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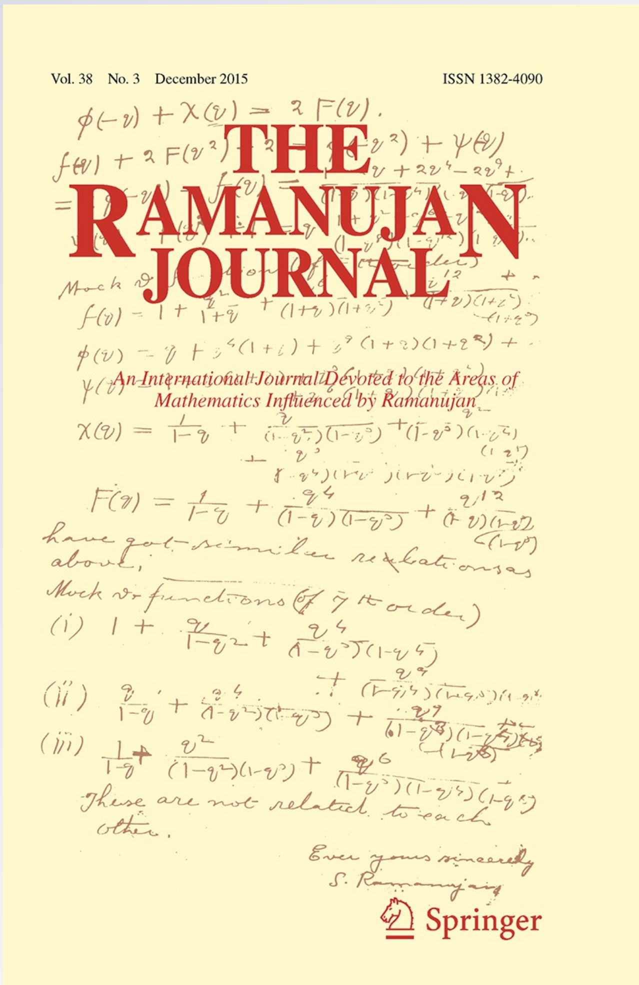



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# A converse to linear independence criteria, valid almost everywhere

**S. Fischler · M. Hussain · S. Kristensen · J. Levesley**

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**Abstract** We prove a weighted analogue of the Khintchine–Groshev theorem, where the distance to the nearest integer is replaced by the absolute value. This is applied to proving the optimality of several linear independence criteria over the field of rational numbers.

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S. Fischler
Equipe d'Arithmétique et de Géométrie Algébrique, Université Paris-Sud,
Bâtiment 425, 91405 Orsay Cedex, France
e-mail: stephane.fischler@math.u-psud.fr

M. Hussain (✉)
School of Mathematical and Physical Sciences, The University of Newcastle,
Callaghan 2308, NSW, Australia
e-mail: mumtaz.hussain@newcastle.edu.au

S. Kristensen
Department of Mathematics, Aarhus University, Ny Munkegade 118, 8000 Aarhus C, Denmark
e-mail: sik@imf.au.dk

J. Levesley
Department of Mathematics, University of York, York YO10 5DD, UK
e-mail: jl107@york.ac.uk

## 1 Introduction

Let $n \geq 1$ and let $\psi_1, \ldots, \psi_n : \mathbb{R}^+ \to \mathbb{R}^+$ be functions tending to zero. We will refer to these functions as approximating functions or error functions. Let $\underline{\psi} = (\psi_1, \ldots, \psi_n)$. An $m \times n$-matrix $X = (x_{ij})_{\substack{1 \leq i \leq n \\ 1 \leq j \leq m}} \in \mathbb{R}^{mn}$ (or the corresponding system of linear forms) is said to be $\underline{\psi}$-approximable if

$$\left|\mathbf{q} \cdot \mathbf{x}^{(i)}\right| = |q_1 x_{1i} + \cdots + q_m x_{mi}| < \psi_i(|\mathbf{q}|), \quad 1 \leq i \leq n, \tag{1}$$

for infinitely many integer vectors $\mathbf{q} = (q_1, \ldots, q_m) \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$. The norm $|\mathbf{q}|$ is the supremum norm here and elsewhere. We will denote the set of $\underline{\psi}$-approximable linear forms inside the set $\mathbb{I}^{mn} := [-\frac{1}{2}, \frac{1}{2}]^{mn}$ by $W_0(m, n, \underline{\psi})$.

The similarity between the $\underline{\psi}$-approximable linear forms studied here and the simultaneously $\psi$-approximable linear forms usually studied in Diophantine approximation is clear. However, in the classical setup one studies the distance to the nearest integer rather than the absolute value.

A major breakthrough in the classical theory was the Khintchine–Groshev theorem [15,21], which establishes a zero-one law for the set of $\underline{\psi}$-approximable matrices depending on the convergence or divergence of a certain series. In the absolute value setting, an analogue of this result was recently obtained by Hussain and Levesley [18]. Their result covers only the case $\psi_1 = \cdots = \psi_n$ with this approximating function being monotonic. The condition of monotonicity was removed by Hussain and Kristensen [17] in the case of a single approximating function (*i.e.* $\psi_1 = \cdots = \psi_n$).

In the present paper, we extend the results of [18] and [17] to the weighted setup, i.e. the case of more than one approximating function. This has applications to linear independence criteria, as we shall see below. Our zero-one law states the following.

**Theorem 1.1** *Let $m > n > 0$ and let $\psi_1, \ldots, \psi_n$ be approximating functions as above. Then, if $(m, n) \neq (2, 1)$,*

$$\mathcal{L}_{mn}(W_0(m, n, \underline{\psi})) = \begin{cases} 0 & \text{if } \sum_{r=1}^{\infty} \psi_1(r) \cdots \psi_n(r) r^{m-n-1} < \infty, \\ 1 & \text{if } \sum_{r=1}^{\infty} \psi_1(r) \cdots \psi_n(r) r^{m-n-1} = \infty, \end{cases}$$

*where $\mathcal{L}_{mn}$ denotes the $mn$-dimensional Lebesgue measure. If $(m, n) = (2, 1)$, the same conclusion holds provided the error function is monotonic.*

The case $m \leq n$ is of less interest, in general, and of no particular interest to us for applications. Briefly, in this case the set $W_0(m, n, \underline{\psi})$ becomes a subset of a lower dimensional set. An easy instance is that of $m = n = 1$, where it is straightforward to prove that the set is in fact a singleton—see, e.g. Lemma 1 in [8] for details. For clarity and further understanding about such cases, we refer to [16, Theorem 1]. This is in contrast to the classical case, where approximation to the nearest integer is considered. Here, the result is independent of the relative sizes of $m$ and $n$.

This setting where linear forms are very small at some points appears in linear independence criteria. To begin with, let us consider the case of one point. Siegel has proved, using essentially a determinant argument, that the existence of $m$ linearly independent linear forms, very small at a given point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m) \in \mathbb{R}^m$, implies a lower bound on the dimension of the $\mathbb{Q}$-vector space spanned by $\xi_1, \ldots, \xi_m$. On the other hand, still in the case of one point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m)$, Nesterenko has derived [22] a similar lower bound on $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_1, \ldots, \xi_m)$ from the existence of just *one* linear form (for each $Q$ sufficiently large), small at $\mathbf{e}_1$ but not too small. The most striking application of his result is the proof by Rivoal [23] and Ball-Rivoal [1] that infinitely many values of Riemann $\zeta$-function at odd integers $s \geq 3$ are irrational.

Both Siegel's and Nesterenko's results have been generalized to several points $\mathbf{e}_1$, …, $\mathbf{e}_n$ as follows (see [12], Proposition 1 and Theorem 3). We denote by $\cdot$ the canonical scalar product on $\mathbb{R}^m$ (which allows us to consider a linear form as the scalar product with a given vector) and by $o(1)$ any sequence that tends to 0 as $Q \to \infty$.

