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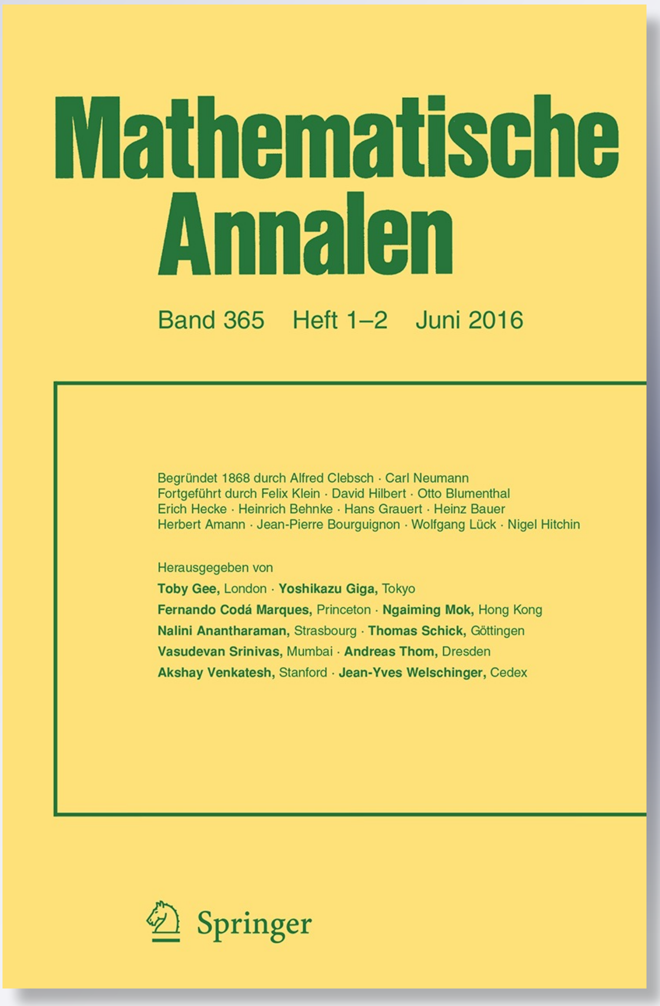

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Connecting interpolation and multiplicity estimates in commutative algebraic groups

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Abstract Let G be a commutative algebraic group embedded in projective space and Γ a finitely generated subgroup of G. From these data we construct a chain of algebraic subgroups of G which is intimately related to obstructions to multiplicity or interpolation estimates used in transcendental number theory and algebraic independence. Let $\gamma_1, \dots, \gamma_l$ denote a family of generators of Γ and, for any $S > 1$, let $\Gamma(S)$ be the set of elements $n_1\gamma_1 + \cdots + n_l\gamma_l$ with integers n_j such that $|n_j| < S$. Then this chain of subgroups controls, for large values of S, the distribution of $\Gamma(S)$ with respect to algebraic subgroups of G. As an application we essentially determine (up to multiplicative constants) the locus of common zeros of all $P \in H^0(\overline{G}, \mathcal{O}(D))$ which vanish to at least some given order at all points of $\Gamma(S)$. When D is very small this result reduces to a multiplicity estimate; when D is very large it is a kind of interpolation estimate.

Mathematics Subject Classification 11J81 (Transcendence (general theory)) · 14L10 (Group varieties) · 11J95 (Results involving abelian varieties) · 14L40 (Other algebraic groups (geometric aspects))

1 Introduction

Let G be a positive dimensional connected commutative algebraic group, embedded in $\mathbb{P}^N$ through the choice of a very ample line bundle on a compactification of G. In most proofs of transcendence or algebraic independence involving G, an important role is played by the evaluation map

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$$H^0(\overline{G}, \mathcal{O}(D)) \to H^0(\overline{G}, \mathcal{O}(D) \otimes \oplus_{\omega \in \Gamma(S)} \mathcal{O}_{\overline{G}}/m_\omega^T); \tag{1}$$

here $\overline{G}$ is the Zariski closure of G in $\mathbb{P}^N$, D is a positive integer so $\mathcal{O}(D)$ is a multiple of the hyperplane bundle, $m_\omega \subset \mathcal{O}_{\overline{G}}$ is the maximal ideal sheaf corresponding to the point ω, and for a positive real number S, $\Gamma(S)$ is the set of all elements $n_1\gamma_1 + \cdots + n_l\gamma_l$ with integers n_j such that $|n_j| < S$. In this setting $\gamma_1, \ldots, \gamma_l$ are fixed elements of G and S is often chosen to be very large. We let Γ denote the $\mathbb{Z}$-module generated by $\gamma_1, \ldots, \gamma_l$. The set $\Gamma(S)$ depends on $\gamma_1, \ldots, \gamma_l \in G$ in addition to Γ and S, making the notation $\Gamma(S)$ rather unpleasant but it is the usual one in this setting. The integers D, S, T are parameters which typically take very large values in transcendence or algebraic independence proofs, except when no multiplicities are involved, that is when $T = 1$.

A crucial step in most transcendence proofs is the *multiplicity estimate*, called a *zero estimate* when $T = 1$. The simplest one in this setting is perhaps the following (see [9] or [17]): if $D < c_1 T S^\mu$ then (1) is injective so that $P = 0$ as soon as $P \in H^0(\overline{G}, \mathcal{O}(D))$ vanishes to order at least T at each point $\omega \in \Gamma(S)$. Here c_1 is a positive constant depending on G, its embedding in $\mathbb{P}^N$, and $\gamma_1, \ldots, \gamma_l$. The real exponent $\mu \geq 0$ is defined by

$$\mu = \mu(\Gamma, G) = \min_{H \subsetneq G} \frac{\operatorname{rk}(\Gamma) - \operatorname{rk}(\Gamma \cap H)}{\dim G - \dim H}$$

where H ranges through the set of all proper connected algebraic subgroups of G.

Instead of a multiplicity estimate and the construction of an auxiliary function, it is possible to use an *interpolation estimate* and an auxiliary functional (see [12, 14–16]). Such a result was proved by Masser [5] when no multiplicities are involved, that is when $T = 1$, and generalized by the first author [3]. It reads as follows: if $D > c_2 T S^{\mu^*}$ then (1) is surjective, where c_2 is a positive constant depending on G, its embedding in $\mathbb{P}^N$, and $\gamma_1, \ldots, \gamma_l$. The real exponent $\mu^* \geq 0$ is defined by

$$\mu^* = \mu^*(\Gamma, G) = \max_{H \neq \{0\}} \frac{\operatorname{rk}(\Gamma \cap H)}{\dim H}$$

where H ranges through the set of all non-zero connected algebraic subgroups of G.

The exponents $\mu(\Gamma, G)$ and $\mu^*(\Gamma, G)$ measure the distribution of Γ (and that of $\Gamma(S)$, if S is sufficiently large) with respect to algebraic subgroups of G. The former appears in early zero estimates [6] and already in [13] (§1.3). It is related to the density coefficient of Γ if $G = \mathbb{G}_a^n$ and $\Gamma \subset (\overline{\mathbb{Q}} \cap \mathbb{R})^n$ (see §1.3.d of [13]), and to Schwarz lemmas (see Chapter 7 of [13] and [11]). The exponent $\mu^*(\Gamma, G)$ is a dual version introduced in [5]. These exponents satisfy the inequalities

$$\mu(\Gamma, G) \leq \frac{\operatorname{rk} \Gamma}{\dim G} \leq \mu^*(\Gamma, G)$$

by definition. A finitely generated $\mathbb{Z}$-module $\Gamma \subset G$ is said to be *well distributed* in G if $\mu(\Gamma, G) = \frac{\operatorname{rk} \Gamma}{\dim G}$ or, equivalently, if $\mu^*(\Gamma, G) = \frac{\operatorname{rk} \Gamma}{\dim G}$ (see [5]).

Elaborating upon ideas of [8], we construct in Sect. 4 a chain of algebraic subgroups $\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_r = G$, with $r \geq 1$, associated to Γ and G. These subgroups satisfy

$$\mu(\Gamma \cap H_j \bmod H_i, H_j/H_i) = \frac{\mathrm{rk}(\Gamma \cap H_j) - \mathrm{rk}(\Gamma \cap H_{j-1})}{\dim H_j - \dim H_{j-1}}$$

and

$$\mu^*(\Gamma \cap H_j \bmod H_i, H_j/H_i) = \frac{\mathrm{rk}\left(\frac{\Gamma\cap H_{i+1}}{\Gamma\cap H_i}\right)}{\dim(H_{i+1}/H_i)}$$

for any i, j such that $0 \leq i < j \leq r$. Here and throughout this text, we let $\Omega \bmod H = \frac{\Omega+H}{H}$ for any subset Ω of G and any algebraic subroup $H \subset G$.

The following properties hold:

- $\mu(\Gamma, G) = \frac{\mathrm{rk}(\Gamma)-\mathrm{rk}(\Gamma\cap H_{r-1})}{\dim G-\dim H_{r-1}}$.
- $\mu^*(\Gamma, G) = \frac{\mathrm{rk}(\Gamma\cap H_1)}{\dim H_1}$.
- Γ is well distributed in G if and only if $r = 1$ so that the chain is simply $\{0\} = H_0 \subsetneq H_1 = G$.
- For any $i \in \{0, \ldots, r-1\}$, $\Gamma \cap H_{i+1} \bmod H_i$ is well distributed in H_{i+1}/H_i.

Moreover, if H is a non-zero connected algebraic subgroup of G such that $\mu^*(\Gamma, G) = \frac{\mathrm{rk}(\Gamma\cap H)}{\dim H}$, then $H \subset H_1$ (see [8], §1.3). In the same way, if H is a proper connected algebraic subgroup of G such that $\mu(\Gamma, G) = \frac{\mathrm{rk}(\Gamma)-\mathrm{rk}(\Gamma\cap H)}{\dim G-\dim H}$, then $H_{r-1} \subset H$.

We think this chain of subgroups can be useful in many problems where the distribution of Γ with respect to algebraic subgroups of G is involved, for instance in studying the points of $\Gamma(S)$ in the spirit of §1.3.d of [13] for the case $G = \mathbb{G}_a^n$ and $\Gamma \subset (\overline{\mathbb{Q}} \cap \mathbb{R})^n$. It may also provide a geometric interpretation closely related to the Seshadri exceptional subvarieties studied in [4]. We use these subgroups here to study the locus $\mathcal{B}_{G,\Gamma(S),T,D}$ of common zeros of all P in the kernel of (1), that is the set of $x \in G$ such that $P(x) = 0$ for any $P \in H^0(\overline{G}, \mathcal{O}(D))$ which vanishes to order at least T at each point of $\Gamma(S)$.

