorem is false if \(M\) is allowed to be non-compact, see the counterexample IV.1.11 in \[Lat91\]. The "only if" part in the non-compact case is still open, but see \[1.2.2\].

For example stabilized compact Legendrian knots in \(J^1 \mathbb{R}\) with 0 rotation number\footnote{Note that if \(M\) is contractible, the rotation class simplifies as there is no choice of class in \([\Lambda, M] = 0\).} admit generating families, whereas:

**Proposition 1.1.17.** Loose (stabilized) Legendrians of \(J^1 M\) cannot have slicewise tame generating families.

**Proof.** Indeed a tame GF allow to compute relative Morse homology of sublevel sets, so that it remains the same above any \(q \in M\). Take a loose chart where we have a birth of critical values of neighboring indices \(i\) and \(i + 1\) immediately followed by a crossing. Assume all this happens between \(z = a\) and \(z = b\) where \(z\) is the vertical direction in the front projection, and in a very small vertical neighborhood of the birth, so no other part of the front lies in the described part. If we compute the Morse homology \(H_{*}(\{f_{q} \leq b\}, \{f_{q} \leq a\})\) for \(q \in M\) under the loose chart, it is constant when we pass the birth (the two critical points cancel each other in the Morse complex). Whereas after the crossing, we get \(H_{*}(\{f_{q'} \leq b\}, \{f_{q'} \leq a\}) \cong H_{*}(\{f_{q} \leq b\}, \{f_{q} \leq a\}) \oplus \mathbb{Z}[i] \oplus \mathbb{Z}[i + 1]\) because the two critical points cannot cancel each other as the lower degree one \(i\) is now above \(i + 1\) and as we chose the levels very close, there cannot be any other critical points getting rid of these. This breaks the invariance, hence the contradiction. \(\Box\)

However in the case of nearby Lagrangians in cotangent bundles of closed manifolds, the following holds:

**Theorem 1.1.18** (Courte-Guillermou 2018). Any closed Legendrian embedded and without Reeb chords in \(J^1 M\) with stably trivial Gauss map, where \(M\) is closed, admits a generating family tame at infinities.  

\footnote{Indeed, in the case of spheres \(S^k\) in \(J^1 \mathbb{R}^k\), as they are (stably) parallelizable (so that the map to \(BO\) is trivial), their regular Legendrian immersion classes are given by \(r(\Lambda) \in \pi_1(U) \cong \mathbb{Z}\), so the rotation class is called the rotation number. Moreover if \(k = 1\), \(\pi_1(U) \cong \pi_1(\mathbb{R}/so)\) so that the stable Gauss map and the rotation number are the same (in fact up to a factor of 2 because \(\mathbb{R}/so\) the oriented Lagrangian Grassmannian is a 2-cover of \(\mathbb{R}/o\)).}
1.1.5 "Standard"

To define what is standard, we have to choose a set of model functions:

**Definition 1.1.19.** A model assigns to a vector bundle over $M$, a subset of families of functions defined on it.

(1.1.8)  
\[ \mu : (E \xrightarrow{\mathcal{P}} M) \mapsto \mu(E) \subset C^\infty(E) \]

This model will have to verify the transversality conditions of generating families. We can define classes of model independently of the base $M$ by general properties: the classic examples are

- the quadratic model $Q$ made of fiber quadratic maps: $Q(E)$ consist of functions which are non-degenerate quadratic forms on each fiber of $E$,
- the linear model $L$ made of fiber linear maps: $L(E)$ where on each fiber of $E$, $f$ restricts to a non-zero linear function,
- the linear-quadratic model $LQ$, these functions must be a direct sum of a linear map and a quadratic form on each fiber.

Now for a GF to be standard means to conform to some model at infinities, precisely:

**Definition 1.1.20.** A generating family $(f, E \xrightarrow{\mathcal{P}} M)$ is **standard at (both) infinities** for a model $\mu$ if outside of a compact subset $K$ of $E$, $f$ coincide with a function in $\mu(E)$.

**Definition 1.1.21.** A generating family $(f, E \xrightarrow{\mathcal{P}} M)$ is **almost standard at (both) infinities** for a model $\mu$ if outside of a compact subset $K$ of $E$, there exists a model function $m \in \mu(E)$ such that $\partial_\eta(f - m)$ is bounded (for some Riemannian metric on $E$).