**Theorem 1.2** *Let $m > n > 0$, and $\mathbf{e}_1, \ldots, \mathbf{e}_n \in \mathbb{R}^m$. Let $\tau_1, \ldots, \tau_n$ be positive real numbers. Assume that one of the following holds:*

(i) *The vectors $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are linearly independent, and for infinitely many integers $Q$ there exist $m$ linearly independent vectors $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} \in \mathbb{Z}^m$ such that, for any $j \in \{1, \ldots, m\}$,*

$$|\mathbf{q}^{(j)}| \leq Q \quad \textit{and} \quad |\mathbf{q}^{(j)} \cdot \mathbf{e}_i| \leq Q^{-\tau_i + o(1)} \quad \textit{for any} \quad i \in \{1, \ldots, n\}.$$

(ii) *The numbers $\tau_1, \ldots, \tau_n$ are pairwise distinct, and for any sufficiently large integer $Q$ there exists $\mathbf{q} \in \mathbb{Z}^m$ such that*

$$|\mathbf{q}| \leq Q \quad \textit{and} \quad |\mathbf{q} \cdot \mathbf{e}_i| = Q^{-\tau_i + o(1)} \quad \textit{for any} \quad i \in \{1, \ldots, n\}.$$

*Then we have*

$$\dim F \geq n + \tau_1 + \ldots + \tau_n$$

*for any subspace $F$ of $\mathbb{R}^m$ which contains $\mathbf{e}_1, \ldots, \mathbf{e}_n$ and is defined over the rationals.*

In the case $n = 1$, under assertion $(i)$ (resp. $(ii)$), this is Siegel's (resp. Nesterenko's) above-mentioned criterion; notice that $\dim_{\mathbb{Q}} \mathrm{Span}_{\mathbb{Q}}(\xi_1, \ldots, \xi_m)$ is equal to the dimension of the smallest subspace $F$ of $\mathbb{R}^m$, defined over the rationals, which contains the point $\mathbf{e}_1 = (\xi_1, \ldots, \xi_m)$. The reader may refer to §8 of [3] for classical facts about subspaces defined over the rationals, to Lemma 1 of [12] for a generalization of this equality, and to [11] (especially pp. 81–82 and 215–216) for more details on Siegel's criterion, including applications.

Note that in Theorem 1.2, $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are always $\mathbb{R}$-linearly independent: this is assumed in $(i)$, and it is an easy consequence of $(ii)$ since $\tau_1, \ldots, \tau_n$ are pairwise distinct (see [12], §3.2). The point is that $\mathrm{Span}_{\mathbb{R}}(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ is not defined over the rationals.

The conclusion of Theorem 1.2 is a lower bound for $\dim F$ (which can be stated as a lower bound for the rank of a family of $m$ vectors in $\mathbb{R}^n$ seen as a $\mathbb{Q}$-vector space, see [12], §3.1, Lemma 1). It is a natural question to ask whether this bound can be improved; we give a negative answer in Theorem 1.3. In the case of Nesterenko's linear independence criterion with only one point, Chantanasiri has given ([7], §3) a very specific example of a point $\mathbf{e}_1 = (\xi_1, \dots, \xi_m)$ for which this bound is optimal (namely when $(\xi_1, \dots, \xi_m)$ is a $\mathbb{Q}$-basis of a real number field of degree $m$). On the contrary, our result deals with generic tuples; it encompasses also Siegel's criterion and the case of several points.

**Theorem 1.3** *Let $m > n > 0$, and $F$ be a subspace of $\mathbb{R}^m$ defined over the rationals. Let $\tau_1, \dots, \tau_n$, $\beta_1, \dots, \beta_n$, $\varepsilon$ be real numbers such that $\tau_1 > 0, \dots, \tau_n > 0$, $\varepsilon > 0$,*

$$\tau_1 + \cdots + \tau_n \leq \dim F - n \text{ and } \beta_1 + \cdots + \beta_n = (1+\varepsilon)(\dim F - 1). \quad (2)$$

*Then for almost all $n$-tuples $(\mathbf{e}_1, \dots, \mathbf{e}_n) \in F^n$ (with respect to Lebesgue measure), the following property holds. For any sufficiently large integer $Q$, there exist $m$ linearly independent vectors $\mathbf{q}^{(1)}, \dots, \mathbf{q}^{(m)} \in \mathbb{Z}^m$ such that, for any $j \in \{1, \dots, m\}$,*

$$|\mathbf{q}^{(j)}| \ll Q \quad (3)$$

*and*

$$Q^{-\tau_i}(\log Q)^{\beta_i - (1+\varepsilon)\dim F} \ll |\mathbf{q}^{(j)} \cdot \mathbf{e}_i| \ll Q^{-\tau_i}(\log Q)^{\beta_i} \quad \text{for any} \quad i \in \{1, \dots, n\}, \quad (4)$$

*where the constants implied in the symbols $\ll$ depend on $m$, $n$, $F$, $\tau_1, \dots, \tau_n$, $\beta_1, \dots, \beta_n$, $\varepsilon$, $\mathbf{e}_1, \dots, \mathbf{e}_n$ but not on $Q$.*

This result will be proved in §2.2, using Theorem 1.1 and Minkowski's theorem on successive minima of a convex body. We also provide some remarks on Theorem 1.3 in §2.1.

Throughout we will use the Vinogradov's notation, *i.e.* for two real quantities $x$ and $y$, we will write $x \ll y$ if there is a constant $C > 0$ such that $x \leq Cy$. In Landau's $O$-notation, this would amount to writing $x = O(y)$. If $x \ll y$ and $y \ll x$, we will write $x \asymp y$.

## 2 A converse to linear independence criteria

### 2.1 Remarks on Theorem 1.3

We gather in this section several remarks on Theorem 1.3.

*Remark 1* In general, Nesterenko's criterion is stated under an assumption slightly different from $(ii)$ in Theorem 1.2: it is assumed that there exist an increasing sequence $(Q_k)_{k\geq 1}$ of positive integers such that $Q_{k+1} = Q_k^{1+o(1)}$ as $k \to \infty$ (where the sequence denoted by $o(1)$ tends to 0 as $k \to \infty$) and a sequence $(\mathbf{q}_k)_{k\geq 1}$ of vectors in $\mathbb{Z}^m$, such that, for any $k$,

$$|\mathbf{q}_k| \leq Q_k \quad \text{and} \quad |\mathbf{q}_k \cdot \mathbf{e}_i| = Q_k^{-\tau_i + o(1)} \quad \text{for any} \quad i \in \{1, \dots, n\}.$$

Requesting also $\tau_1, \ldots, \tau_n$ to be pairwise distinct, this is actually equivalent to assumption $(ii)$ of Theorem 1.2. In precise terms, if there is such a sequence $(Q_k)$, then for any $Q$ sufficiently large one may choose the integer $k$ such that $Q_k \le Q < Q_{k+1}$, and let $\mathbf{q} = \mathbf{q}_k$. The converse is easy too: if assumption $(ii)$ of Theorem 1.2 holds, then one can choose *any* increasing sequence $(Q_k)_{k\ge 1}$ of positive integers such that $Q_{k+1} = Q_k^{1+o(1)}$ (for instance, $Q_k = \beta^k$ with an arbitrary $\beta > 1$), and let $\mathbf{q}_k$ be the vector corresponding to $Q = Q_k$.

This remark shows that $\tau_r(\underline{\xi}) = \tau_r'(\underline{\xi}) = \tau_r''(\underline{\xi})$ for any $\underline{\xi}$ in the notation of §4.3 of [13]. With the same notation, Theorem 1.3 (with $F = \mathbb{R}^m$ and $n = 1$) implies that this Diophantine exponent is equal to $m - 1$ for almost all $\underline{\xi} = \mathbf{e}_1 \in \mathbb{R}^m$ (with respect to Lebesgue measure); this answers partly a question asked at the end of [13].

*Remark 2* In the setting of Theorem 1.3, if $\mathbf{e}_1, \ldots, \mathbf{e}_n$ are $\mathbb{Q}$-linearly independent and belong to $F \cap \overline{\mathbb{Q}}^m$, then applying Schmidt's Subspace Theorem instead of Theorem 1.1 in the proof yields the same conclusion as that of Theorem 1.3, except that Eq. (4) is weakened to $|\mathbf{q}^{(j)} \cdot \mathbf{e}_i| = Q^{-\tau_i + o(1)}$.