To state our result, we let

$$\mu_i = \mu(\Gamma \cap H_{i+1} \bmod H_i, H_{i+1}/H_i) = \frac{\mathrm{rk}(\Gamma \cap H_{i+1}) - \mathrm{rk}(\Gamma \cap H_i)}{\dim H_{i+1} - \dim H_i}$$

for any $i \in \{0, \ldots, r-1\}$. As stated previously, we have also $\mu_i = \mu^*(\Gamma \cap H_{i+1} \bmod H_i, H_{i+1}/H_i)$ since $\Gamma \cap H_{i+1} \bmod H_i$ is well distributed in H_{i+1}/H_i. Moreover $\mu_0 = \mu^*(\Gamma, G)$ and $\mu_{r-1} = \mu(\Gamma, G)$; we shall prove that

$$\mu_0 > \mu_1 > \cdots > \mu_{r-1}. \tag{2}$$

For convenience we write $\mu_{-1} = +\infty$ and $\mu_r = -\infty$. In loose terms, the series of inequalities (2) can be understood as follows. The algebraic subgroup H_1 contains the largest possible proportion of Γ (with respect to its dimension), so that the proportion of $\Gamma \bmod H_1$ contained in H_2/H_1 has to be smaller with respect to $\dim(H_2/H_1)$; otherwise H_2 would contradict the maximality of H_1. This argument is made precise in Proposition 4.4 below and the associated remarks.

Our main result reads as follows.

Theorem 1.1 *For any $\varepsilon > 0$ with $0 < \varepsilon < 1$ there exists a positive constant c_3, depending only on the embedding of G in $\mathbb{P}^N$, ε, and $\gamma_1, \dots, \gamma_l$, with the following property. For any positive integers D and T, if S is a sufficiently large positive integer (in terms of $G \hookrightarrow \mathbb{P}^N$, ε, $\gamma_1, \dots, \gamma_l$) and*

$$c_3 S^{\mu_i} T \le D \le c_3^{-1} S^{\mu_{i-1}} T$$

for some $i \in \{0, \dots, r\}$, then we have

$$\Gamma((1-\varepsilon)S) + H_i \subset \mathcal{B}_{G,\Gamma(S),T,D} \subset \Gamma(S) + H_i.$$

With $i = r$ this is the above-mentioned multiplicity estimate because $\mathcal{B}_{G,\Gamma(S),T,D} = G$. With $i = 0$ it follows from an interpolation estimate since such an estimate gives sections which separate jets at the points of $\Gamma(S) \cup \{x\}$ for any $x \notin \Gamma(S)$. This result establishes a bridge between multiplicity and interpolation estimates. It is a partial answer to a question asked by Michel Waldschmidt to the first author: what can be said about the evaluation map (1) if D is too large to apply a multiplicity estimate but too small to apply an interpolation estimate? Of course this question remains wide open: for instance no non-trivial lower bound on the rank of this linear map is known for these values of D. However we hope that Theorem 1.1 can be useful to produce new transcendence proofs.

If X is a smooth projective variety, $\eta \in X$ a very general point, and L an ample line bundle on X then the analogue of (1) has been studied closely (see [1,7]):

$$H^0(X, L^{\otimes D}) \longrightarrow H^0(X, L^{\otimes D} \otimes \mathcal{O}_X/m_\eta^T).$$

The main idea is that once T/D excedes the Seshadri constant of L at η, then the map ceases to be surjective. This failure is estimated in [1,7], and it is this extra information which allows a quantitative improvement for the lower bound of the Seshadri constant of L at η. These techniques have been formalized in a broader setting in [2].

Another motivation for Theorem 1.1 is its relation to a conjecture of the second author (see Sect. 2.2). In Conjecture 1.1.9 of [8] a sequence of subgroups analogous to our $(H_i)_{0 \le i \le r}$ is alluded to and it is conjectured that these subgroups appear as the base locus of a linear series as in Theorem 1.1. Because the methods employed in that paper are restricted to working on a compactification of G, with no auxiliary constructions such as projections to quotient groups, it was not possible to bound from above the size of the base loci in question as is done here.

When D lies between $c_3^{-1} S^{\mu_i} T$ and $c_3 S^{\mu_i} T$, for some $i \in \{0, \dots, r-1\}$, Theorem 1.1 applied with these bounds yields

$$\Gamma((1-\varepsilon)S) + H_i \subset \mathcal{B}_{G,\Gamma(S),T,D} \subset \Gamma(S) + H_{i+1}$$

since $\mathcal{B}_{G,\Gamma(S),T,D}$ is a non-increasing function of D when the subset $\Gamma(S)$ and the order of vanishing T are held constant. It would be interesting to have more information on $\mathcal{B}_{G,\Gamma(S),T,D}$ for these critical values of D, but new ideas are needed. Indeed the proof of Theorem 1.1 is based on applying the special cases $i = 0$ and $i = r$ to sub-quotients of G obtained from the chain of subgroups $(H_i)_{0\le i\le r}$. This strategy, remniscent of that used by Masser [5] to prove his interpolation estimate, is responsible for the constant c_3.

In this paper we shall prove Theorem 1.1 in a more general form: for any $S_1,\ldots,S_l \in \mathbb{R}$ we consider the set $\Gamma(\underline{S})$ of all points $n_1\gamma_1 + \cdots + n_l\gamma_l$ with integers n_j such that $|n_j| < S_j$. Here $\underline{S}$ denotes the tuple $(S_1,\ldots,S_l)$, and we let $\lambda\underline{S} = (\lambda S_1,\ldots,\lambda S_l)$ for any $\lambda > 0$. Up to a permutation of $\gamma_1,\ldots,\gamma_l$, we may assume that $S_1 \ge \cdots \ge S_l$. This assumption will be useful to define the subgroups H_i which depend in this case on $S_1,\ldots,S_l$ and $\gamma_1,\ldots,\gamma_l$ (whereas they depend only on Γ and G if $S_1 = \cdots = S_l$). The distribution of $\Gamma(\underline{S})$ with respect to algebraic subgroups of G is no longer measured simply by exponents like μ, μ^* and the μ_i (see for instance §3 of [3]).

For any subset Ω of G, we let $\mathcal{B}_{G,\Omega,T,D}$ denote the set of $x \in G$ such that $P(x) = 0$ for any $P \in \mathcal{R}(G)_D$ which vanishes to order at least T at each point of Ω; here and throughout this text, we let $\mathcal{R}(G)_D = H^0(\overline{G}, \mathcal{O}(D))$ as soon as G is a commutative algebraic group embedded in a projective space, and we call *homogeneous polynomial of degree D* any element of $\mathcal{R}(G)_D$. The base field is $\mathbb{C}$, though any algebraically closed field of characteristic zero could be considered, for instance its p-adic analog $\mathbb{C}_p$; see also [5], §1.

The structure of this text is as follows. We state in Sect. 2 our main result and explain the connection with a conjecture of the second author. We gather in Sect. 3 the main tools in the proof, namely the multiplicity and interpolation estimates we rely on, and also a counting lemma which provides an asymptotic estimate for the cardinality of the image of $\Gamma(\underline{S})$ in sub-quotients of G. Then we construct in Sect. 4 the chain of algebraic subgroups $(H_i)_{0\le i\le r}$ and study its properties. This section might be of independent interest, and is logically independent from the previous ones. Finally in Sect. 5 we prove our main result and gather in Sect. 6 some remarks and comments on possible generalizations.

2 Statement of the results

Throughout this section we let G be a connected commutative algebraic group embedded in projective space $\mathbb{P}^N$. Suppose $\gamma_1,\ldots,\gamma_l \in G$ and let Γ denote the subgroup generated by $\gamma_1,\ldots,\gamma_l$. Let $S_1 \ge \cdots \ge S_l \ge 1$ be real numbers, and recall that $\Gamma(\underline{S})$ is the set of all points $n_1\gamma_1 + \cdots + n_l\gamma_l$ with integers n_j such that $|n_j| < S_j$; here $\underline{S}$ denotes the tuple $(S_1,\ldots,S_l)$.

Using this data we shall construct in Sect. 4 a chain of algebraic subgroups $\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_r = G$, with $r \ge 1$.

We let Γ_j denote the subgroup generated by $\gamma_1,\ldots,\gamma_j$, setting $\Gamma_0 = \{0\}$, and we put

$$\mathfrak{S}_i = \left(\prod_{j=1}^{l} S_j^{\mathrm{rk}\left(\frac{\Gamma_j \cap H_{i+1}}{\Gamma_j \cap H_i}\right) - \mathrm{rk}\left(\frac{\Gamma_{j-1} \cap H_{i+1}}{\Gamma_{j-1} \cap H_i}\right)} \right)^{1/(\dim H_{i+1} - \dim H_i)}$$

for any $i \in \{0, \ldots, r-1\}$. Then we shall prove that

$$1 \leq \mathfrak{S}_{r-1} \leq \mathfrak{S}_{r-2} \leq \cdots \leq \mathfrak{S}_1 \leq \mathfrak{S}_0, \tag{3}$$

as an immediate consequence of Proposition 4.4 and Eq. (17) in Sect. 4.

2.1 The main result

Our main result is twofold. The first one is proved using interpolation estimates, whereas the second one is based on multiplicity estimates.

Theorem 2.1 *There exists a positive constant c_4, depending only on $G \hookrightarrow \mathbb{P}^N$, $\gamma_1, \ldots, \gamma_l$ but not on $S_1 \geq \cdots \geq S_l$, such that*

$$\mathcal{B}_{G,\Gamma(\underline{S}),T,D} \subset \Gamma(\underline{S}) + H_i$$

for any positive integers D, T such that $D > c_4 \mathfrak{S}_i T$ with $i \in \{0, \ldots, r-1\}$.