We will investigate the classical cases of standardness in detail in section 1.2.

**Remark 1.1.22.** The "(both) infinities" precision only matters when $M$ is non-compact. In this case we could relax these definitions by replacing the compactness condition on the subset $K$ by asking that its intersections $K \cap E_x$ with each fiber of $E$ are compact for all $x \in M$.

Note that the model characterizes the kind of behaviour at infinities that the generated Legendrian $\Lambda$ have. For $f$ a GF standard at infinities for the model $L$, also called "linear at infinities" (GFLI), it implies that

- $p(\Sigma f) \subset M$ is compact (indeed $m$ is linear so it has no fiber critical points) so that $\pi_M(\Lambda)$ is also compact (where $\pi_M : J^1M \to M$),
- $f(\Sigma f) \subset \mathbb{R}$ is compact (indeed outside of $K \cap E_x$, there are no critical points, so the fiber critical values are contained in $f(K)$),

so that the front of $\Lambda$ is a compact subset of $J^0M = M \times \mathbb{R}$ (and thus $\Lambda$ is a compact Legendrian in $J^1M$ as the "slopes" are then bounded above). \[\text{7}\]

For GF quadratic at infinities (GFQI), we have

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7If $f$ is asked to be linear at one infinity, then $\Lambda$ is only compact in restriction to compact subsets $K$ of $M$, i.e. $\Lambda \cap J^1K$ is compact, whereas $\Lambda$ may be non-compact.
\[ p(\Sigma f) \subset M \text{ is the zero section of } E \text{ outside a compact subset of } M \text{ (indeed } \mu \text{ is quadratic so its fiber critical points are exactly } M \times \{0\} \subset E) , \]

\[ f(\Sigma f) \subset \mathbb{R} \text{ is compact (indeed outside of } K \cap E_x \cup M \times \{0\} \text{, there are no critical points, so the fiber critical values are contained in } f(K \cup M \times \{0\}) , \]

so that \( \Lambda \) coincide outside of a compact subset \( p(K) \subset M \) with the zero section \( J_1^0 \) of \( J^1_M \), and \( \Lambda \cap J_1^1 K \) is compact (with boundary).

The significance of this is that the type of standard GF we need to consider depends on the type of Legendrians we are looking at. This can be summarized by looking at the Morse homology of a function \( f_x(f \text{ restricted to the fiber } E_x) \) which is a continuous locally constant function in \( x \in M \).

For linear \( H_*(f_x) = 0 \), and for quadratic \( H_*(f_x) = \mathbb{Z} \) if \( * = \text{ind } Q_x, 0 \) otherwise.

**Example 1.1.23.** Sphere models. This corresponds to the case \( H_*(f_x) = \mathbb{Z} \) if \( * = i \) or \( i + k, 0 \) otherwise. We could easily make such functions with fiber bundle \( E \to M \) where the fiber is an homotopy sphere, but we can also make it with a Morse-Bott function on a vector bundle. We construct the Morse-Bott function associated to \( S^k \) by first showing it for \( k = 0 \), and then spinning the result.

Let \( f : \mathbb{R}^2 \to \mathbb{R} \) be defined by

\[
(1.1.9) \quad f(x, y) = x^2 y - y^3 - y.
\]

This is the universal unfolding of the singularity \( D^-_4 \) called "monkey saddle" i.e. \( x^2 y - y^3 \). As this is an homogeneous polynomial with only critical point at zero, it is a fibration at infinity, and \( f \) obtained from it by lower degree term is also a fibration at infinity (this is the discussion following 1.1.9). Moreover \( f \) has only two critical points \( (\pm 1, 0) \), which are saddle points, on the same level set \( \{ f = 0 \} \). It is symmetric relative to the \( y \)-axis (by the reflection \( x \mapsto -x \)), so we can \( S^k \)-spin \( f \) around this axis to get a Morse-Bott function with critical set \( S^k \) of critical value zero and of index 1. We get \( f_k \) a fibration at infinity (just spin the compact subset of the criterion 1.1.9). Now modify \( f_k \) by any small optimal Morse function on \( S^k \), and stabilize by some fixed quadratic form, then the set of Morse functions obtained is the \( k \)-sphere model prototype. See figure 2 for non-compact Legendrians needing the \( S^k \) model.