In the rest of this section, we shall focus on the special case $m = 2, n = 1, F = \mathbb{R}^2$. By homogeneity we may restrict to vectors $\mathbf{e}_1 = (\xi, -1)$ with $\xi \in \mathbb{R}$. Since non-zero linear forms in $\xi$ and $-1$ with integer coefficients are bounded from below in absolute value if $\xi$ is a rational number, we assume $\xi$ to be irrational. Recall that the irrationality exponent of $\xi$, denoted by $\mu(\xi)$, is the supremum (possibly $+\infty$) of the set of $\mu > 0$ such that there exist infinitely many $p, q \in \mathbb{Z}$ with $q > 0$ such that $|\xi - \frac{p}{q}| \le q^{-\mu}$. Then the first question related to Theorem 1.3 is to know for which $\tau > 0$ the following holds:

$$\begin{array}{c}\text{For any } Q \text{ there exists } \mathbf{q} = (q_1, q_2) \in \mathbb{Z}^2 \setminus \{(0,0)\} \text{ such that} \\ |\mathbf{q}| \le Q \text{ and } |q_1 \xi - q_2| = Q^{-\tau + o(1)}.\end{array} \tag{5}$$

Lemma 1 and Theorem 2 of [13] imply (using Remark 1 above) that (5) holds if, and only if, $\tau < \frac{1}{\mu(\xi)-1}$ (except maybe for $\tau = \frac{1}{\mu(\xi)-1}$: this case is not settled in [13]). This result can be thought of as a transference principle, in the style of Jarník's [20] (see also [4]), but different from it because $|q_1\xi - q_2|$ is required to be equal to $Q^{-\tau+o(1)}$, but not less than that. It would be interesting to generalize this property to arbitrary values of $m$ and $n$: questions in this respect are asked (in the case $n = 1$) in §4 of [13]. This result shows also that the conclusion of Theorem 1.3 does not hold for *any* $\mathbf{e}_1, \ldots, \mathbf{e}_n$: property (5) fails to hold for $\tau = 1$ if $\mu(\xi) > 2$.

If $\xi$ is generic (with respect to Lebesgue measure), then $\mu(\xi) = 2$ and the question left open in [13] is whether property (5) holds for $\tau = 1$. Theorem 1.3 answers this question: it does, and the error term $Q^{o(1)}$ can be bounded between powers of $\log Q$. Moreover, Theorem 1.3 provides, for any $Q$, two linearly independent vectors $\mathbf{q}$ as in (5): as far as we know, no result in the style of [13] provides this conclusion for a non-generic $\xi$.

In the same situation (namely with $m = 2$, $n = 1$, $F = \mathbb{R}^2$, and a generic $\xi$), Theorem 1.3 with $\tau_1 = 1$ and $\beta_1 > 1$ provides (for any $Q$) two linearly independent vectors $\mathbf{q} = (q_1, q_2) \in \mathbb{Z}^2$ such that $|\mathbf{q}| \ll Q$ and

$$Q^{-1}(\log Q)^{-\beta_1} \ll |q_1\xi - q_2| \ll Q^{-1}(\log Q)^{\beta_1}. \tag{6}$$

The lower bound on $|q_1\xi - q_2|$ is natural since for infinitely many $Q$ there exists $\mathbf{q}$ such that $|\mathbf{q}| \le Q$ and $Q^{-1}(\log Q)^{-\beta_1} \ll |q_1\xi - q_2| \ll Q^{-1}(\log Q)^{-1}$. The upper bound in Eq. (6) could seem too large, since Dirichlet's pigeonhole principle yields (for any $Q$) a non-zero $\mathbf{q}$ such that $|\mathbf{q}| \le Q$ and $|q_1\xi - q_2| \ll Q^{-1}$. However, it is possible (by adapting the proof of Theorem 1.3) to prove that, for infinitely many $Q$, all vectors $\mathbf{q} \in \mathbb{Z}^2$ such that $|\mathbf{q}| \ll Q$ and $|q_1\xi - q_2| \ll Q^{-1}$ are collinear. To obtain two linearly independent such vectors, one needs (for infinitely many $Q$) to let $|q_1\xi - q_2|$ increase a little more, at least up to $Q^{-1}\log Q$: the upper bound in Eq. (6) is optimal (except that the case $\beta_1 = 1$ could probably be considered, upon multiplying by a power of $\log\log Q$).

## 2.2 Proof of Theorem 1.3

Before proving Theorem 1.3, let us outline the strategy in the case where $F = \mathbb{R}^m$ and $\tau_1 + \cdots + \tau_n = \dim F - n$ (from which we shall deduce the general case). We fix a small positive real number $\alpha$, and let $Q$ be sufficiently large. The convex body $\mathcal{C} \subset \mathbb{R}^m$ defined by

$$|\mathbf{q}| \le Q(\log Q)^{\alpha} \text{ and } |\mathbf{q}\cdot\mathbf{e}_i| \le Q^{-\tau_i}(\log Q)^{\beta_i+\alpha} \quad \text{for any } i \in \{1,\ldots,n\}$$

has Lebesgue measure essentially equal to a power of $\log Q$. There are non-zero integer points $\mathbf{q}$ inside $\mathcal{C}$, but not "too far away inside": if $\mathbf{q}$ is such a point and $\mu > 0$ is such that $\mu\mathbf{q} \in \mathcal{C}$, then $\mu$ is less than some power of $\log Q$ (otherwise the scalar products $|\mathbf{q}\cdot\mathbf{e}_i|$ would be too small, in contradiction with the convergent case of Theorem 1.1). This gives a lower bound on the first successive minimum $\lambda_1$ of $\mathcal{C}$. Using Minkowski's convex body theorem, we deduce an upper bound on the last successive minimum $\lambda_m$. This concludes the proof, except for the lower bound in Eq. (4) for which the argument is similar: if $|\mathbf{q}^{(j)}\cdot\mathbf{e}_i|$ is too small for some $i, j$, then $(\mathbf{e}_1,\ldots,\mathbf{e}_n)$ is not generic (using again the convergent case of Theorem 1.1).

Let us now come to a detailed proof of Theorem 1.3, starting with the following remark:

*Remark 3* The general case of Theorem 1.3 follows from the special case where the inequality in Eq. (2) is an equality, that is $\tau_1+\cdots+\tau_n = \dim F - n$. Indeed in general, we have $\tau_1+\cdots+\tau_n = \eta(\dim F - n)$ with $0 < \eta \le 1$, and applying the special case with $\tau_1/\eta$, …, $\tau_n/\eta$ and $Q^\eta$ yields the desired conclusion.

As a first step, let us assume that Theorem 1.3 holds if $F = \mathbb{R}^m$ and deduce the general case. Since $F$ is defined over $\mathbb{Q}$, there exists a basis $(\mathbf{u}_1,\ldots,\mathbf{u}_d)$ of $F$ consisting of vectors of $\mathbb{Z}^m$ (where $d = \dim F$; notice that Eq. (2) implies $d > n$). Let $\Phi : \mathbb{R}^d \to \mathbb{R}^m$ be the linear map which sends the canonical basis of $\mathbb{R}^d$ to $(\mathbf{u}_1,\ldots,\mathbf{u}_d)$. The special case of Theorem 1.3 applies to $\mathbb{R}^d$ (with the same parameters); it provides a subset $\tilde{A} \subset (\mathbb{R}^d)^n$ of full Lebesgue measure, and for any $(\tilde{\mathbf{e}}_1,\ldots,\tilde{\mathbf{e}}_n) \in \tilde{A}$ and any $Q$ sufficiently large, $d$ linearly independent vectors $\tilde{\mathbf{q}}^{(1)},\ldots,\tilde{\mathbf{q}}^{(d)} \in \mathbb{Z}^d$. Then we let $A \subset F^n$ denote the set of all $n$-tuples $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ given by $\mathbf{e}_1 = \Phi(\tilde{\mathbf{e}}_1)$, ..., $\mathbf{e}_n = \Phi(\tilde{\mathbf{e}}_n)$ with $(\tilde{\mathbf{e}}_1, \ldots, \tilde{\mathbf{e}}_n) \in \tilde{A}$; this subset $A$ has full Lebesgue measure in $F^n = (\mathrm{Im}\,\Phi)^n$.