Theorem 2.2 *For any ε with $0 < \varepsilon < 1$ there exists a positive constant c_5, depending only on $G \hookrightarrow \mathbb{P}^N$, ε, $\gamma_1, \ldots, \gamma_l$ but not on $S_1 \geq \cdots \geq S_l$, such that*

$$\Gamma((1-\varepsilon)\underline{S}) + H_i \subset \mathcal{B}_{G,\Gamma(\underline{S}),T,D}$$

for any positive integers D, T such that $D < c_5^{-1} \mathfrak{S}_{i-1} T$ with $i \in \{1, \ldots, r\}$.

Theorem 2.2 is closely related to Lemma 1.5.3 in [8]. This latter result assumes that $S_1 = S_2 = \cdots = S_l$ and it only treats the case $i = 1$. It is stated for H_1 alone rather than $\Gamma((1-\varepsilon)\underline{S}) + H_1$ but it applies to these translates of H_1. Subgroups closely related to the sequence $H_2, \ldots, H_r$ appear in Conjecture 1.1.9 of [8]. The techniques of [8] are completely different from the present paper. In particular all constructions take place on X: no embeddings or quotient maps are used. The end result is that the results of [8] are quantitatively stronger (the constants are sharp in the same way as those of Philippon's multiplicity estimates) but they apply in very few cases.

The cases where D is very small or very large in comparison with T and the $\mathfrak{S}_i$ will be dealt with in Sect. 3.1. If $D < c_5^{-1} \mathfrak{S}_{r-1} T$ then Theorem 2.2 asserts that $\mathcal{B}_{G,\Gamma(\underline{S}),T,D} = G$; this is a multiplicity estimate, stated below as Proposition 3.1. In a "dual" way, if $D > c_4 \mathfrak{S}_0 T$ then Theorem 2.1 means that $\mathcal{B}_{G,\Gamma(\underline{S}),T,D} = \Gamma(\underline{S})$ since the inclusion $\Gamma(\underline{S}) \subset \mathcal{B}_{G,\Gamma(\underline{S}),T,D}$ holds trivially. We shall derive this result, stated as Proposition 3.3, from an interpolation estimate, namely Proposition 3.2.

When $r = 1$, we do not prove anything more—we could probably refine our result in this case, to make the constants c_4 and c_5 explicit, but we are not able to do it in general (see Sect. 6). When $r \geq 2$, our proof procedes by applying these results in sub-quotients of G coming from the algebraic subgroups H_i.

If $r \geq 2$ and

$$c_4 \mathfrak{S}_i T < D < c_5^{-1} \mathfrak{S}_{i-1} T$$

for some $i \in \{1, \ldots, r-1\}$, which happens for some integers D provided Card $\Gamma(\underline{S})$ is sufficiently large in terms of $G \hookrightarrow \mathbb{P}^N$, ε, $\gamma_1, \ldots, \gamma_l$, then

$$\Gamma((1-\varepsilon)\underline{S}) + H_i \subset \mathcal{B}_{G,\Gamma(\underline{S}),T,D} \subset \Gamma(\underline{S}) + H_i.$$

Therefore Theorem 1.1 follows from Theorems 2.1 and 2.2, since when $S_1 = \cdots = S_l = S$ we have

$$\mathfrak{S}_i = S^{\mu_i} \quad \text{with } \mu_i = \frac{\operatorname{rk}(\Gamma \cap H_{i+1}) - \operatorname{rk}(\Gamma \cap H_i)}{\dim H_{i+1} - \dim H_i}.$$

Remark 2.3 We shall prove in Lemma 3.4 below (Sect. 3.2) that $\mathfrak{S}_i^{\dim H_{i+1} - \dim H_i}$ is equal to the cardinality of $(\Gamma(\underline{S}) \cap H_{i+1})$ mod H_i, up to a multiplicative constant depending only on $\gamma_1, \ldots, \gamma_l$. Therefore $\mathfrak{S}_i$ might be replaced by the ($\dim H_{i+1} - \dim H_i$)-th root of this cardinality in Theorems 2.1 and 2.2 up to changing the values of the constants c_4 and c_5. The assumption $D < c_5 \mathfrak{S}_{i-1} T$ in Theorem 2.2 is the one needed to apply a multiplicity estimate in H_i/H_{i-1} in order to guarantee that no non-zero polynomial of degree D on H_i/H_{i-1} vanishes to order at least T at each point of $(\Gamma(\underline{S}) \cap H_i)$ mod H_{i-1}. Of course c_5 should take here a suitable value in terms of a projective embedding of H_i/H_{i-1}. The same remarks apply to the assumption $D > c_4 \mathfrak{S}_i T$ in Theorem 2.1 needed to apply an interpolation estimate (or Proposition 3.3 below) on $(\Gamma(\underline{S}) \cap H_{i+1})$ mod H_i in the algebraic group H_{i+1}/H_i (see Sect. 3.1).

2.2 Connection to a conjecture of the second author

Following [8] we let

$$\alpha_j = \sup\{\alpha \in \mathbb{Q},\ \dim \mathcal{B}_{G,\Gamma(\underline{S}),k\alpha,k} < j \quad \text{for any } k \text{ sufficiently large}\}$$

where $j \in \{1, \ldots, n\}$ and $n = \dim G$. Theorems 2.1 and 2.2, applied with $\varepsilon = 1/2$, yield

$$c_4^{-1} \mathfrak{S}_i^{-1} \leq \alpha_j \leq c_5 \mathfrak{S}_i^{-1}$$

where $i \in \{0, \ldots, r-1\}$ is chosen so that $\dim H_i < j \leq \dim H_{i+1}$. Consequently

$$c_4^{-n} \prod_{i=0}^{r-1} \mathfrak{S}_i^{-(\dim H_{i+1} - \dim H_i)} \leq \prod_{j=1}^{n} \alpha_j \leq c_5^n \prod_{i=0}^{r-1} \mathfrak{S}_i^{-(\dim H_{i+1} - \dim H_i)}.$$

Thus

$$c_4^{-n}\left[\prod_{j=1}^{l} S_j^{\operatorname{rk}\Gamma_j-\operatorname{rk}\Gamma_{j-1}}\right]^{-1} \le \prod_{j=1}^{n}\alpha_j \le c_5^n\left[\prod_{j=1}^{l} S_j^{\operatorname{rk}\Gamma_j-\operatorname{rk}\Gamma_{j-1}}\right]^{-1}.$$

Using Lemma 3.4 below with $H' = G$ and $H'' = \{0\}$ we obtain a positive constant c_6, depending only on $G \hookrightarrow \mathbb{P}^N$ and $\gamma_1, \ldots, \gamma_l$, such that

$$c_6^{-1} \le (\operatorname{Card}\Gamma(\underline{S}))\prod_{j=1}^{n}\alpha_j \le c_6.$$

Of course the important point here is that c_6 does not depend on $S_1, \ldots, S_l$. In parallel to Conjecture 1.1.4 of [8], it seems natural to ask whether

$$\frac{\deg_{\mathcal{O}(1)}(\overline{G})}{n!} \le (\operatorname{Card}\Gamma(\underline{S}))\prod_{j=1}^{n}\alpha_j \le \deg_{\mathcal{O}(1)}(\overline{G}),$$

where $n = \dim G$ and $\overline{G}$ is the Zariski closure of $G \hookrightarrow \mathbb{P}^N$. The upper bound can be proved using intersection theory and the definition of the α_i, as in §1.2 of [8].

3 Prerequisites

In this section we state the interpolation and multiplicity estimates we rely on, and apply them to the extremal cases $i = 0$ (in Theorem 2.1) and $i = r$ (in Theorem 2.2). Then we state and prove in Sect. 3.2 a lemma that provides an asymptotic estimate for the cardinality of the image of $\Gamma(\underline{S})$ in sub-quotients of G.

3.1 Interpolation and multiplicity estimates

We shall use the following notation: given a finite subset Ω of a commutative algebraic group G and a positive integer k, we let $\Omega[k]$ denote the set of all sums $\omega_1 + \cdots + \omega_k$ where $\omega_1, \ldots, \omega_k$ are (not necessarily distinct) elements of Ω. We denote by $\Omega\{k\}$ the set $\Omega[k] - \Omega[k]$, that is the set of all elements $x - y$ with $x, y \in \Omega[k]$.

The following is a weak form of the multiplicity estimate, Theorem 2.1, from [9].

Proposition 3.1 *Let G be a connected commutative algebraic group, embedded in projective space $\mathbb{P}^M$. Then there is a positive constant c_7, depending only on G and on this embedding, with the following property. Let Ω be a finite subset of G, and suppose D, T are positive integers such that, for every connected algebraic subgroup $H \subsetneq G$,*

$$\operatorname{Card}(\Omega \bmod H)\, T^{\dim(G/H)} > c_7 D^{\dim(G/H)}. \tag{4}$$

Then no non-zero $P \in \mathcal{R}(G)_D$ vanishes to order at least T at every point of $\Omega[\dim G]$. *In other words,*

$$\mathcal{B}_{G,\Omega,T,D} = G.$$

We shall deduce the statement "dual" to Proposition 3.1, namely Proposition 3.3, from the following interpolation estimate (which is Corollary 1.2 of [4]).

Proposition 3.2 *Let G be a connected commutative algebraic group, embedded in projective space $\mathbb{P}^M$. Then there is a positive constant c_8, depending only on G and on this embedding, with the following property. Let Ω be a finite subset of G, and suppose D, T are positive integers such that, for any translate $x + H$ of a non-zero connected algebraic subgroup H of G,*

$$\mathrm{Card}((\Omega \cap (x + H))[\dim(H)])\, T^{\dim(H)} < c_8 D^{\dim(H)}.$$

Then the evaluation map

$$\mathcal{R}(G)_D = H^0(\overline{G}, \mathcal{O}(D)) \to H^0(\overline{G}, \mathcal{O}(D) \otimes \oplus_{\omega\in\Omega} \mathcal{O}_{\overline{G}}/m_\omega^T)$$

is surjective, where $\overline{G}$ is the Zariski closure of G in $\mathbb{P}^M$ and $m_\omega \subset \mathcal{O}_{\overline{G}}$ is the maximal ideal sheaf corresponding to the point ω.

This result is essentially as precise as Philippon's multiplicity estimate (namely Theorem 2.1 of [9]), and even slightly more. A less precise estimate (in the style of [5] or [3]) would not be sufficient to deduce the following result, which we shall use later in this text.