**Remark 1.1.24 (Courte-Guillermou).** Twisted generating families i.e. the function don’t have to extend globally, for example due to some Maslov index obstruction - of the kind we want to talk about in these pages.

**Proposition 1.1.25.**

(a) If the model \( \mu \) is tame at infinity, then so is any GF (almost) modeled on \( \mu \) at infinities.

(b) If the prototype of a model (i.e a typical function of the family) is tame at infinity, then the GFs modeled on it are tame at one infinity (in the fiber).

As a linear function is a fibration, \( \mu \in \mathcal{L}(E) \) is a fibration, and so is \( p \times \mu \). Consequently GFLIs are tame at infinity.

A quadratic function is a fibration at infinity : it verifies the criterion 1.1.9 for \( K \) a small ball (its derivative is bounded below) around zero in \( \mathbb{R}^N \). To check it directly would involve \( K \) being a Morse chart around zero, and \( K' \) a small interval in \( \mathbb{R} \). Consequently GFQIs are tame at (one ?) infinity.

However not all tame generating families can be made (almost) standard:

\[
\]
Figure 1: Some level sets of the function $f$ from 1.1.9. Its only critical points are two saddles at $(\pm 1, 0)$.

Figure 2: The Morse homology for each function is $\mathbb{Z}$ in degree $i$ and $i+k$. The existence of a graded normal (generalised) ruling (i.e. "Barannikov partition of critical points") is equivalent to the existence of a slicewise tame generating family... maybe even standard at infinities.
Proposition 1.1.26. Legendrians which are not null-cobordant cannot have standard at infinities generating families.

For example, the Legendrian oriented cobordism group of $J^1\mathbb{R}^3$ is $\mathbb{Z}/3$ generated by a 3-sphere with non-trivial stable framing coming from the framed cobordism group (isomorphic to $\pi_3^S = \mathbb{Z}/24$). This sphere has trivial Gauss map (as $\pi_3(U) \rightarrow \pi_3(U/o)$ is the zero map), so has a semi-local gf, but cannot have a tame gf because of Proposition [1.1.26]. But three copies of this sphere is a Legendrian which may admit tame gfs! Question : is this sphere stabilized ? If yes the three copies of this sphere don’t have a tame gf either.

Proof. If the gf is standard at infinities, we simple interpolate linearly the compact part with zero. This defines a Legendrian filling with a conical singularity. Perturbing the resulting gf a bit, we get an embedded Legendrian. The almost standard case reduces to the standard case by fibered diffeomorphism.

Remark 1.1.27. This is not true for Legendrians with more general tame generating family: Counterexample, the bad gf of the flying saucer, due to $\pi_2(G/o) = \mathbb{Z}/2$. This one cannot have a tame-gf-filling. It cannot be standard, because a compatible model must be a family of functions without critical points parametrized by $\mathbb{R}^2$, whereas this bad gf contains a non-trivial loop of functions without critical points.

1.2 Classical types of GFs

The aim of these standards is double : achieve tameness at infinity, and define a good equivalence relation so that equivalences classes of GF define an invariant by Legendrian isotopy.

The crucible of all equivalence relations of GFs are

(a) Fibered diffeomorphisms:

Let $E' \xrightarrow{p'} M$ be another vector bundle and let $\phi : E' \rightarrow E$ be a diffeomorphism preserving the fibers (but a priori it is not a vector bundle map, we don’t ask it to be linear on fibers). Then $f' := f \circ \phi$ is a new generating family of $\Lambda$. Indeed such a transformation obviously preserves the generating family axioms, i.e. transversality, and the fact that it generates precisely the same Legendrian $\Lambda$.

(b) Stabilisations:

This is where the kind of stabilisations we should allow depends on the kind of standard model. Nevertheless, the general procedure reads as follows:

Given a generating family $(f, E \xrightarrow{p} M)$, we let $S(f)$ be a set of smooth families of functions on the bundle $E$ i.e. functions on $\tilde{E} \xrightarrow{\tilde{p}} E$, that we call the set of stabilisation data for $f$.