Let us denote by $\Omega \in M_d(\mathbb{R})$ the matrix in the basis $(\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of the scalar product of $\mathbb{R}^m$ restricted to $F$. This means that, for any $\mathbf{x}, \mathbf{y} \in \mathbb{R}^d$ we have $\Phi(\mathbf{x}) \cdot \Phi(\mathbf{y}) = {}^t\mathbf{x}\Omega\mathbf{y}$, where $\mathbf{x}$ and $\mathbf{y}$ are seen as column vectors (indeed, they are the vectors of coordinates in the basis $(\mathbf{u}_1, \ldots, \mathbf{u}_d)$ of $\Phi(\mathbf{x})$ and $\Phi(\mathbf{y})$, respectively). This matrix $\Omega$ has integer coefficients (given by $\mathbf{u}_k \cdot \mathbf{u}_\ell$ for $1 \leq k, \ell \leq d$) and a non-zero determinant, so that $(\det \Omega)\Omega^{-1}$ is a matrix with integer coefficients.

Let $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A$, and $Q$ be sufficiently large. We let

$$\mathbf{q}^{(j)} = \Phi\Big((\det \Omega)\Omega^{-1}\tilde{\mathbf{q}}^{(j)}\Big) \quad \text{for any } j \in \{1, \ldots, d\},$$

so that

$$\mathbf{q}^{(j)} \cdot \mathbf{e}_i = {}^t\Big((\det \Omega)\Omega^{-1}\tilde{\mathbf{q}}^{(j)}\Big)\Omega\tilde{\mathbf{e}}_i = (\det \Omega)\tilde{\mathbf{q}}^{(j)} \cdot \tilde{\mathbf{e}}_i \quad \text{for any } i \in \{1, \ldots, n\}$$

because $\Omega$ is symmetric. Therefore, Eqs. (3) and (4) hold for $j \leq d$; moreover $\mathbf{q}^{(1)}$, ..., $\mathbf{q}^{(d)}$ are linearly independent vectors in $\mathbb{Z}\mathbf{u}_1 + \cdots + \mathbb{Z}\mathbf{u}_d \subset F \cap \mathbb{Z}^m$ (because the coefficients of $(\det \Omega)\Omega^{-1}$ are integers).

Since $F^\perp$ is a subspace of $\mathbb{R}^m$ defined over the rationals (because $F$ is), there exists a basis $(\mathbf{v}_{d+1}, \ldots, \mathbf{v}_m)$ of $F^\perp$ consisting of vectors of $\mathbb{Z}^m$. Then we let

$$\mathbf{q}^{(d+1)} = \mathbf{v}_{d+1} + \mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} = \mathbf{v}_m + \mathbf{q}^{(1)}.$$

Then $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)}$ are linearly independent vectors in $\mathbb{Z}^m$, and for any $j \in \{d+1, \ldots, m\}$ and any $i \in \{1, \ldots, n\}$, we have $\mathbf{q}^{(j)} \cdot \mathbf{e}_i = \mathbf{q}^{(1)} \cdot \mathbf{e}_i$ so that Eq. (4) holds. Since $\mathbf{v}_{d+1}, \ldots, \mathbf{v}_m$ can be chosen independently of $Q$, we have also $|\mathbf{q}^{(j)}| \ll |\mathbf{q}^{(1)}| \ll Q$ so that Eq. (3) holds too. This concludes the proof that the full generality of Theorem 1.3 follows from the special case where $F = \mathbb{R}^m$.

From now on, we assume that $F = \mathbb{R}^m$ and prove Theorem 1.3 in this case. We fix a real number $\alpha$ such that $0 < \alpha < \varepsilon/(n-1)$.

Let $A_0$ denote the set of all $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in (\mathbb{R}^m)^n$ such that the system of inequalities

$$|\mathbf{q} \cdot \mathbf{e}_i| \leq |\mathbf{q}|^{-\tau_i}(\log|\mathbf{q}|)^{\beta_i - (1+\varepsilon)(1+\tau_i)} \quad \text{for any} \quad i \in \{1, \ldots, n\} \tag{7}$$

holds for only finitely many $\mathbf{q} \in \mathbb{Z}^m$; here and below, $\log|\mathbf{q}|$ should be understood as 1 if $|\mathbf{q}| \leq 1$, which is a completely harmless convention.

For any $i_0 \in \{1, \ldots, n\}$, let $A_{i_0}$ denote the set of all $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in (\mathbb{R}^m)^n$ such that the system of inequalities

$$\begin{cases} |\mathbf{q} \cdot \mathbf{e}_i| \leq |\mathbf{q}|^{-\tau_i}(\log|\mathbf{q}|)^{\beta_i + \alpha} & \text{for any} \quad i \in \{1, \ldots, n\}, i \neq i_0, \\ |\mathbf{q} \cdot \mathbf{e}_{i_0}| \leq |\mathbf{q}|^{-\tau_{i_0}}(\log|\mathbf{q}|)^{\beta_{i_0} - m(1+\varepsilon)} \end{cases} \tag{8}$$

holds for only finitely many $\mathbf{q} \in \mathbb{Z}^m$.

Using Eq. (2), Remark 3, and the assumption $(n-1)\alpha < \varepsilon$, the convergent case of Theorem 1.1 (with $(x_{1i}, \dots, x_{mi}) = \mathbf{e}_i$) implies that $A_i \cap \mathbb{I}^{nm}$ has full Lebesgue measure for any $i \in \{0, \dots, n\}$. Moreover $kA_i \subset A_i$ for any $k \geq 1$, so that $A_i = \cup_{k\in\mathbb{N}} k(A_i \cap \mathbb{I}^{nm})$ has full Lebesgue measure. At last, let $A_\infty$ denote the set of all $(\mathbf{e}_1, \dots, \mathbf{e}_n) \in (\mathbb{R}^m)^n$ such that $\mathbf{q} \cdot \mathbf{e}_i \neq 0$ for any $\mathbf{q} \in \mathbb{Z}^m \backslash \{0\}$ and any $i \in \{1, \dots, n\}$. Then we let $A = A_0 \cap A_1 \cap \dots \cap A_n \cap A_\infty$, and $A$ has full Lebesgue measure in $(\mathbb{R}^m)^n$.

Let $(\mathbf{e}_1, \dots, \mathbf{e}_n) \in A$, and $Q$ be sufficiently large. Let $\mathcal{C}$ denote the set of all $\mathbf{q} \in \mathbb{R}^m$ such that

$$|\mathbf{q}| \leq Q(\log Q)^\alpha \quad \text{and} \quad |\mathbf{q} \cdot \mathbf{e}_i| \leq Q^{-\tau_i}(\log Q)^{\beta_i+\alpha} \quad \text{for any } i \in \{1, \dots, n\}. \tag{9}$$

Then $\mathcal{C}$ is convex, compact, and symmetric with respect to the origin. Its Lebesgue measure satisfies $\mathcal{L}_m(\mathcal{C}) \asymp (\log Q)^{(1+\varepsilon)(m-1)+m\alpha}$, using both equalities of Eq. (2) (thanks to Remark 3) with $\dim F = m$.