Proposition 3.3 *Let G be a connected commutative algebraic group, embedded in projective space $\mathbb{P}^M$. Then there is a positive constant c_9, depending only on G and on this embedding, with the following property. Let Ω be a finite subset of G, and suppose D, T are positive integers such that, for every non-zero connected algebraic subgroup H of G,*

$$\mathrm{Card}(\Omega\{n\} \cap H)\, T^{\dim H} < c_9 D^{\dim H} \tag{5}$$

where $n = \dim G$. Then

$$\mathcal{B}_{G,\Omega,T,D} = \Omega.$$

It should be noticed that only algebraic subgroups H appear in this result, whereas translates are needed in Proposition 3.2. This is due to the fact that $\Omega\{n\}$ (i.e., the set of all elements $x - y$ with $x, y \in \Omega[n]$) is used instead of $\Omega[n]$.

Proof of Proposition 3.3 The inclusion $\Omega \subset \mathcal{B}_{G,\Omega,T,D}$ holds trivially. Let $g \in G\backslash\Omega$ and put $\Omega' = \Omega \cup \{g\}$. Let $H' = x + H$ be any translate of a non-zero connected algebraic subgroup H of G. Then $(\Omega' \cap H')[\dim H] \subset \cup_{i=0}^{\dim H} \mathcal{E}_i$ where

$$\mathcal{E}_i = \{ig + \gamma,\ \gamma \in \Omega[\dim H - i]\} \cap H'.$$

If $\mathcal{E}_i \neq \emptyset$, substracting a fixed element of $\mathcal{E}_i$ yields an injective map

$$\mathcal{E}_i \to \Omega\{\dim H - i\} \cap H \subset \Omega\{n\} \cap H,$$

so that Card $\mathcal{E}_i \leq \mathrm{Card}(\Omega\{n\} \cap H)$, and this inequality holds also if $\mathcal{E}_i = \emptyset$. Therefore we have

$$\begin{aligned}\mathrm{Card}((\Omega' \cap H')[\dim(H)])\, T^{\dim(H)} &\leq (n+1)\mathrm{Card}(\Omega\{n\} \cap H)\, T^{\dim(H)} \\ &< (n+1)c_9 D^{\dim H}.\end{aligned}$$

Choosing $c_9 = c_8/(n+1)$, Proposition 3.2 provides $P \in \mathcal{R}(G)_D$ which vanishes to order at least T at each point of Ω and does not vanish at g. This proves that $g \notin \mathcal{B}_{G,\Omega,T,D}$, and concludes the proof of Proposition 3.3.

3.2 A counting lemma

The following lemma is very useful for estimating the number of points of $\Gamma(\underline{S})$ in sub-quotients of G. The fundamental idea is that $S_1, \ldots, S_l$ will be assumed to be sufficiently large, in terms of $\gamma_1, \ldots, \gamma_l$, so that this number of points can be estimated asymptotically in terms of ranks of $\mathbb{Z}$-modules. Recall that Γ_j denotes the $\mathbb{Z}$-module generated by $\gamma_1, \ldots, \gamma_j$, with $\Gamma_0 = \{0\}$.

Lemma 3.4 *Let H', H'' be algebraic subgroups of a commutative algebraic group G, such that $H'' \subset H'$. Let $\gamma_1, \ldots, \gamma_l \in G$ and let Γ be the subgroup generated by $\gamma_1, \ldots, \gamma_l$. Then there exist positive constants c_{10} and c_{11} with the following properties:*

- *c_{10} depends only on $\gamma_1, \ldots, \gamma_l$ and on H' (but not on H'').*
- *c_{11} depends only on $\gamma_1, \ldots, \gamma_l$ and on H'' (but not on H').*
- *For any real numbers $S_1 \geq \cdots \geq S_l \geq 1$ we have*

$$c_{10}\mathcal{N}_{H',H''}(\underline{S}) < \mathrm{Card}(\Gamma(\underline{S}) \cap H' \bmod H'') < c_{11}\mathcal{N}_{H',H''}(\underline{S})$$

where

$$\mathcal{N}_{H',H''}(\underline{S}) = \prod_{j=1}^{l} S_j^{\mathrm{rk}\left(\frac{\Gamma_j \cap H'}{\Gamma_j \cap H''}\right) - \mathrm{rk}\left(\frac{\Gamma_{j-1} \cap H'}{\Gamma_{j-1} \cap H''}\right)}.$$

In the special case $H'' = \{0\}$, Lemma 3.4 reduces to Lemma 1.5 of [3], except that in [3] the constant c_{11} may depend on H'.

We did not try to make the constants c_{10} and c_{11} explicit since it is not needed in our application. However it is critical that c_{10} does not depend on H'' and that c_{11} does not depend on H'.

To illustrate this situation, let us consider the case where $l = 1$ and γ_1 is not torsion in G. Let H be a connected algebraic subgroup of G which contains $N\gamma_1$ for some $N \geq 1$, but no $k\gamma_1$ with $1 \leq k \leq N - 1$. Then $\mathrm{Card}(\Gamma(S_1) \cap H) = 2M + 1$, where $M \geq 0$ is the largest integer such that $MN < S_1$, and $\mathrm{Card}(\Gamma(S_1) \bmod H) = N$ if $S_1 > N$. Taking $H' = H$ and $H'' = \{0\}$ we see that c_{10} has to depend on H', since N may take arbitrarily large values in terms of $\gamma_1, \ldots, \gamma_l$. In the same way, taking $H' = G$ and $H'' = H$ shows that c_{11} has to depend on H''.

Using Lemma 4.8 and the notation of Sect. 2, Lemma 3.4 proves that $\mathfrak{S}_i^{\dim H_{i+1} - \dim H_i}$ is equal to the cardinality of $(\Gamma(\underline{S}) \cap H_{i+1}) \bmod H_i$, up to a multiplicative constant depending only on $\gamma_1, \ldots, \gamma_l$.

Remark 3.5 An immediate consequence of Lemma 3.4 is that for any $x \in G$

$$\mathrm{Card}(\Gamma(\underline{S}) \cap (x + H') \bmod H'') < c_{12}\mathcal{N}_{H',H''}(\underline{S})$$

where c_{12} depends only on $\gamma_1, \ldots, \gamma_l$ and on H'' but neither on H' nor on x. Indeed, subtracting a fixed element of $\Gamma(\underline{S}) \cap (x + H')$ yields an injective map $\Gamma(\underline{S}) \cap (x + H') \to \Gamma(2\underline{S}) \cap H'$.

Remark 3.6 For any $\lambda \geq 1$, applying Lemma 3.4 with $\lambda\underline{S} = (\lambda S_1, \ldots, \lambda S_l)$ yields

$$c_{10}\lambda^{\mathrm{rk}\left(\frac{\Gamma\cap H'}{\Gamma\cap H''}\right)}\mathcal{N}_{H',H''}(\underline{S}) < \mathrm{Card}(\Gamma(\lambda\underline{S}) \cap H' \bmod H'') < c_{11}\lambda^{\mathrm{rk}\left(\frac{\Gamma\cap H'}{\Gamma\cap H''}\right)}\mathcal{N}_{H',H''}(\underline{S}). \tag{6}$$

This will be used several times in the proof of Lemma 3.4, without explicit reference. Moreover the first inequality in Eq. (6) holds for any $\lambda > 0$.

Proof of Lemma 3.4 Since the result is trivial when $l = 0$, we may assume by induction that it holds for Γ_{l-1}. Notice that the value of $\mathcal{N}_{H',H''}(\underline{S})$ relative to $\Gamma(\underline{S})$ is the same as the one relative to $\Gamma_{l-1}(S_1, \ldots, S_{l-1})$ if $\mathrm{rk}\left(\frac{\Gamma\cap H'}{\Gamma\cap H''}\right) = \mathrm{rk}\left(\frac{\Gamma_{l-1}\cap H'}{\Gamma_{l-1}\cap H''}\right)$, and it is S_l times bigger otherwise.

The lower bound on $\mathrm{Card}(\Gamma(\underline{S}) \cap H' \bmod H'')$ follows at once from the inclusion $\Gamma_{l-1}(S_1, \ldots, S_{l-1}) \subset \Gamma(\underline{S})$ if $\mathrm{rk}\left(\frac{\Gamma\cap H'}{\Gamma\cap H''}\right) = \mathrm{rk}\left(\frac{\Gamma_{l-1}\cap H'}{\Gamma_{l-1}\cap H''}\right)$. Otherwise we have $\mathrm{rk}(\Gamma \cap H') = 1 + \mathrm{rk}(\Gamma_{l-1} \cap H')$ and $\mathrm{rk}(\Gamma \cap H'') = \mathrm{rk}(\Gamma_{l-1} \cap H'')$. In this case, there exist $m_1, \ldots, m_l \in \mathbb{Z}$, with $m_l \geq 1$, such that $\widetilde{\gamma} = m_1\gamma_1 + \cdots + m_l\gamma_l$ belongs to $\Gamma \cap H'$ and has infinite order in $\frac{\Gamma\cap H'}{\Gamma_{l-1}\cap H'}$. Letting $M = \max(|m_1|, \ldots, |m_l|)$, the elements $\gamma_0 + n\widetilde{\gamma}$, where $|n| < S_l/2M$ and γ_0 ranges through a system of representatives of $\Gamma_{l-1}(S_1/2, \ldots, S_{l-1}/2) \cap H' \bmod H''$, belong to $\Gamma(\underline{S}) \cap H'$ since $S_l \leq S_i$ for any $i \in \{1, \ldots, l-1\}$. If two of them are equal modulo H'', say $\gamma_0 + n\widetilde{\gamma} \in \gamma_0' + n'\widetilde{\gamma} + H''$ with $(\gamma_0, n) \neq (\gamma_0', n')$, then $(n - n')\widetilde{\gamma} + (\gamma_0 - \gamma_0') \in \Gamma \cap H''$. Now $\Gamma_{l-1} \cap H''$ has finite index, say N, in $\Gamma \cap H''$ so that $N(n - n')\widetilde{\gamma} + N(\gamma_0 - \gamma_0') \in \Gamma_{l-1} \cap H'' \subset \Gamma_{l-1} \cap H'$ and $N(n - n')\widetilde{\gamma} \in \Gamma_{l-1} \cap H'$. Since the image of $\widetilde{\gamma}$ in $\frac{\Gamma\cap H'}{\Gamma_{l-1}\cap H'}$ has infinite order, we have $n = n'$ and $\gamma_0 \in \gamma_0' + H''$ which is a contradiction. Therefore the elements given above are pairwise distinct, concluding the proof of the lower bound on $\mathrm{Card}(\Gamma(\underline{S}) \cap H' \bmod H'')$.