Let $(g, \tilde{E} \xrightarrow{\tilde{p}} E) \in S(f)$ be a stabilisation datum for $f$. The stabilisation of $f$ by $g$ is the new family of functions $f + g : E' \rightarrow \mathbb{R}$. Explicitly if $\tilde{e} \in \tilde{E}$, then $(f + g)(\tilde{e}) = f \circ \tilde{p}(\tilde{e}) + g(\tilde{e})$.
$S(f)$ must be carefully chosen, so that $f + g$ is indeed a new GF for $\Lambda$. Some stabilisation data sets can depend only on $\Lambda$ or even $M$ rather than explicitly on the GF $f$, but these may be not general enough to obtain a "good" equivalence relation.

Remark 1.2.1. For Lagrangians, the addition of a constant to $f$ is also an elementary action preserving the Lagrangian. Really in the Lagrangian case, we should talk of exact (or closed) 1-forms rather than functions.

1.2.1 The quadratic case

This relies on Théret’s work.

Let $E$ be a vector bundle over $M$. The most general quadratic model is $Q(E)$. These are families $g : \tilde{E} \to \mathbb{R}$ where $\tilde{E}$ is a vector bundle over $E$ such that $g_e : \tilde{E}_e \to \mathbb{R}$ is a non-degenerate quadratic form for all $e \in E$. We call them fibered quadratic at infinities GFFQI.

The classical model is the case $E = M$ denoted by $Q(M)$. The corresponding GFs are usually called "quadratic at infinities" but we will call them basic quadratic at infinities: GFBQI.

An useful submodel is $Q_{\text{trivial}}(M)$ made only of the $Q$ on a trivial vector bundle: $E = M \times \mathbb{R}^N$ : GFCQI.

The most used submodel $Q(\{pt\})$ is still more restricted : it is made of families which are constant in $q$, i.e $Q_q = A$ for some quadratic form $A$ on $\mathbb{R}^N$. The notation is coherent as the families of functions on a point, which are quadratic forms, are exactly the datum of a unique quadratic form $A$. Usually families of functions modeled at infinities on $Q(\{pt\})$ are called special quadratic at infinities or simply "quadratic at infinities" (which is a source of confusion), but we will call them more explicitly constant quadratic : GFCQI.

Now we can investigate, for some classical equivalence relations induced by a choice of a stabilisation data set, the relations between the models. These non-degenerate quadratic stabilisation data sets only depend on $E$ the domain of $f$:

1. $SQ(E) := Q(E)$ "fibered non-degenerate stabilisation"

Families modeled on $Q(E) \supset Q_{\text{trivial}}(E) \supset Q(M) \supset Q_{\text{constant}}(M) \supset Q(\{pt\})$ are all $SQ(E)$-equivalent.

2. $SQ(M) := Q(M)$ "basic non-degenerate stabilisation"

Families modeled on $Q(M)$, $Q_{\text{trivial}}(M)$, $Q_{\text{constant}}(M)$ are all $SQ(M)$-equivalent.

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3. $\mathcal{SQ}(\{pt\}) := \mathcal{Q}(\{pt\})$ "constant non-degenerate stabilisation"

Families modeled on $\mathcal{Q}(M) \supset \mathcal{Q}_{\text{trivial}}(M) \supset \mathcal{Q}_{\text{constant}}(M)$ are not $\mathcal{SQ}(\{pt\})$-equivalent.

The obstruction lies in $[M, BO]$ for and in $[E, BO]$.

**Theorem 1.2.2** (Kragh 2018). Any embedded long Legendrian $\Lambda$ in $J^1\mathbb{R}^n$ without Reeb chords has a generating family constant quadratic at infinity (in the fiber). Moreover the stable Gauss map of $\Lambda$ is trivial.

A priori this GF is not constant quadratic at infinity in the base. Besides, if the $(n - 1)$-loop of at infinity is stably-non-trivial among functions with a single Morse critical point, $\Lambda$ cannot be isotopic to the zero-section. By the (hard) computation of homotopy groups of this space, these loops are always trivial in dimensions $1, 2, 3, 5, 6, 7$.

The last point of 1.2.2 is not a consequence of Giroux’s theorem, as $\Lambda$ is non-compact. However when the generating family can be chosen to be constant quadratic at both infinities, then we can close $\Lambda$ by restricting it on a big disc $D^n$, and identifying its boundary $\partial D^n = S^{n-1}$ on which the family is a constant quadratic form, so that it defines a GFQI of $\Lambda^+$ in $J^1S^n$. In this case, the stable Gauss map has to be trivial by 1.1.15.