For any $j \in \{1, \dots, m\}$, let $\lambda_j$ denote the infimum of the set of all positive real numbers $\lambda$ such that $\mathbb{Z}^m \cap \lambda\mathcal{C}$ contains $j$ linearly independent vectors, where $\lambda\mathcal{C} = \{\lambda\mathbf{q},\ \mathbf{q} \in \mathcal{C}\}$. These $\lambda_j$ are the successive minima of the convex body $\mathcal{C}$ with respect to the lattice $\mathbb{Z}^m$; Minkowski's theorem (see for instance [6], Chapter VIII) yields $\frac{2^m}{m!} \leq \lambda_1 \dots \lambda_m \mathcal{L}_m(\mathcal{C}) \leq 2^m$, so that

$$\lambda_1 \dots \lambda_m \asymp (\log Q)^{-(1+\varepsilon)(m-1)-m\alpha}. \tag{10}$$

Since $(\mathbf{e}_1, \dots, \mathbf{e}_n) \in A_0$, for any $\mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$ there exists $i \in \{1, \dots, n\}$ (which depends on $\mathbf{e}_1, \dots, \mathbf{e}_n$ and $\mathbf{q}$) such that

$$|\mathbf{q} \cdot \mathbf{e}_i| \gg |\mathbf{q}|^{-\tau_i}(\log |\mathbf{q}|)^{\beta_i-(1+\varepsilon)(1+\tau_i)}, \tag{11}$$

where the constant implied in the symbol $\gg$ is small enough to take into account the finitely many $\mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$ that satisfy Eq. (7); we have used here that $\mathbf{q} \cdot \mathbf{e}_i \neq 0$ for any $\mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$ and any $i$, because $(\mathbf{e}_1, \dots, \mathbf{e}_n) \in A_\infty$.

Let us deduce from this property that $\lambda_1 \gg (\log Q)^{-(1+\varepsilon+\alpha)}$. With this aim in view, we let $\lambda > 0$ be such that $Q^{-1/2} \leq \lambda \leq 1$ and $\lambda\mathcal{C} \cap \mathbb{Z}^m \neq \{\mathbf{0}\}$; we are going to prove that $\lambda \gg (\log Q)^{-(1+\varepsilon+\alpha)}$. There exists $\mathbf{q}' \in \mathcal{C}$ such that $\mathbf{q} = \lambda\mathbf{q}' \in \mathbb{Z}^m$ and $\mathbf{q} \neq \mathbf{0}$. Then Eq. (11) provides an integer $i \in \{1, \dots, n\}$ such that, using Eq. (9),

$$|\mathbf{q}|^{-\tau_i}(\log |\mathbf{q}|)^{\beta_i-(1+\varepsilon)(1+\tau_i)} \ll |\mathbf{q} \cdot \mathbf{e}_i| = \lambda|\mathbf{q}' \cdot \mathbf{e}_i| \leq \lambda Q^{-\tau_i}(\log Q)^{\beta_i+\alpha}.$$

Since we have also $|\mathbf{q}| = \lambda|\mathbf{q}'| \leq \lambda Q(\log Q)^\alpha$ and $Q^{-1/2} \leq \lambda \leq 1$ (so that $\log(\lambda Q) \asymp \log Q$), this yields

$$\begin{aligned}\lambda^{-\tau_i} Q^{-\tau_i}(\log Q)^{\beta_i-(1+\varepsilon)(1+\tau_i)-\alpha\tau_i} &\asymp (\lambda Q)^{-\tau_i}(\log(\lambda Q))^{\beta_i-(1+\varepsilon)(1+\tau_i)-\alpha\tau_i} \\ &\ll \lambda Q^{-\tau_i}(\log Q)^{\beta_i+\alpha},\end{aligned}$$

thereby proving that $\lambda \gg (\log Q)^{-(1+\varepsilon+\alpha)}$. This concludes the proof that $\lambda_1 \gg (\log Q)^{-(1+\varepsilon+\alpha)}$; since $\lambda_1 \leq \dots \leq \lambda_m$ by definition of the successive minima, this implies $\lambda_j \gg (\log Q)^{-(1+\varepsilon+\alpha)}$ for any $j \in \{1, \ldots, m\}$. Plugging this lower bound for $j \le m-1$ into Eq. (10) yields $\lambda_m \ll (\log Q)^{-\alpha}$: there exist linearly independent vectors $\mathbf{q}^{(1)}, \ldots, \mathbf{q}^{(m)} \in \mathbb{Z}^m$ such that

$$|\mathbf{q}^{(j)}| \ll Q \quad \text{and} \quad |\mathbf{q}^{(j)} \cdot \mathbf{e}_i| \ll Q^{-\tau_i} (\log Q)^{\beta_i} \quad \text{for any} \quad i \in \{1, \ldots, n\}. \tag{12}$$

Therefore, these vectors satisfy Eq. (3) and the upper bound in Eq. (4). To prove the lower bound in Eq. (4), we start by noticing that Eq. (12) yields

$$|\mathbf{q}^{(j)} \cdot \mathbf{e}_i| \le |\mathbf{q}^{(j)}|^{-\tau_i} (\log |\mathbf{q}^{(j)}|)^{\beta_i+\alpha} \text{ for any } i \in \{1, \ldots, n\} \text{ and any } j \in \{1, \ldots, m\} \tag{13}$$

provided $Q$ is large enough. Now let $i_0 \in \{1, \ldots, n\}$. Since $(\mathbf{e}_1, \ldots, \mathbf{e}_n) \in A_{i_0}$ and $\mathbf{q} \cdot \mathbf{e}_{i_0} \neq 0$ for any $\mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$, the second inequality in (12) shows that no $\mathbf{q}^{(j)}$ belongs to the finite subset of $\mathbb{Z}^m$ defined by (8), provided $Q$ is sufficiently large. Therefore, Eq. (13) yields

$$|\mathbf{q}^{(j)} \cdot \mathbf{e}_{i_0}| > |\mathbf{q}^{(j)}|^{-\tau_{i_0}} (\log |\mathbf{q}^{(j)}|)^{\beta_{i_0}-m(1+\varepsilon)} \gg Q^{-\tau_{i_0}} (\log Q)^{\beta_{i_0}-m(1+\varepsilon)}$$

since $|\mathbf{q}^{(j)}| \ll Q$. This concludes the proof of Theorem 1.3.

## 3 Proof of Theorem 1.1

### 3.1 Convergence case for any choice of $m$ and $n$

In order to prove the convergence case, we will exhibit a family of covers of $W_0(m, n, \underline{\psi})$. The covers will be the natural ones, *i.e.* the cover of $W_0(m, n, \underline{\psi})$ by the sets of solutions to (1) for each individual non-zero $\mathbf{q}$. To demonstrate this, consider the $(m-1)n$-dimensional plane defined for $\mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$ by

$$H(\mathbf{q}) := \left\{X = (\mathbf{x}^{(1)}, \cdots, \mathbf{x}^{(n)}) \in \mathbb{I}^{mn} : \mathbf{q}X = \mathbf{0}\right\} = \prod_{i=1}^{n} \left(\mathbb{I}^{mn} \cap R_{\mathbf{q}}^{(i)}\right),$$

where each $R_{\mathbf{q}}^{(i)}$ is the $(m-1)$-dimensional hyperplane $\{\mathbf{x}^{(i)} \in \mathbb{R}^m : \mathbf{q} \cdot \mathbf{x}^{(i)} = \mathbf{0}\}$. Given the approximating functions $\underline{\psi}$ and the sets $H(\mathbf{q})$, define the $\psi_i(|\mathbf{q}|)/|\mathbf{q}|_2$-neighbourhood of $R_{\mathbf{q}}^{(i)}$ in $\mathbb{I}^{mn}$ as

$$\Delta_{\mathbf{q},i} = \left\{\mathbf{x}^{(i)} \in \mathbb{I}^m : \text{dist}\left(\mathbf{x}^{(i)}, R_{\mathbf{q}}^{(i)}\right) < \frac{\psi_i(|\mathbf{q}|)}{|\mathbf{q}|_2}\right\},$$

where $\text{dist}\left(\mathbf{x}^{(i)}, R_{\mathbf{q}}^{(i)}\right) := \inf\{|\mathbf{x}^{(i)} - \mathbf{y}|_2 : \mathbf{y} \in R_{\mathbf{q}}^{(i)}\}$ and $|\mathbf{q}|_2 = \sqrt{\mathbf{q} \cdot \mathbf{q}}$ is the $L^2$-norm. Define $\Delta_{\mathbf{q}} = \prod_{i=1}^{n} \Delta_{\mathbf{q},i}$, and then it is straightforward to verify that $X \in W_0(m, n, \underline{\psi})$ if and only if $X \in \Delta_{\mathbf{q}}$ for infinitely many $\mathbf{q} \in \mathbb{Z}^m \backslash \{\mathbf{0}\}$. Clearly

$$\mathcal{L}_{mn}(\Delta_{\mathbf{q}}) \ll 2^n \psi_1(|\mathbf{q}|) \cdots \psi_n(|\mathbf{q}|)\, |\mathbf{q}|_2^{-n} \asymp \psi_1(|\mathbf{q}|) \cdots \psi_n(|\mathbf{q}|)\, |\mathbf{q}|^{-n}, \tag{14}$$

where the implied constants depend on $m$ and $n$, as $m^{-n/2}|\mathbf{q}|^{-n} \le |\mathbf{q}|_2^{-n} \le |\mathbf{q}|^{-n}$. Next, we will need to estimate the number of $\mathbf{q} \in \mathbb{Z}^m \setminus \{\mathbf{0}\}$ of a given norm, $r$ say. This is however easily seen to be at most $2m(2r+1)^{m-1}$, and so comparable with $r^{m-1}$.