To prove the upper bound, we distinguish between three cases.

(a) If $\mathrm{rk}(\Gamma \cap H') = \mathrm{rk}(\Gamma_{l-1} \cap H')$, the upper bound holds trivially if we also have $\Gamma(\underline{S}) \cap H' = \Gamma_{l-1}(S_1, \dots, S_{l-1}) \cap H'$. Otherwise there exist $m_1, \dots, m_l \in \mathbb{Z}$, with $m_l \geq 1$, such that $\widetilde{\gamma} = m_1\gamma_1 + \cdots + m_l\gamma_l$ belongs to $\Gamma \cap H'$. Letting N denote the index of $\Gamma_{l-1} \cap H'$ in $\Gamma \cap H'$, we have $N\widetilde{\gamma} = Nm_1\gamma_1 + \cdots + Nm_l\gamma_l \in \Gamma_{l-1} \cap H'$ so that $Nm_l\gamma_l \in \Gamma_{l-1}$. Therefore the image of γ_l in Γ/Γ_{l-1} has finite order: let ω denote this order, which depends only on $\gamma_1, \dots, \gamma_l$. There exist $r_1, \dots, r_l \in \mathbb{Z}$ such that $r_1\gamma_1 + \cdots + r_l\gamma_l = 0$ and $r_l = \omega \geq 1$. Letting $R = \max(|r_1|, \dots, |r_l|)$, we have

$$\Gamma(\underline{S}) \subset \bigcup_{n=0}^{r_l-1} n\gamma_l + \Gamma_{l-1}((R+1)S_1, \dots, (R+1)S_{l-1})$$

since $S_l \leq S_i$ for any $i \in \{1, \dots, l-1\}$. The upper bound follows at once.

(b) If $\mathrm{rk}(\Gamma \cap H') = 1 + \mathrm{rk}(\Gamma_{l-1} \cap H')$ and $\mathrm{rk}(\Gamma \cap H'') = \mathrm{rk}(\Gamma_{l-1} \cap H'')$, there exist $m_1, \dots, m_l \in \mathbb{Z}$ such that $\widetilde{\gamma} = m_1\gamma_1 + \cdots + m_l\gamma_l \in \Gamma \cap H'$ and $m_l \geq 1$; we choose these integers with the least possible value of m_l. Then for any $\gamma = n_1\gamma_1 + \cdots + n_l\gamma_l \in \Gamma(\underline{S}) \cap H'$, n_l is a multiple of m_l and we have $\gamma - r\widetilde{\gamma} \in \Gamma_{l-1} \cap H'$ where $r = n_l/m_l$ is such that $|r| < S_l$. Letting $\gamma' = \widetilde{\gamma} - m_l\gamma_l = m_1\gamma_1 + \cdots + m_{l-1}\gamma_{l-1}$ we obtain

$$\Gamma(\underline{S}) \subset \bigcup_{r=-S_l}^{S_l} rm_l\gamma_l + [\Gamma_{l-1}(S_1, \dots, S_{l-1}) \cap (r\gamma' + H')].$$

Using Remark 3.5 this concludes the proof of the upper bound in this case.

(c) If $\mathrm{rk}(\Gamma \cap H') = 1 + \mathrm{rk}(\Gamma_{l-1} \cap H')$ and $\mathrm{rk}(\Gamma \cap H'') = 1 + \mathrm{rk}(\Gamma_{l-1} \cap H'')$, there exist $m_1, \dots, m_l \in \mathbb{Z}$ such that $\widetilde{\gamma} = m_1\gamma_1 + \cdots + m_l\gamma_l \in \Gamma \cap H''$ and $m_l \geq 1$. Let $\gamma = n_1\gamma_1 + \cdots + n_l\gamma_l \in \Gamma(\underline{S}) \cap H'$, and let $q, r \in \mathbb{Z}$ be such that $n_l = qm_l + r$ with $|r_l| < m_l$ and $|q| \leq \frac{|n_l|}{m_l} < S_l$. Then we have $\gamma - q\widetilde{\gamma} = (n_1 - qm_1)\gamma_1 + \cdots + (n_{l-1} - qm_{l-1})\gamma_{l-1} + r\gamma_l$ so that, letting $M = \max(|m_1|, \dots, |m_l|)$,

$$\begin{aligned}&\Gamma(\underline{S}) \cap H' \bmod H'' \\ &\quad \subset \bigcup_{r=-M}^{M} r\gamma_l + [\Gamma_{l-1}((M+1)S_1, \dots, (M+1)S_{l-1}) \cap (-r\gamma_l + H')] \bmod H''.\end{aligned}$$

Using Remark 3.5 this concludes the proof of Lemma 3.4.

4 A chain of algebraic subgroups

Throughout this section we fix a connected commutative algebraic group G, real numbers $S_1, \dots, S_l$ and elements $\gamma_1, \dots, \gamma_l \in G$; we assume that $S_1 \geq \cdots \geq S_l \geq 1$. With this data we associate in Sect. 4.1 a chain of connected algebraic subgroups $(H_i)_{0 \leq i \leq r}$ of G. We study its properties throughout this section, with a special emphasis on its connection to the distribution of $\Gamma(\underline{S})$ with respect to algebraic subgroups of G (Sect. 4.4), and on the case where $S_1 = \cdots = S_l$ as in the introduction (Sect. 4.5). Examples are given in Sect. 4.2 to illustrate this construction.

4.1 Construction and first properties

For any connected algebraic subgroup K of G we let

$$\varphi_{\underline{S}}(K) = \sum_{j=1}^{l} (\mathrm{rk}(\Gamma_j \cap K) - \mathrm{rk}(\Gamma_{j-1} \cap K)) \log S_j,$$

where Γ_j is the subgroup of Γ generated by $\gamma_1, \ldots, \gamma_j$, $\Gamma_0 = \{0\}$, and $\underline{S} = (S_1, \ldots, S_l)$. With this definition, $\mathrm{Card}(\Gamma(\underline{S}) \cap K)$ is essentially equal to $\exp \varphi_{\underline{S}}(K)$ by Lemma 3.4 above, with $H'' = \{0\}$, so that $\mathcal{N}_{K,\{0\}}(\underline{S}) = \exp \varphi_{\underline{S}}(K)$. We refer to §3 of [3] for a related construction.

In the special case where $S_1 = \cdots = S_l = S > 1$, we have $\varphi_{\underline{S}}(K) = \mathrm{rk}(\Gamma \cap K) \log S$ and the chain (H_i) we construct here does not depend on S nor on the choice of $\gamma_1, \ldots, \gamma_l$ (see Lemma 4.7 and Sect. 4.5) so one may actually assume that $S = e = \exp(1)$ and $\varphi_{\underline{S}}(K) = \mathrm{rk}(\Gamma \cap K)$. The starting point of our construction is the existence [8], in this case, of a maximal element H_1 with respect to inclusion among the non-zero connected algebraic subgroups H such that $\mu^*(\Gamma, G) = \frac{\mathrm{rk}(\Gamma \cap H)}{\dim H}$. Applying this construction again in G/H_1 with $\Gamma \bmod H_1 = (\Gamma + H_1)/H_1$ yields a maximal connected algebraic subgroup H_2/H_1 of G/H_1, with $H_1 \subsetneq H_2$. Repeating this argument leads to a chain of algebraic subgroups of G, which we construct now in the general case where $S_1, \ldots, S_l$ are not assumed to be equal.

Proposition 4.1 *There exists a unique chain* $\{0\} = H_0 \subsetneq H_1 \subsetneq \cdots \subsetneq H_r = G$ *of connected algebraic subgroups of* G, *with* $r \geq 1$, *such that*:

- *For any* $i \in \{0, \ldots, r-1\}$ *and any connected algebraic subgroup* K *such that* $\dim K > \dim H_i$, *we have*

$$\frac{\varphi_{\underline{S}}(K) - \varphi_{\underline{S}}(H_i)}{\dim K - \dim H_i} \leq \frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i}. \tag{7}$$

- *If equality holds in Eq.* (7) *then* $H_i \subset K \subset H_{i+1}$.

Remark 4.2 In the proof of Proposition 4.1 we shall prove actually a stronger property of these subgroups, namely that for any $i \in \{0, \ldots, r-1\}$ and any connected algebraic subgroup K we have

$$\begin{aligned}&[\dim H_{i+1} - \dim H_i]\varphi_{\underline{S}}(K) - [\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)] \dim K \\ &\quad + \varphi_{\underline{S}}(H_{i+1}) \dim H_i - \varphi_{\underline{S}}(H_i) \dim H_{i+1} \leq 0,\end{aligned} \tag{8}$$

and if equality holds then $H_i \subset K \subset H_{i+1}$. This inequality can be also be written as

$$[\varphi_{\underline{S}}(K) - \varphi_{\underline{S}}(H_i)][\dim H_{i+1} - \dim H_i] \leq [\dim K - \dim H_i][\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)].$$

If $\dim K > \dim H_i$ it is equivalent to Eq. (7). If $\dim K < \dim H_i$ it yields

$$\frac{\varphi_{\underline{S}}(H_i) - \varphi_{\underline{S}}(K)}{\dim H_i - \dim K} > \frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i}. \tag{9}$$

In the case where $S_1 = \cdots = S_l$, reasoning as in [8] one can prove the existence of a minimal element H_{r-1} with respect to inclusion among the connected algebraic subgroups $H \subsetneq G$ such that $\mu(\Gamma, G) = \frac{\mathrm{rk}(\Gamma) - \mathrm{rk}(\Gamma \cap H)}{\dim G - \dim H}$. Applying this property again in H_{r-1} with $\Gamma \cap H_{r-1}$ provides $H_{r-2} \subsetneq H_{r-1}$. The following immediate consequence of Eq. (8) asserts that the chain of connected algebraic subgroups of G constructed by iterating this process (and generalizing it to allow $S_1, \ldots, S_l$ not to be equal) is the same as above.

Proposition 4.3 *For any $i \in \{0, \ldots, r-1\}$ and any connected algebraic subgroup K such that* $\dim K < \dim H_{i+1}$ *we have*

$$\frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(K)}{\dim H_{i+1} - \dim K} \geq \frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i}. \tag{10}$$

Moreover if equality holds then $H_i \subset K \subset H_{i+1}$.