### 1.2.2 The linear case

#### 2 A twist-spinnable generating family which cannot be GF-surgered

#### 2.1 The non-trivial stable loop of quadratic forms

Let $p + q = n$ be integers. We want to define the space of non-degenerate quadratic forms of signature $(p, q)$ on $\mathbb{R}^n$. Every two such quadratic forms can be mapped to one another by conjugation by an invertible matrix, i.e. an element of $GL(n)$. In particular they always reduce to $Q(x) = x_1^2 + \cdots + x_p^2 - x_{p+1}^2 - \cdots - x_{p+q}^2$. Let $O(p, q) \subset GL(n)$ denote the indefinite orthogonal group (and $O(n) := O(n, 0) = O(0, n)$ the definite one) i.e. the stabiliser of a $(p, q)$-quadratic form.

Note that $O(p) \times O(q)$ is a maximal compact subgroup of $O(p, q)$, so that the former is a deformation retract of the latter. The same can be said of $O(n)$ and $GL(n)$. As $O(p) \times O(q)$ is a subgroup of $O(n)$, we can consider the left coset $O(n)/O(p) \times O(q)$ with the quotient topology. By the previous considerations, this is homotopy equivalent to the left coset $GL(n)/O(p, q)$. Any one of these may be considered (up to homotopy) as the space of quadratic forms of signature $(p, q)$ on $\mathbb{R}^n$.

Now consider the tower of inclusions $O(1) \subset O(2) \subset O(3) \subset \ldots$ which define the infinite dimensional orthogonal group $O$ (the orthogonal spectrum). Indeed the inclusion $O(n) \subset O(n+1)$ is $(n - 1)$-connected because the quotient is the n-sphere $S^n$. So this induces bijections on $\pi_k$ for $k < n - 1$ so that the homotopy groups of $O$ - which really are the stable homotopy groups of this tower - are well defined. We can use this tower, and the previous topological quotients, to define the spectrum of indefinite quadratic forms denoted by $O/O \times O$.

Equivalently, we can build this space by considering the stabilisation map on a $(p, q)$-quadratic form:

$$O(n)/O(p) \times O(q) \rightarrow O(n + 2)/O(p + 1) \times O(q + 1)$$

$$(Q : \mathbb{R}^n \rightarrow \mathbb{R}) \mapsto (Q \oplus s : \mathbb{R}^{n+2} \rightarrow \mathbb{R})$$
where \( s(y, z) = yz \) is the standard \((1, 1)\)-quadratic form so that \((Q \oplus s)(x, y, z) = Q(x) + yz\).

This map preserves the signature \( p - q \) and from this point of view it is clear that in the limit \( \pi_0(O/O \times O) = \pi_0(BO \times \mathbb{Z}) = \mathbb{Z} \) is the signature of a stable quadratic form. Moreover, this space has non-trivial fundamental group:

**Proposition 2.1.1.** The following loop of quadratic forms generates \( \pi_1(O(2)/O(1) \times O(1)) = \mathbb{Z} : \)

\[
(2.1.2) \quad (Q_t(x, y) = (x^2 - y^2) \cos(2\pi t) + xy \sin(2\pi t))_{t \in [0, 1]}
\]

Moreover its stabilisation is never trivial, and thus generates \( \pi_1(O/O \times O) = \mathbb{Z}/2 \).

**Remark 2.1.2.** This non-trivial stable loop exists because of a non-trivial symmetry of the quadratic function \( x^2 - y^2 \) (here a \( \mathbb{Z}/2 \) symmetry). It is thus obtained through a \( \pi \) rotation in the \((x, y)\)-plane. If instead we considered degenerate quadratic forms such as the monkey saddle \( x^3 - 3xy^2 \) which has a \( \mathbb{Z}/3 \) symmetry, this would generate other evil twins for families of functions with singularities modelled on the monkey saddle.