We now estimate the Lebesgue measure of $W_0(m,n,\underline{\psi})$ under the assumption of convergence. For each $N \ge 1$,

$$\begin{aligned}\mathcal{L}_{mn}\left(W_0(m,n,\underline{\psi})\right) &\le \mathcal{L}_{mn}\Big(\bigcup_{r\ge N}\bigcup_{|\mathbf{q}|=r}\Delta_{\mathbf{q}}\Big) \le \sum_{r\ge N}\sum_{|\mathbf{q}|=r}\mathcal{L}_{mn}\left(\Delta_{\mathbf{q}}\right)\\ &\ll \sum_{r\ge N}\sum_{|\mathbf{q}|=r}\psi_1(r)\cdots\psi_n(r)r^{-n}\\ &\ll \sum_{r\ge N}\psi_1(r)\cdots\psi_n(r)r^{m-n-1}.\end{aligned}$$

We have used (14) and the counting estimates. The final sum is the tail of a convergent series, which tends to zero as $N$ tends to infinity.

### 3.2 Divergence case

We give a general approach to the problem in question which has been adapted from the one used in [17]. In the case $(m,n) \ne (2,1)$, we will not need the assumption of monotonicity of the approximating functions. However, under the divergence assumption, we assume without loss of generality that $\psi_i(r) < 1/2$ for all $r \in \mathbb{N}$ and all $i = 1,\cdots,n$. Specifically, we need this assumption when the approximating functions are non-monotonic, see [2, §2]. This will be clear from the proof below.

Initially, we will remove a null-set from the set of matrices $M_{m\times n}(\mathbb{I})$, namely the set consisting of the matrices for which the $n\times n$-matrix $X$ formed by the first $n$ rows is singular. Evidently, the condition that the determinant of the matrix $X$ is zero defines a hyper-surface in the space $M_{m\times n}(\mathbb{I})$, which is of measure zero. Denote the subset of $M_{m\times n}(\mathbb{I})$ with this hyper-surface removed by $M'_{m\times n}(\mathbb{I})$ and the corresponding set of $\underline{\psi}$-approximable linear forms inside the set $M'_{m\times n}(\mathbb{I})$ by $\hat{W}_0(m,n,\underline{\psi})$. Evidently,

$$\hat{W}_0(m,n,\underline{\psi}) \subseteq W_0(m,n,\underline{\psi}) \tag{15}$$

and clearly, under the divergence sum condition $\sum_{r=1}^{\infty}\psi_1(r)\cdots\psi_n(r)r^{m-n-1} = \infty$,

$$\mathcal{L}_{mn}\left(W_0(m,n,\underline{\psi})\right) = 1 \quad \text{if} \quad \mathcal{L}_{mn}\left(\hat{W}_0(m,n,\underline{\psi})\right) = 1.$$

For each $\mathbf{q} \in \mathbb{Z}^{m-n}$, let

$$B_{\mathbf{q}} = \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n\\ |\mathbf{p}|\le|\mathbf{q}|}} \left\{A \in M'_{m\times n}(\mathbb{I}) : |(\mathbf{p},\mathbf{q})A|_i < \psi_i(|\mathbf{q}|) \text{ for any } i = 1,\ldots,n\right\}.$$

Writing each $A \in M'_{m\times n}(\mathbb{I})$ as $\binom{I_n}{\tilde{A}}X$, where $X$ is the invertible $n\times n$ matrix formed by the first $n$ rows of $A$ and $I_n$ is the $n\times n$ identity matrix, we find the related set

$$B'_{\mathbf{q}}(X) = \bigcup_{\substack{\mathbf{p}\in\mathbb{Z}^n \\ |\mathbf{p}|\le|\mathbf{q}|}} \left\{\tilde{A} \in M_{(m-n)\times n}(\mathbb{I}) : \left|\mathbf{p}X + \mathbf{q}\tilde{A}X\right|_i < \psi_i(|\mathbf{q}|) \text{ for any } i = 1, \ldots, n\right\}.$$

Finally, set $B'_{\mathbf{q}} = B'_{\mathbf{q}}(I_n)$.

Let $\varepsilon > 0$ be fixed and sufficiently small. We will be more explicit later. From now on, we restrict ourselves to considering matrices $A \in M'_{m\times n}(\mathbb{I})$ for which the determinant of the matrix $X$ consisting of the first $n$ rows of $A$ has absolute value $\ge \varepsilon$. Evidently, this determinant is also $\le n!$. This immediately implies that $X$ is invertible with $(n!)^{-1} \le \left|\det(X^{-1})\right| \le \varepsilon^{-1}$.

**Lemma 3.1** *For each $X \in M_n(\mathbb{I})$ with $|\det(X)| \ge \varepsilon$, and each $\mathbf{q}, \mathbf{q}_1, \mathbf{q}_2 \in \mathbb{Z}^{m-n}$,*

$$\mathcal{L}_{(m-n)n}(B'_{\mathbf{q}}(X)) \asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_{\mathbf{q}}) \tag{16}$$

*and*

$$\mathcal{L}_{(m-n)n}(B'_{\mathbf{q}_1}(X) \cap B'_{\mathbf{q}_2}(X)) \asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_{\mathbf{q}_1} \cap B'_{\mathbf{q}_2}); \tag{17}$$

*here the implied constants may depend on $m$, $n$, $\varepsilon$, $\psi_1$, ..., $\psi_n$, but not on $\mathbf{q}$, $\mathbf{q}_1$, $\mathbf{q}_2$.*

*Proof* Let us prove Eq. (17), assuming that $\mathbf{q}_1$ and $\mathbf{q}_2$ are not collinear (otherwise it reduces to Eq. (16), which can be proved along the same lines). Let $\mathbf{p}_1, \mathbf{p}_2 \in \mathbb{Z}^n$ be such that $|\mathbf{p}_1| \le |\mathbf{q}_1|$ and $|\mathbf{p}_2| \le |\mathbf{q}_2|$. The affine map

$$f : M_{(m-n)\times n}(\mathbb{R}) \to \mathbb{R}^n \times \mathbb{R}^n, \qquad \tilde{A} \mapsto (\mathbf{p}_1 + \mathbf{q}_1\tilde{A}, \mathbf{p}_2 + \mathbf{q}_2\tilde{A})$$

is surjective because $\mathbf{q}_1$ and $\mathbf{q}_2$ are not collinear. Let $\mathcal{R}_{\mathbf{q}}$ denote the set of all $\mathbf{x} = (x_1, \ldots, x_n) \in \mathbb{R}^n$ such that $|x_i| < \psi_i(|\mathbf{q}|)$ for any $i = 1, \ldots, n$. Then for any invertible $X \in M_n(\mathbb{I})$, the Lebesgue measure of $M_{(m-n)\times n}(\mathbb{I}) \cap f^{-1}(\mathcal{R}_{\mathbf{q}_1}X^{-1} \times \mathcal{R}_{\mathbf{q}_2}X^{-1})$ is proportional (up to a positive constant depending only on $m$ and $n$) to $\mathcal{L}_{2n}((\mathcal{R}_{\mathbf{q}_1}X^{-1} \cap |\mathbf{q}_1|\mathbb{I}^n) \times (\mathcal{R}_{\mathbf{q}_2}X^{-1} \cap |\mathbf{q}_2|\mathbb{I}^n))$; the coefficient of proportionality depends on $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{p}_1$, $\mathbf{p}_2$, $m$, $n$, but not on $X$. Assuming $|\det(X)| \ge \varepsilon$, the latter measure is $\asymp_\varepsilon \mathcal{L}_{2n}(\mathcal{R}_{\mathbf{q}_1} \times \mathcal{R}_{\mathbf{q}_2})$. Denoting by $B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2}(X)$ the set of all $\tilde{A} \in M_{(m-n)\times n}(\mathbb{I})$ such that $|\mathbf{p}_1X + \mathbf{q}_1\tilde{A}X|_i < \psi_i(|\mathbf{q}_1|)$ and $|\mathbf{p}_2X + \mathbf{q}_2\tilde{A}X|_i < \psi_i(|\mathbf{q}_2|)$ for any $i$, we deduce that