Assuming again $S_1 = \cdots = S_l$, H_1 is maximal such that $\frac{\mathrm{rk}(\Gamma \cap H_1)}{\dim H_1} = \mu^*(\Gamma, G) = \max_{H \neq 0} \frac{\mathrm{rk}(\Gamma \cap H)}{\dim H}$. In particular we have $\frac{\mathrm{rk}(\Gamma \cap H_1)}{\dim H_1} > \frac{\mathrm{rk}(\Gamma \cap H_2)}{\dim H_2}$ since $H_1 \subsetneq H_2$. Now $\frac{\mathrm{rk}(\Gamma \cap H_2)}{\dim H_2}$ lies between $\frac{\mathrm{rk}(\Gamma \cap H_1)}{\dim H_1}$ and $\frac{\mathrm{rk}(\Gamma \cap H_2) - \mathrm{rk}(\Gamma \cap H_1)}{\dim H_2 - \dim H_1}$ because its numerator is the sum of both numerators and the same property holds for the denominators, so that $\frac{\mathrm{rk}(\Gamma \cap H_2) - \mathrm{rk}(\Gamma \cap H_1)}{\dim H_2 - \dim H_1} < \frac{\mathrm{rk}(\Gamma \cap H_1)}{\dim H_1}$. Generalizing this result to all subgroups H_i and removing the assumption $S_1 = \cdots = S_l$, we obtain the following.

Proposition 4.4 *For any $i \in \{1, \ldots, r-1\}$ we have*

$$\frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i} < \frac{\varphi_{\underline{S}}(H_i) - \varphi_{\underline{S}}(H_{i-1})}{\dim H_i - \dim H_{i-1}}.$$

This proposition follows immediately from Eq. (9) by taking $K = H_{i-1}$. It is the key point in the proof of Eqs. (2) and (3) above.

Remark 4.5 With each connected algebraic subgroup K of G we may associate the point $M_K = (\dim K, \varphi_{\underline{S}}(K)) \in \mathbb{R}^2$. Then our construction yields a convex polygon $M_{H_0} M_{H_1} \ldots M_{H_r} N$, where $N = (\dim G, 0)$; in particular $M_{H_0} = M_{\{0\}} = (0, 0)$ and $M_{H_r} = M_G = (\dim G, \varphi_{\underline{S}}(G))$. For any K the point M_K is either inside this polygon or on an edge; if it lies on the segment $[M_{H_i} M_{H_{i+1}}]$ with $0 \leq i \leq r-1$ then $H_i \subset K \subset H_{i+1}$. Indeed Eq. (7) means that the line $(M_{H_i} M_K)$ has slope less than or equal to that of $(M_{H_i} M_{H_{i+1}})$, if $\dim K > \dim H_i$. This means that M_K is below the line $(M_{H_i} M_{H_{i+1}})$, which is expressed by Eq. (8). A similar statement, if $\dim K < \dim H_{i+1}$, is provided by Proposition 4.3. Namely, the slope of $(M_K M_{H_{i+1}})$ is less than or equal to that of $(M_{H_i} M_{H_{i+1}})$. Lastly the slope of $(M_{H_i} M_{H_{i+1}})$ is a decreasing function of i, as Proposition 4.4 states.

Remark 4.6 As the proof below shows, the algebraic subgroups H_i are constructed in terms of the function $\varphi_{\underline{S}}$ only; accordingly they depend only on $\Gamma_1, \dots, \Gamma_l$ instead of the specific choice of $\gamma_1, \dots, \gamma_l$. Multiplying the function $\varphi_{\underline{S}}$ with a fixed positive real number does not change the construction either (since this number cancels out everywhere): see Lemma 4.7 below.

Proof of Proposition 4.1 and Remark 4.2 To begin with, we notice hat

$$\varphi_{\underline{S}}(K) = \sum_{j=1}^{l} \mathrm{rk}(\Gamma_j \cap K) \log(S_j/S_{j+1})$$

for any connected algebraic subgroup K of G; here we let $S_{l+1} = 1$. For any j and any connected algebraic subgroups K, K' of G we have $(\Gamma_j \cap K) + (\Gamma_j \cap K') \subset \Gamma_j \cap (K + K')$ so that

$$\mathrm{rk}(\Gamma_j \cap (K \cap K')) + \mathrm{rk}(\Gamma_j \cap (K + K')) \geq \mathrm{rk}(\Gamma_j \cap K) + \mathrm{rk}(\Gamma_j \cap K')$$

and

$$\varphi_{\underline{S}}(K \cap K') + \varphi_{\underline{S}}(K + K') \geq \varphi_{\underline{S}}(K) + \varphi_{\underline{S}}(K') \tag{11}$$

since $\log(S_j/S_{j+1}) \geq 0$ for any j. We shall also use the fact that

$$\dim(K \cap K') + \dim(K + K') = \dim(K) + \dim(K'). \tag{12}$$

Now let us construct H_i and prove the results at the same time, by induction on i. If the algebraic subgroups $H_0, \dots, H_i$ satisfy the desired properties with $i \geq 0$ and $H_i \neq G$, we define H_{i+1} to be a connected algebraic subgroup of G of dimension greater than $\dim H_i$ for which $\frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i}$ is maximal. If there are several connected algebraic subgroups K of G with $\dim K > \dim H_i$ for which $\frac{\varphi_{\underline{S}}(K) - \varphi_{\underline{S}}(H_i)}{\dim(K) - \dim H_i}$ is equal to this maximal value, then we choose H_{i+1} with maximal dimension among them. If there are several such subgroups K with this maximal dimension, we choose H_{i+1} arbitrarily among them (but it will follow from Proposition 4.1 that this situation can not happen).

In this way Eq. (7) holds for any K such that $\dim K > \dim H_i$. Now let $\chi_i(K)$ denote the left handside of Eq. (8); notice that

$$\chi_i(H_i) = \chi_i(H_{i+1}) = 0. \tag{13}$$

Actually if we associate with each connected algebraic subgroup K of G the point $M_K = (\dim K, \varphi_{\underline{S}}(K))$ as in Remark 4.5 above, then $\chi_i(K) = 0$ means that M_K lies on the line $(M_{H_i} M_{H_{i+1}})$.

By definition of H_{i+1} we have

$$\begin{cases} \chi_i(K) \leq 0 \quad \text{if } \dim K > \dim H_i \\ \text{if equality holds then } \dim K \leq \dim H_{i+1}. \end{cases} \tag{14}$$

To conclude the proof of Eq. (8) for any K, let us prove also that

$$\begin{cases} \chi_i(K) \leq 0 \quad \text{if } \dim K \leq \dim H_i \\ \text{if equality holds then } K = H_i. \end{cases} \tag{15}$$

If $i = 0$ this is a triviality. If $i \geq 1$, $\dim K = \dim H_i$ and $K \neq H_i$ then $\chi_i(K) = (\dim H_{i+1} - \dim H_i)(\varphi_{\underline{S}}(K) - \varphi_{\underline{S}}(H_i)) < 0$ using Eq. (7) with $i-1$. If $i \geq 1$ and $\dim K < \dim H_i$, notice that Eq. (8) with $i-1$ reads

$$\frac{\varphi_{\underline{S}}(H_i) - \varphi_{\underline{S}}(K)}{\dim H_i - \dim K} \geq \frac{\varphi_{\underline{S}}(H_i) - \varphi_{\underline{S}}(H_{i-1})}{\dim H_i - \dim H_{i-1}}.$$

Combining this inequality with Proposition 4.4 (which holds for i, since it follows from Eq. (9) by taking $K = H_{i-1}$), we obtain $\chi_i(K) < 0$; this completes the proof of (15) and that of Eq. (8) for any K.

Now let K be a connected algebraic subgroup of G such that $\dim K > \dim H_i$ and $\chi_i(K) = 0$. Let us prove that $H_i \subset K$ and $K \subset H_{i+1}$; this will conclude the proofs of the equality cases in Eqs. (7) and (8), and will also prove that $H_i \subset H_{i+1}$ since one may take $K = H_{i+1}$.

With this aim in view, we notice, using (14), that $\dim K \leq \dim H_{i+1}$ and that for any K' we have

$$\chi_i(K \cap K') + \chi_i(K + K') \geq \chi_i(K) + \chi_i(K') \tag{16}$$

using Eqs. (11) and (12). Recall that $\chi_i(H_i) = \chi_i(H_{i+1}) = \chi_i(K) = 0$ thanks to Eq. (13) and our assumption on K, and that $\chi_i(K'') \leq 0$ for any K'' using (14) and (15). With $K' = H_i$ we obtain in this way $\chi_i(K \cap H_i) = \chi_i(K + H_i) = 0$, so that (15) yields $K \cap H_i = H_i$ and finally $H_i \subset K$. In a similar way, with $K' = H_{i+1}$ we get $\chi_i(K \cap H_{i+1}) = \chi_i(K + H_{i+1}) = 0$, so that (14) yields $\dim(K + H_{i+1}) = \dim H_{i+1}$ and thus $K \subset H_{i+1}$. This concludes the proof that $H_i \subset K \subset H_{i+1}$.

It remains to check that only one chain (H_i) satisfies the conclusions of Proposition 4.1. Indeed given two chains (H_i) and (H'_i), let i be the least index such that $H'_{i+1} \neq H_{i+1}$. Applying Eq. (7) twice yields

$$\frac{\varphi_{\underline{S}}(H'_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H'_{i+1} - \dim H_i} = \frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i};$$

then using twice the equality case in this inequality we obtain $H'_{i+1} = H_{i+1}$. This contradicts the definition of i, and concludes the proof of Proposition 4.1 and Remark 4.2.