In fact homotopy groups of \( O/O \times O \) - being the same as \( BO \times \mathbb{Z} \) - verify the real Bott periodicity, so that we have (see chapters 1 and 2 of [Lat91] for an elementary proof or part IV of [Mil63] for the classical exposition of the original proof of Bott):

\[
(2.1.3) \quad \pi_k(O/O \times O) = \begin{cases}
\mathbb{Z} & \text{if } k \equiv 0 \\
\mathbb{Z}/2 & \text{if } k \equiv 1 \\
\mathbb{Z}/2 & \text{if } k \equiv 2 \\
0 & \text{if } k \equiv 3 \\
\mathbb{Z} & \text{if } k \equiv 4 \\
0 & \text{if } k \equiv 5 \\
0 & \text{if } k \equiv 6 \\
0 & \text{if } k \equiv 7 \\
\end{cases} \quad (\text{mod 8}).
\]

### 2.2 A non-surgerable Legendrian torus

#### 2.2.1 Making the Legendrian unknot standard generating family standard at infinity

The usual generating family of functions for a Legendrian unknot in \( J^1\mathbb{R}^+ \) with coordinate \( r \) on the base \( \mathbb{R}^+ \) (as we want to spin it to make a torus) is given by (with \( z \in \mathbb{R} \) the extra parameter):

\[
f_r(z) = \frac{z^3}{3} - (g(r) - 1)z
\]

where \( g(r) = 2 - (r - 2)^2 \leq 2 \). So that we have :

\[
\partial_z f_r(z) = z^2 - (g(r) - 1) = 0 \iff z^2 = g(r) - 1 = 1 - (r - 2)^2
\]

which has 2 solutions for \( 1 < r < 3 \) and no solution outside of these bounds (see [4]). This is also true for the flattened version of \( g \) to zero outside \([\frac{1}{2}, 3 + \frac{1}{2}]\) still denoted \( g \) (see [5]).
Figure 3: Some level sets of the loop of quadratic forms \((f_1)_t\) on \(\mathbb{R}^2\), for \(t\) from 0 to 1. To be read as usual from left to right and top to bottom.
Lemma 2.2.1. The family \((f_r)_{r \in \mathbb{R}^+}\) can be made linear at infinity by a fibered diffeomorphism \(\phi : \mathbb{R}^+ \times \mathbb{R} \to \mathbb{R}^+ \times \mathbb{R}\) such that \(\tilde{f} = f \circ \phi^{-1}\) defines our standard generating family \((\tilde{f}_r)_{r \in \mathbb{R}^+}\).

Proof. First apply the (inverse of the) fibered diffeomorphism \((r,z) \mapsto (r,f_0(z) = z^3 + z)\), so that the newly obtained family of functions \(\tilde{f}_r\) verifies \(\tilde{f}_r(z) = z\) for \(r\) outside \([\frac{1}{2}, 3 + \frac{1}{2}]\), i.e. it is linear at infinity in the base.

Second apply the "path method" of Moser (following the proof of Proposition 11 of [Thé96]) to find a fibered isotopy \((\phi_t)_{t \in [0,1]}\) from \(\phi_0 = \text{Id}\) on \(\mathbb{R}^+ \times \mathbb{R}\) to \(\phi_1\) so that \(\tilde{f} = \tilde{f} \circ \phi_1\) is linear at infinity in all fibers (especially over the interesting region \(r \in [\frac{1}{2}, 3 + \frac{1}{2}]\)), i.e. outside a big enough compact subset \(K\) of \(\mathbb{R}^+ \times \mathbb{R}\), we want \(\tilde{f}(z) = z\), while \(\phi_1\) is the identity inside a smaller compact subset \(K'\). Writing \(S_t(r,z) = z + t(\tilde{f}_r(z) - z)\) these conditions read

\[
S_t \circ \phi_t(r,z) = z \text{ outside } K,
\]

\[
\phi_t(r,z) = (r,z) \text{ in } K' \subset K.
\]

Deriving this with respect to \(t\), writing \((0,X_t(r,z))\) for the vector field generating \(\phi_t\), we get

\[
\partial_z S_t \circ X_t(r,z) + \partial_z S_t = 0 \text{ outside } K,
\]

\[
X_t(r,z) = 0 \text{ in } K' \subset K.
\]

First condition gives

\[
(1 + t\partial_z(\tilde{f}_r(z) - z))X_t(r,z) = z - \tilde{f}_r(z) \text{ outside } K
\]