$$\mathcal{L}_{(m-n)n}(B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2}(X)) \asymp_\varepsilon \mathcal{L}_{(m-n)n}(B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2}(I_n)), \tag{18}$$

where the implied constant may depend on $m$, $n$, $\varepsilon$, but not on $\mathbf{q}_1$, $\mathbf{q}_2$, $\mathbf{p}_1$, $\mathbf{p}_2$ since the above-mentioned coefficient of proportionality cancels out.

To conclude we have to deal with the possible existence of $\mathbf{p}$, $\mathbf{p}'$, $\mathbf{q} \in \mathbb{Z}^n$ such that $\mathbf{p} \ne \mathbf{p}'$ and $\frac{1}{2}(\mathbf{p}-\mathbf{p}')X \in \mathcal{R}_{\mathbf{q}}$. This implies that some non-trivial $\mathbb{Z}$-linear combination of the rows of $X$ is very close to $\mathbf{0}$, so that

$$|\det(X)| \le 2(n-1)! \sum_{i=1}^{n} \psi_i(|\mathbf{q}|). \tag{19}$$

Now let $\varepsilon > 0$ be fixed, and assume that $|\det(X)| \geq \varepsilon$. Equation (19) provides a real number $C_\varepsilon > 0$ such that elements $\mathbf{p}, \mathbf{p}', \mathbf{q} \in \mathbb{Z}^n$ with $\mathbf{p} \neq \mathbf{p}'$ and $\frac{1}{2}(\mathbf{p} - \mathbf{p}')X \in \mathcal{R}_\mathbf{q}$ may exist only if $|\mathbf{q}| < C_\varepsilon$. If both $\mathbf{q}_1$ and $\mathbf{q}_2$ are less than $C_\varepsilon$, then Eq. (17) holds trivially: $\mathbf{q}_1$ and $\mathbf{q}_2$ may take only a finite number of values (depending on $\varepsilon$), and the left-hand side of Eq. (17) is a positive and continuous function of the variable $X$ ranging through the compact subset of $M_n(\mathbb{I})$ defined by $|\det(X)| \geq \varepsilon$.

Let us consider the case where $|\mathbf{q}_1| \geq C_\varepsilon > |\mathbf{q}_2|$ (the other cases are similar). Then all implied constants may depend on $\mathbf{q}_2$ and not only on $\varepsilon$. Moreover, by construction of $C_\varepsilon$, if $\mathbf{p}_1' \neq \mathbf{p}_1$ then $B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2}(X) \cap B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1',\mathbf{p}_2'}(X) = \emptyset$ so that

$$\mathcal{L}_{(m-n)n}(B'_{\mathbf{q}_1}(X) \cap B'_{\mathbf{q}_2}(X)) = \mathcal{L}_{(m-n)n}\left( \bigcup_{\substack{\mathbf{p}_1,\mathbf{p}_2 \in \mathbb{Z}^n \\ |\mathbf{p}_1| \leq |\mathbf{q}_1|,\ |\mathbf{p}_2| \leq |\mathbf{q}_2|}} B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2}(X) \right)$$
$$\asymp_{\varepsilon,\mathbf{q}_2} \max_{\substack{\mathbf{p}_2 \in \mathbb{Z}^n \\ |\mathbf{p}_2| \leq |\mathbf{q}_2|}} \sum_{\substack{\mathbf{p}_1 \in \mathbb{Z}^n \\ |\mathbf{p}_1| \leq |\mathbf{q}_1|}} \mathcal{L}_{(m-n)n}(B''_{\mathbf{q}_1,\mathbf{q}_2,\mathbf{p}_1,\mathbf{p}_2}(X)).$$

Using Eq. (18) this concludes the proof of Lemma 3.1. □

**Lemma 3.2** *For each pair* $\mathbf{q}, \mathbf{q}'$,

$$\mathcal{L}_{mn}(B_\mathbf{q}) \asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_\mathbf{q}) \tag{20}$$

*and*

$$\mathcal{L}_{mn}(B_\mathbf{q} \cap B_{\mathbf{q}'}) \asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_\mathbf{q} \cap B'_{\mathbf{q}'}), \tag{21}$$

*where the implied constants may depend on $m$, $n$, $\varepsilon$, $\psi_1$, …, $\psi_n$, but not on* $\mathbf{q}, \mathbf{q}'$.

*Proof* This follows on integrating out the $X$ and applying Lemma 3.1. Indeed,

$$\mathcal{L}_{mn}(B_\mathbf{q}) = \lim_{\varepsilon \to 0} \int_{\substack{X \in M_n(\mathbb{I}) \\ \varepsilon \leq |\det(X)|}} \int_{\tilde{A} \in M_{(m-n)\times n}(\mathbb{I})X^{-1}} \mathbf{1}_{B_\mathbf{q}}\left( \begin{pmatrix} I_n \\ \tilde{A} \end{pmatrix} X \right) \mathrm{d}\tilde{A}\,\mathrm{d}X,$$

where $\mathbf{1}_{B_\mathbf{q}}$ denotes the characteristic function of $B_\mathbf{q}$. Let us prove that the inner integral is $\asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_q(X))$.

First, we deal with the case when $m - n > 1$. For simplicity, we consider first the case $m = 3, n = 1$ and extend subsequently. We are integrating over the set $M_{2\times 1}(\mathbb{I})X^{-1}$, which is a square of area between 1 and $\varepsilon^{-2}$, since $X$ in this case is just a number between $\varepsilon$ and 1. Consider the intersection with each fundamental domain for the standard lattice $\mathbb{Z}^2$. Except for lower order terms arising at the boundary of $M_{2\times 1}(\mathbb{I})X^{-1}$, each such intersection will have measure $\mathcal{L}_{(m-n)n}(B'_\mathbf{q}(X))$. The number of such contributing fundamental domains is bounded from below by 1 and from above by $\varepsilon^{-2}$. Hence, the result follows in this case.

To get the full result for $m - n > 1$, the set $M_{(m-n)\times n}(\mathbb{I})X^{-1}$ still covers at least $\frac{1}{n}M_{(m-n)\times n}(\mathbb{I})$, as the entries of $X$ are between $-1/2$ and $1/2$. For $|\mathbf{q}|$ large enough, the measure of the intersection of $B'_{\mathbf{q}}(X)$ with this set is $\asymp \frac{1}{n^{(m-n)n}}\mathcal{L}_{(m-n)n}(B'_{\mathbf{q}}(X))$, and the result follows. The upper bound again follows as the determinant of $X$ is bounded from below.

When $m - n = 1$, the set consists of neighbourhoods of single points, and we simply count the contributions as usual. We have now shown that the inner integral is $\asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_q(X))$.