4.2 Examples in powers of $\mathbb{G}_m$

In this section we illustrate the construction of Sect. 4.1 with two examples. Let $G = \mathbb{G}_m^3$ and let $x_1, \ldots, x_6, y_5, y_6, z_7$ be multiplicatively independent non-zero complex numbers. Let

$$\begin{aligned}\gamma_i &= (x_i, 1, 1) \in G(\mathbb{C}) = \mathbb{C}^{\star 3} \quad \text{for } 1 \leq i \leq 4,\\ \gamma_5 &= (x_5, y_5, 1),\\ \gamma_6 &= (x_6, y_6, 1),\\ \gamma_7 &= (1, 1, z_7).\end{aligned}$$

Then $\gamma_1, \ldots, \gamma_7$ are $\mathbb{Z}$-linearly independent in G so that $\operatorname{rk}\Gamma = 7$, where Γ is the $\mathbb{Z}$-module spanned by these seven elements. We let

$$H_0 = \{(1, 1, 1)\}, \quad H_1 = \mathbb{G}_m \times \{1\}^2, \quad H_2 = \mathbb{G}_m^2 \times \{1\}, \quad H_3 = \mathbb{G}_m^3 = G.$$

We consider the case where $S_1 = \cdots = S_7 = S > 1$: since the value of S is not relevant here (see Lemma 4.7 and Sect. 4.5), we assume for simplicity $S = e$ so that $\varphi_{\underline{S}}(K) = \operatorname{rk}(\Gamma \cap K)$ for any connected algebraic subgroup K of G. Then it is not difficult to check that $\{0\} = H_0 \subsetneq H_1 \subsetneq H_2 \subsetneq H_3 = G$ is the chain constructed in Sect. 4.1. With the notation of Remark 4.5, we have

$$M_{H_0} = (0, 0), \quad M_{H_1} = (1, 4), \quad M_{H_2} = (2, 6), \quad M_{H_3} = (3, 7).$$

Together with $N = (3, 0)$, these points make up a convex polygon inside which lie all points $M_K = (\dim K, \operatorname{rk}(\Gamma \cap K))$ corresponding to connected algebraic subgroups K of G. The rational numbers $\mu_i = \frac{\operatorname{rk}(\Gamma\cap H_{i+1}) - \operatorname{rk}(\Gamma\cap H_i)}{\dim H_{i+1} - \dim H_i}$, defined in the introduction and used in the statement of Theorem 1.1, are the slopes of the edges $[M_{H_i} M_{H_{i+1}}]$: we have

$$\mu_0 = 4, \quad \mu_1 = 2, \quad \mu_2 = 1.$$

As noticed in the introduction, $\mu(\Gamma, G) = \mu_2 = 1$ and $\mu^*(\Gamma, G) = \mu_0 = 4$.

Finally let $H' = \mathbb{G}_m \times \{1\} \times \mathbb{G}_m$: we have $\dim H' = 2$ and $\operatorname{rk}(\Gamma \cap H') = 5$, so that $\frac{\operatorname{rk}(\Gamma\cap H')}{\dim H'} = 5/2 > 7/3 = \frac{\operatorname{rk}\Gamma}{\dim G}$: the $\mathbb{Z}$-module Γ is "more densely distributed" in H' than in G. Howerer H' does not belong to the chain (H_i).

Let us move now to a second example. Let $G = \mathbb{G}_m^4$ and let $t_1, \ldots, t_{13}$ be multiplicatively independent non-zero complex numbers. Suppose Γ is the $\mathbb{Z}$-module spanned by the following 13 elements:

$$\begin{aligned}\gamma_i &= (t_i, 1, 1, 1) \quad \text{for } 1 \leq i \leq 7,\\ \gamma_i &= (1, t_i, 1, 1) \quad \text{for } 8 \leq i \leq 10,\\ \gamma_i &= (1, 1, t_i, 1) \quad \text{for } 11 \leq i \leq 12,\\ \gamma_i &= (1, 1, 1, t_i) \quad \text{for } i = 13.\end{aligned}$$

Setting $S_1 = \cdots = S_{13} = e$, the chain of algebraic subgroups constructed in Sect. 4.1 is the following:

$$\begin{aligned}H_0 = \{1\}^4, \quad H_1 = \mathbb{G}_m \times \{1\}^3, \quad H_2 = \mathbb{G}_m^2 \times \{1\}^2,\\ H_3 = \mathbb{G}_m^3 \times \{1\}, \quad H_4 = \mathbb{G}_m^4 = G.\end{aligned}$$

The hexagon of Remark 4.5 has vertices

$$M_{H_0} = (0, 0), \quad M_{H_1} = (1, 7), \quad M_{H_2} = (2, 10), \quad M_{H_3} = (3, 12), \quad M_{H_4} = (4, 13)$$

and $N = (4, 0)$. The slopes of its edges are 0, ∞, and the rational numbers μ_i given by

$$\mu_0 = \mu^*(\Gamma, G) = 7, \quad \mu_1 = 3, \quad \mu_2 = 2, \quad \mu_3 = \mu(\Gamma, G) = 1.$$

Letting $H' = \mathbb{G}_m \times \{1\}^2 \times \mathbb{G}_m$, we have $\frac{\mathrm{rk}(\Gamma \cap H')}{\dim H'} = 8/2 = 12/3 = \frac{\mathrm{rk}(\Gamma \cap H_3)}{\dim H_3}$. However H' is not contained in H_3 despite what is announced in [8], p. 479, l. 11. Indeed in §1.3 of this paper, the argument producing the chain of obstruction subgroups does not show unicity except for the first subgroup, H_1, in the chain. The example here shows that these subgroups H with specific critical values of $\frac{\mathrm{rk}(\Gamma \cap H)}{\dim H}$ are not in general unique.

4.3 Independence and finiteness results

Let us come back to the general setting of Sect. 4.1. It is clear from the construction that the subgroups $H_0, \dots, H_r$ and the integer r depend on $S_1, \dots, S_l$. However there is a transformation under which they are invariant:

Lemma 4.7 *The subgroups $H_0, \dots, H_r$ and the integer r remain the same if $S_1, \dots, S_l$ are replaced with $S_1^\alpha, \dots, S_l^\alpha$ for some $\alpha > 0$.*

An important consequence of this lemma is that if $S_1 = \cdots = S_l = S > 1$ as in the introduction, then $H_0, \dots, H_r$ do not depend on S (see Sect. 4.5).

Proof of Lemma 4.7 Upon replacing $S_1, \dots, S_l$ with $S_1^\alpha, \dots, S_l^\alpha$, the function $\varphi_{\underline{S}}$ is multiplied by $\alpha > 0$ so that Eq. (7) is still valid: since the chain of subgroups constructed above is unique (see Proposition 4.1 and Remark 4.6), it remains the same.

Throughout the proof of Theorems 2.1 and 2.2, many constants will appear that depend on the subgroups $(H_i)_{0 \le i \le r}$. The following lemma shows that such a constant can be made independent from these subgroups, by increasing it if necessary.

Lemma 4.8 *There exists a finite set $\mathcal{E}$, which depends on $\gamma_1, \dots, \gamma_l$ but not on $S_1, \dots, S_l$, such that all subgroups $H_0, \dots, H_r$ belong to $\mathcal{E}$.*

The idea behind this lemma is simply that the construction of H_i involves only $\dim H_i$ and the ranks of $\Gamma_j \cap H_i$, which take only finitely many values (see §3.1.1 of [3] for an analogous situation). Moreover there is no connected algebraic subgroup $H' \neq H_i$ such that $\dim H' = \dim H_i$ and $\mathrm{rk}(\Gamma_j \cap H') = \mathrm{rk}(\Gamma_j \cap H_i)$ for any j, so that H_i can take only finitely many values. In the notation of Remark 4.5, there is no $H' \neq H_i$ such that $M_{H'} = M_{H_i}$, even though in general there may exist connected algebraic subgroups $H' \neq H''$ such that $M_{H'} = M_{H''}$: the connected algebraic subgroups H_i are uniquely determined by the vertices of the polygon $M_{H_0} M_{H_1} \dots M_{H_r} N$. Let us make these ideas more precise now.

Proof of Lemma 4.8 Let us denote by $\mathcal{S}$ the set of all connected algebraic subgroups of G, and for $K \in \mathcal{S}$ let $\psi(K) = (\dim K, \mathrm{rk}(\Gamma_1 \cap K), \ldots, \mathrm{rk}(\Gamma_l \cap K)) \in \mathbb{Z}^{l+1}$. Let $\mathcal{E}$ denote the set of all $H \in \mathcal{S}$ such that $\psi^{-1}(\psi(H)) = \{H\}$. Then $\mathcal{E}$ is a finite set, because $\psi(\mathcal{S})$ clearly is and $\psi_{|\mathcal{E}} \colon \mathcal{E} \to \psi(\mathcal{S})$ is an injective map. Now for any subgroup H_{i+1} in a chain corresponding to some $S_1, \ldots, S_l$, we have $\psi^{-1}(\psi(H_{i+1})) = \{H_{i+1}\}$ because equality holds in Eq. (7) for any $K \in \psi^{-1}(\psi(H_{i+1}))$; therefore $H_{i+1} \in \mathcal{E}$. Since $H_0 = \{0\} \in \mathcal{E}$, this concludes the proof of Lemma 4.8.

4.4 Applications to the distribution of Γ

The chain of algebraic subgroups $(H_i)_{0\le i\le r}$ constructed in Sect. 4.1 is useful to study the distribution of $\Gamma(\underline{S})$ with respect to algebraic subgroups of G. Several results of this kind have been stated in the introduction when $S_1 = \cdots = S_l$, and will be proved in Sect. 4.5. In the general case where $S_1, \ldots, S_l$ are not assumed to be equal, the same results hold except that they have to be stated differently: exponents like $\mu(\Gamma, G)$ and $\mu^*(\Gamma, G)$ are no longer available. We shall neither state nor prove the corresponding generalizations of all results stated in the introduction, but only the ones that will be used in the proof of Theorems 2.1 and 2.2.

To begin with, let us generalize the fact that $\Gamma(S) \cap H_{i+1} \bmod H_i$ is well-distributed in H_{i+1}/H_i. Recall that $\mathfrak{S}_i$ has been defined at the beginning of Sect. 2.

Lemma 4.9 *There exists a positive constant c_{13}, which depends only on $G, \gamma_1, \ldots, \gamma_l$ but not on $S_1, \ldots, S_l$, with the following property: for any $i \in \{0, \ldots, r-1\}$ and any connected algebraic subgroup H such that $H_i \subsetneq H$, we have*

$$\mathrm{Card}(\Gamma(2n\underline{S}) \cap H \bmod H_i) < c_{13}\mathfrak{S}_i^{\dim(H/H_i)}$$

where $n = \dim G$.