(2.2.1)
This has a solution if and only if the following expression never vanishes
\[(1 + t\partial_z(f_r(z) - z)) = 1 + t(\partial_z(f_r(f_0^{-1}(z)) - 1) = 1 + t(f'_r(f_0^{-1}(z)) - 1)\]
which gives, denoting \(f_0^{-1}(z)\) by \(z'\),
\[= 1 + t\left(\frac{z'^2 - (g(r) - 1)}{z'^2 + 1}\right) = 1 - t\frac{g(r)}{z'^2 + 1}\]
For \(z'^2\) big enough, so \(z^2\) big enough, this is always bounded below, so that we can solve for \(X_t(r, z)\) in \([2.2.1]\). Note that outside the interesting region \([\frac{1}{2}, 3 + \frac{1}{2}] \times [-R, R]\) with \(z'^2 \geq R^2\) implying \(\frac{g(r)}{z'^2 + 1} \leq \frac{1}{2}\), and \(K' = [\frac{1}{2}, 3 + \frac{1}{2}] \times [-R, R]\), so that we can interpolate between \(X_t(r, z)\) and 0 on \(K \setminus K'\). The resulting vector field on \(\mathbb{R}^+ \times \mathbb{R}\) is integrable because it is everywhere bounded above by 2 (the maximum of \(g(r)\)).

2.2.2 Stabilizing by the non-trivial loop

Consider the stabilisation of \(\tilde{f}\) depending on \(t\) by \(Q_t\), denoted by \(F_{r,t} : \mathbb{R}^3 \to \mathbb{R}\):
\[F_{r,t}(x, y, z) = \tilde{f}_r(z) + \chi(r - 2)Q_t(x, y)\]
where \(\chi\) is a positive bump function such that:
\[\chi(r) = 0 \iff r \leq \frac{1}{2}\] or \(r \geq \frac{7}{2}\)
\[\chi(r) = 1 \iff 1 \leq r \leq 3\]
which makes \(F\) linear at infinity in the base \(\mathbb{R}^+ \times S^1\). Now to make \(F\) linear at infinity in the fibers we need to build another fibered diffeomorphism \(\psi\) with \(\tilde{F} = F \circ \psi\). This could be done along the same techniques as lemma \([2.2.1]\) but we will refer to lemma 3.8 of \([ST13]\).

Now this family is a loop of generating families which is spinnable, because it is constantly in \(r, t\) the linear function \(z\) near \(r = 0\). The twist-spun Legendrian is a GF-Legendrian torus \((\Lambda, \tilde{F})\) in \(J_1(\mathbb{R}^2)\) where \(\tilde{F}\) is the spun family.

**Proposition 2.2.2.** \((\Lambda, \tilde{F})\) has a circle of cusps in its front projection, which is a GF-attaching region for a 2-handle with core disk = \(\{r \leq 1\}\). However this surgery cannot be made GF-compatibility i.e. the family \(\tilde{F}\) cannot be modified in this region to become a generating family for the Legendrian two-dimensional flying saucer.

**Proof.** Suppose the surgery can be made. Then \(\tilde{F}\) becomes \(\tilde{F}_1\) where they only differ above the disk = \(\{r \leq 1 + \epsilon\}\) for some small \(\epsilon > 0\). Consider the pairs of its critical points over the circle \(\{r = \text{constant}\}\), and denote by \(c_{r,t}\) the one which has positive critical value (sent to the top sheet). When \(t\) goes once around the circle, \((c_{r,t})_{t \in S^1}\) parametrizes a loop in the domain of \(\tilde{F}_1 : \mathbb{R}^2 \times \mathbb{R}^3 \to \mathbb{R}\). Then the Hessian \(d^2_{c_{r,t}}\tilde{F}_1\), which is well defined as \(\partial_{c_{r,t}}\tilde{F}_1 = 0\) by definition of a critical point, describes a loop of non-degenerate quadratic forms. For \(r > 1 + \epsilon\), this loop is, by construction and as \(\tilde{F}_1 = \tilde{F}\), up to a constant \(d^2_{c_{r,t}}\tilde{F}_1 = Q_t(x, y) - z^2\). This is exactly the
non-trivial loop of $O(3)/O(1) \times O(2)$. Now $\tilde{F}_1$ is a generating family for the flying saucer, so the loop of critical points must continue to exist until $r = 0$. However, making $r$ go from $1 + 2\epsilon$ to $0$ in the loop of Hessians, we define an homotopy from $d_{\epsilon_{i},2}\tilde{F}_1$ to $d_{\epsilon_{i},0}\tilde{F}_1 = d_{\epsilon_{0},0}\tilde{F}_1$ which is the constant loop. This is the expected contradiction.