Equation (20) follows at once, since using Eq. (16) we have

$$\int_{\substack{X\in M_n(\mathbb{I})\\ \varepsilon\le|\det(X)|}} \mathcal{L}_{(m-n)n}(B'_q(X))\mathrm{d}X \asymp_\varepsilon \int_{\substack{X\in M_n(\mathbb{I})\\ \varepsilon\le|\det(X)|}} \mathcal{L}_{(m-n)n}(B'_{\mathbf{q}})\mathrm{d}X \asymp_\varepsilon \mathcal{L}_{(m-n)n}(B'_{\mathbf{q}}).$$

The case of intersections follows similarly, this time using (17) instead of (16). □

At this point, proving the divergence case of the theorem is a relatively straightforward matter. Indeed, a form of the divergence case of the Borel–Cantelli lemma [25, Lemma 5] states that if $(A_n)$ is a sequence of sets in a probability space with probability measure $\mu$ such that $\sum \mu(A_n) = \infty$, then

$$\mu\left(\bigcap_{k=1}^{\infty}\bigcup_{n=k}^{\infty} A_n\right) \ge \limsup_{N\to\infty} \frac{\left(\sum_{n=1}^{N}\mu(A_n)\right)^2}{\sum_{m,n=1}^{N}\mu(A_m\cap A_n)}. \tag{22}$$

If one can prove that for a sufficiently large set of pairs $(A_n, A_m)$, the denominator on the right-hand side is $\ll \mu(A_m)\mu(A_n)$ whenever $m \ne n$, it follows from Eq. (22) that the measure of the set on the left-hand side is strictly positive. Even if this does not hold, one could hope for it to be true on average, so that the resulting right-hand side would be positive. This is a standard technique in metric Diophantine approximation, with the property on the sets $A_n$ being called quasi-independence or in the latter case quasi-independence on average. It follows from Lemma 3.2, that if a classical Khintchine–Groshev theorem can be established using quasi-independence on average, then the measure of the absolute value set is positive under the appropriate divergence assumption.

In the classical setup, one usually proves Khintchine–Groshev type results using a variant of this lemma. Here, one applies the lemma with some subset of the family $B'_{\mathbf{q}}$ in place of $A_n$. In the simplest case, when $n = 1$ and $m = 3$, the family can be chosen to be those $\mathbf{q} = (p, \tilde{\mathbf{q}}) \in \mathbb{Z}\times\mathbb{Z}^2$ with the entries of $\tilde{\mathbf{q}}$ co-prime and the last entry positive. This will ensure that the corresponding sets $\cup_p B'_{(p,\tilde{\mathbf{q}})}$ are stochastically independent and hence quasi-independent. The fact that we take a union over $p$'s is critical. This gives a pleasing description of the sets involved as neighbourhoods of geodesics winding around a torus and provides a simple argument for the stochastic independence of the sets. For details on this case, see [9]. In that paper, the case $m - n > 1$ is fully described. For the case when $m - n = 1$, more delicate arguments are required. Below, we give references to work, where the refining procedure is carried out in each individual case.

For our purposes, in order to prove Theorem 1.1, using Lemma 3.2 we will translate the right-hand side of inequality (22) to a statement on the 'classical' sets $B'_{\mathbf{q}}$ with the corresponding limsup set. In the case $m - n > 2$, the required upper bound on the intersections on average was established in [24] without the monotonicity assumption. For $m - n = 2$, the bound is found in [19] and for $m - n = 1$, this is the result of [14]. In the last case, the monotonicity is critical in the case $m = 2, n = 1$, as otherwise we could exploit the Duffin–Schaeffer counterexample [10] to arrive at a counterexample to the present statement.

Having established that the measure is positive, it remains to prove that the measure is full. To accomplish this, we apply an inflation argument due to Cassels [5], but tweaked to the absolute value setup. We pick a slowly decreasing function $\tau(r)$ which tends to 0, such that the functions $\psi_i'(r) = \tau(r)\psi_i(r)$ satisfy the divergence assumption of the theorem.

One can show that the origin $0 \in \mathrm{Mat}_{mn}(\mathbb{R})$ is a point of metric density for the set $W_0(m, n; \underline{\psi}')$. This uses two properties. One is the fact that 0 is an inner point of each set of matrices satisfying (1) for a fixed $\mathbf{q}$. The other is the fact that the error function depends only on $|\mathbf{q}|$, so the parallelepiped of matrices satisfying Eq. (1) does not change shape but only orientation as $\mathbf{q}$ varies over integer vectors with the same height $|\mathbf{q}|$. Since the distribution of angles of integer vectors of the same height becomes uniform as the height increases, this implies that the origin must be a point of metric density.

We will apply the Lebesgue Density Theorem to prove that $W_0(m, n; \underline{\psi})$ is full. Suppose to the contrary that this is not the case, and let $A_0$ be a point of metric density for the complement of $W_0(m, n; \underline{\psi})$, so that

$$\frac{\mathcal{L}_{mn}\left(W_0(m, n; \underline{\psi}) \cap B(A_0, \delta)\right)}{\mathcal{L}_{mn}\left(B(A_0, \delta)\right)} \to 0, \tag{23}$$

as $\delta \to 0$. We will derive a contradiction to this statement.

Let $r_0 = |A_0| + \delta$. We will later fix a number $r > 0$, depending only on $m$, $n$, $A_0$, and $\delta$. Suppose that $\tilde{A} \in W_0(m, n; \underline{\psi}')$ and let $t > 0$. Then the matrix $A = t\tilde{A}$ satisfies

$$|\mathbf{q}A|_i = t\,|\mathbf{q}\tilde{A}|_i < t\,\psi_i'(|\mathbf{q}|),$$

for infinitely many $\mathbf{q}$. This implies that $A \in W_0(m, n; \underline{\psi})$, since $t$ is fixed and $\tau$ tends to 0. Now let

$$S_{r,\delta,A_0} = \left\{A \in \mathbb{I}^{mn} : |A|_2 < r \text{ and there is a } t > 0 \text{ such that } tA \in B(A_0, \delta)\right\}$$

where $|A|_2$ denotes the Euclidean norm of $A$. Using the scaling properties of the Lebesgue measure, it is not difficult to deduce that

$$\mathcal{L}_{mn}\left(W_0(m, n; \underline{\psi}) \cap B(A_0, \delta)\right) \geq Cr^{-mn}\delta\mathcal{L}_{mn}\left(W_0(m, n; \underline{\psi}') \cap S_{r,\delta,A_0}\right), \tag{24}$$

where $C > 0$ depends only on $m$, $n$, and $A_0$.

We proceed with estimating $r^{-mn}\mathcal{L}_{mn}\left(W_0(m, n; \underline{\psi}') \cap S_{r,\delta,A_0}\right)$. Recall that 0 is a point of metric density for $W_0(m, n; \underline{\psi}')$, so

$$\frac{\mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi}')\cap B(0,r)\right)}{\mathcal{L}_{mn}\left(B(0,r)\right)}\to 1,$$

as $r$ tends to zero. Hence, for $\epsilon > 0$, we may pick $r$ so small that

$$\mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi}')^c\cap B(0,r)\right) < \epsilon\mathcal{L}_{mn}\left(B(0,r)\right)\leq \epsilon C' r^{mn},$$

where $C' > 0$ depends only on $m$ and $n$. On the other hand,

$$\mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi}')^c\cap S_{r,\delta,A_0}\right)\leq \mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi}')^c\cap B(0,r)\right),$$

by simple inclusion, so that

$$\mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi}')\cap S_{r,\delta,A_0}\right)\geq \mathcal{L}_{mn}\left(S_{r,\delta,A_0}\right)-\epsilon C' r^{mn}\geq C'' r^{mn}\delta^{mn-1}-\epsilon C' r^{mn},$$

where again $C'' > 0$ is some constant depending only on $m$, $n$, and $A_0$. Fixing $\epsilon = \frac{C''}{2C'}\delta^{mn-1}$, we find that

$$r^{-mn}\mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi}')\cap S_{r,\delta,A_0}\right)\geq \tfrac{1}{2}C''\delta^{mn-1}.$$

Inserting this estimate into Eq. (24), we see that

$$\frac{\mathcal{L}_{mn}\left(W_0(m,n;\underline{\psi})\cap B(A_0,\delta)\right)}{\mathcal{L}_{mn}\left(B(A_0,\delta)\right)}\geq \frac{\frac{1}{2}CC''\delta^{mn}}{\mathcal{L}_{mn}\left(B(A_0,\delta)\right)} > C_0 > 0,$$

where the final constant once again depends only on $m$, $n$, and $A_0$. This contradicts Eq. (23) and completes the proof.

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