\[
\text{Remark 2.2.3.} \text{ This quadratic non-trivial loop cannot be removed by a fibered diffeomorphism. Indeed such a diffeomorphism must send isomorphically Hessians on Hessians at a critical point. More precisely, if the diffeo is denoted by } \Psi, \text{ then } d_{\Psi,r,t} \text{ conjugates } d_{\epsilon_{i},r}\tilde{F} \text{ and } d_{\epsilon_{i},r}(\tilde{F} \circ \Psi), \text{ so that they describe the same loop of quadratic forms. Moreover as this loop is stable, a stabilisation by a constant quadratic form (i.e. } \tilde{F} + Q : \mathbb{R}^2 \times \mathbb{R}^3 \times \mathbb{R}^N \rightarrow \mathbb{R} \text{ where } Q \text{ is a non-degenerate quadratic form on } \mathbb{R}^N \text{) again yields the same loop. Thus it is characteristic of the GF-equivalence class } \tilde{[F]} \text{.}
\]

\[
\text{Remark 2.2.4.} \text{ The previous construction can easily be generalized to any dimension by } S^k\text{-spinning in place of the } S^1\text{-spinning, and using a family of quadratic forms parametrized by the } k\text{-sphere } S^k. \text{ Note that similarly to } \pi_1(O(1) \times O), \text{ all the non-zero elements of homotopy groups of } O(1) \times O \text{ (shown in } 2.1.3 \text{) are representable by spheres of quadratic forms, so that taking such a family in this construction, we can build many examples of generating families for some } S^k \times S^1 \text{ with non-surgerable GF-attaching regions.}
\]

### 3 Evil twin of a generating family

Consider any closed (i.e. compact without boundary) Legendrian in $J^1(M^2)$, where $M$ is a surface possible non-compact or with boundary (then we can ask that the GF is linear near the boundary), with a linear at (both) infinity generating family $(\Lambda, f)$. Suppose the complement of its base projection $\pi(\Lambda) \subset M$ contains two disconnected open sets $U_1$ and $U_2$ (see figure 6). Suppose also that one of these is a open disk which is a GF-attaching region for a 2-handle. Then we can do the same trick as above. Stabilizing $f$ (adding two extra variables to the parameter space) by $Q_1$ multiplied by a bump function $\chi$ defined on $M$ which is 0 on $W_1$ and $W_2$, where $W_i \subset V_i \subset U_i$ with $W_i$ and $V_i$ closed, and 1 on $M - V_1 - V_2$, we get an evil twin $(\Lambda, F^1)$ which cannot any more be surgered, despite the existence of a GF-attaching region. Moreover, by the previous remark, these two generating families cannot be equivalent.

Clearly the equivalence class of this evil twin is independent of the choices made for $W_i$, $V_i$, and the bump function $\chi$.

If $\Lambda$ has more GF-attaching regions for 2-handles with empty projection to the base, we can do this trick on each region, and get non-equivalent generating families.

Note that if you do twice the trick on the same hole in the projection, you get back to the original generating family (up to equivalence). Suppose the first loop of quadratic forms is done in the stabilization directions $x_1, y_1$, and the second loop in two other directions $x_2, y_2$. This is equivalent to the concatenation of the loops in the same plane $(x_1, y_1)$, thus realizing the group addition in $\pi_1(O(1) \times O)$. Indeed by a fibered diffeomorphism we can make the first loop constant on a half-circle (let’s say the top one), and the second loop constant on the bottom half-circle. Then by a global action of the orthogonal group, we can make the second loop in the direction $x_1, y_1$ (basically doing $\frac{\pi}{2}$ rotations in the planes $x_1, x_2$ and $y_1, y_2$). Now we get twice the bad loop, which is stably homotopic to the zero loop of quadratic forms.
Figure 6: Schematics view of the $W_i \subset V_i \subset U_i$, $i = 1, 2$ and the associated bump function $\chi$. The projection of $\Lambda$ on the base $M^2$ is in the darkest blue (think of $\Lambda$ as a Legendrian torus for example). The complement of $\pi(\Lambda)$ has two connected components: the open sets $U_1, U_2$ (one bounded, the other unbounded if $M = \mathbb{R}^2$), whereas the successive tubes between $W_i \subset V_i \subset U_i$ are in lighter and lighter blue. The very thick black line is the projection of a circle of cusps, which is surgerable, GF-compactly before the trick, but non-GF compatibly after.
References