This lemma asserts that in applying Proposition 3.3 to $\Gamma(2n\underline{S}) \cap H_{i+1} \bmod H_i$ in the algebraic group H_{i+1}/H_i, it is enough to check assumption (5) with $H = H_{i+1}/H_i$ (with a smaller value of c_9, though), so that this proposition applies as soon as $D > c_{14}\mathfrak{S}_i T$ for some constant c_{14}. Indeed $\mathfrak{S}_i$ is given by

$$\mathfrak{S}_i = \exp\left[\frac{\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i)}{\dim H_{i+1} - \dim H_i}\right], \tag{17}$$

and Lemma 3.4 shows that $\mathfrak{S}_i^{\dim H_{i+1} - \dim H_i}$ is equal, up to a multiplicative constant, to the cardinality of $(\Gamma(\underline{S}) \cap H_{i+1}) \bmod H_i$: the conclusion of Lemma 4.9 is an equality for $H = H_{i+1}$, except for the value of the constant c_{13}.

We prove Lemma 4.9 for $\Gamma(2n\underline{S})$ because it will be applied in this way in the proof of Theorem 2.2. The value $2n$ could be replaced with any other constant c_{15} and then c_{13} would depend on c_{15}. Notice also that the chain of algebraic subgroups associated (as in Sect. 4.1) with the parameters $2nS_1, \ldots, 2nS_l$ might be distinct from the chain $(H_i)_{0\le i\le r}$ associated with $S_1, \ldots, S_l$ (which appears in Lemma 4.9).

Proof of Lemma 4.9 Lemma 3.4 applied to $\Gamma(2n\underline{S})$, $H' = H$ and $H'' = H_i$ yields

$$\mathrm{Card}(\Gamma(2n\underline{S}) \cap H \bmod H_i) < c_{16} \exp(\varphi_{\underline{S}}(H) - \varphi_{\underline{S}}(H_i))$$

where c_{16} depends only on $\gamma_1, \ldots, \gamma_l$ and n, using Remark 3.6 and Lemma 4.8. Since

$$\mathfrak{S}_i^{\dim(H/H_i)} = \exp\left[\frac{\dim H - \dim H_i}{\dim H_{i+1} - \dim H_i}(\varphi_{\underline{S}}(H_{i+1}) - \varphi_{\underline{S}}(H_i))\right],$$

Lemma 4.9 follows using Eq. (7) of Proposition 4.1.

The next lemma corresponds, when $S_1 = \cdots = S_l$, to the result $\mu(\Gamma \cap H_i, H_i) = \frac{\mathrm{rk}(\Gamma\cap H_i) - \mathrm{rk}(\Gamma \cap H_{i-1})}{\dim H_i - \dim H_{i-1}}$.

Lemma 4.10 *For any $\varepsilon > 0$ there exists a positive constant c_{17}, which depends only on ε, G, $\gamma_1, \ldots, \gamma_l$ but not on $S_1, \ldots, S_l$, with the following property: for any $i \in \{1, \ldots, r\}$ and any connected algebraic subgroup H such that $H \subsetneq H_i$, we have*

$$\mathrm{Card}\left(\Gamma\left(\frac{\varepsilon}{\dim H_i}\underline{S}\right) \cap H_i \bmod H\right) > c_{17}\mathfrak{S}_{i-1}^{\dim(H_i/H)}.$$

This lemma asserts that in applying Proposition 3.1 to $\Omega = \Gamma(\frac{\varepsilon}{\dim H_i}\underline{S}) \cap H_i$ in the algebraic group H_i, it is enough to check assumption (4) with $H = H_{i-1}$ (with a larger value of c_7, though), so that this proposition applies as soon as $D < c_{18}\mathfrak{S}_{i-1}T$ for some constant c_{18}. Indeed $\mathfrak{S}_{i-1}^{\dim(H_i/H_{i-1})}$ is equal, up to a multiplicative constant, to the cardinality of $(\Gamma(\frac{\varepsilon}{\dim H_i}\underline{S}) \cap H_i) \bmod H_{i-1}$: the conclusion of Lemma 4.10 is an equality for $H = H_{i-1}$, up to the value of c_{17}.

As for Lemma 4.9 above, we prove Lemma 4.10 for $\Gamma(\frac{\varepsilon}{\dim H_i}\underline{S})$ because it will be applied in this way in the proof of Theorem 2.1. The value $\frac{\varepsilon}{\dim H_i}$ could be replaced with another constant.

Proof of Lemma 4.10 Applying Lemma 3.4 to $\Gamma(\frac{\varepsilon}{\dim H_i}\underline{S})$ with $H' = H_i$ and $H'' = H$ yields, using Remark 3.6 and Lemma 4.8:

$$\mathrm{Card}\left(\Gamma\left(\frac{\varepsilon}{\dim H_i}\underline{S}\right) \cap H_i \bmod H\right) > c_{19} \exp(\varphi_{\underline{S}}(H_i) - \varphi_{\underline{S}}(H))$$

where c_{19} depends only on $\gamma_1, \ldots, \gamma_l$ and ε. Since

$$\mathfrak{S}_{i-1}^{\dim(H_i/H)} = \exp\left[\frac{\dim H_i - \dim H}{\dim H_i - \dim H_{i-1}}(\varphi_{\underline{S}}(H_i) - \varphi_{\underline{S}}(H_{i-1}))\right],$$

Equation (10) in Proposition 4.3 enables one to conclude the proof of Lemma 4.9.

4.5 The case $S_1 = \cdots = S_l$

In this subsection we assume that $S_1 = \cdots = S_l = S > 1$ as in the introduction. We shall deduce the results announced there from those proved previously in the general setting.

To begin with, we have $\varphi_{\underline{S}}(K) = \text{rk}(\Gamma \cap K) \log S$ for any connected algebraic subgroup K of G. In particular $\varphi_{\underline{S}}(K)$ does not depend on $\gamma_1, \dots, \gamma_l$, but only on Γ, S and K. It is easily seen that the factor $\log S$ cancels out in all inequalities like Eq. (7), so that the chain of algebraic subgroups $(H_i)_{0 \le i \le r}$ depends only on Γ but neither on $\gamma_1, \dots, \gamma_l$ nor on S (the latter point is also a consequence of Lemma 4.7).

On the other hand Eq. (7) is equivalent to

$$\frac{\text{rk}(\Gamma \cap K) - \text{rk}(\Gamma \cap H_i)}{\dim K - \dim H_i} \le \frac{\text{rk}(\Gamma \cap H_{i+1}) - \text{rk}(\Gamma \cap H_i)}{\dim H_{i+1} - \dim H_i}.$$

Since $H_0 = \{0\}$, Proposition 4.1 asserts that $\frac{\text{rk}(\Gamma \cap K)}{\dim K} \le \frac{\text{rk}(\Gamma \cap H_1)}{\dim H_1}$, and if equality holds then $K \subset H_1$. This implies $\mu^*(\Gamma, G) = \frac{\text{rk}(\Gamma \cap H_1)}{\dim H_1}$. In the same way, $H_r = G$ so that Proposition 4.3 yields $\frac{\text{rk}(\Gamma) - \text{rk}(\Gamma \cap K)}{\dim G - \dim K} \ge \frac{\text{rk}(\Gamma) - \text{rk}(\Gamma \cap H_{r-1})}{\dim G - \dim H_{r-1}}$, and if equality holds then $H_{r-1} \subset K$. In particular we have $\mu(\Gamma, G) = \frac{\text{rk}(\Gamma) - \text{rk}(\Gamma \cap H_{r-1})}{\dim G - \dim H_{r-1}}$.

Letting $\mu_i = \frac{\text{rk}(\Gamma \cap H_{i+1}) - \text{rk}(\Gamma \cap H_i)}{\dim H_{i+1} - \dim H_i}$ as in the introduction, with $i \in \{0, \dots, r-1\}$, Proposition 4.4 asserts that $\mu_{r-1} < \dots < \mu_1 < \mu_0$; in the notation of Remark 4.5 this is clear because $\mu_i \log S$ is the slope of the line $(M_{H_i} M_{H_{i+1}})$. Lemma 3.4 proves that $\text{Card}((\Gamma(\underline{S}) \cap H_{i+1}) \bmod H_i)$ is equal, up to a multiplicative constant depending only on $\gamma_1, \dots, \gamma_l$, to $S^{\text{rk}(\Gamma \cap H_{i+1}) - \text{rk}(\Gamma \cap H_i)} = S^{\mu_i (\dim H_{i+1} - \dim H_i)}$; the quantity denoted by $\mathfrak{S}_i$ in this paper equals S^{μ_i} in this case.

Let us fix i, j such that $0 \le i < j \le r$. Then the chain of algebraic subgroups associated with $\Gamma \cap H_j \bmod H_i$ in the algebraic group H_j / H_i is $\{0\} = \frac{H_i}{H_i} \subsetneq \frac{H_{i+1}}{H_i} \subsetneq \cdots \subsetneq \frac{H_{j-1}}{H_i} \subsetneq \frac{H_j}{H_i}$. This proves the equalities

$$\mu(\Gamma \cap H_j \bmod H_i, H_j / H_i) = \frac{\text{rk}(\Gamma \cap H_j) - \text{rk}(\Gamma \cap H_{j-1})}{\dim H_j - \dim H_{j-1}}$$

and

$$\mu^*(\Gamma \cap H_j \bmod H_i, H_j / H_i) = \frac{\text{rk}\left(\frac{\Gamma \cap H_{i+1}}{\Gamma \cap H_i}\right)}{\dim(H_{i+1}/H_i)}.$$

5 Proof of the main result

In this section we prove Theorems 2.1 and 2.2. The strategy is to apply the special cases where D is very large or very small, proved in Sect. 3.1, to sub-quotients of G. We deduce Theorem 2.2 from a multiplicity estimate in the algebraic group H_i.

The proof of Theorem 2.1 is more complicated: it involves interpolation estimates in H_{i+1}/H_i and in G/H_{i+1}.

The addition law and the translations on G play a key role in the proof and we begin by establishing our notation to represent these. Let a and b be integers such that there exists a complete system of addition laws on G of bi-degree (a, b). This means that the addition law on G (embedded in $\mathbb{P}^N$) is represented, on every element of a suitable open cover, by a family of bi-homogeneous polynomials of bi-degree (a, b). We may assume (see [6], p. 493) that some open set U in this cover contains Γ and the point γ introduced below at the beginning of the proof of Theorem 2.1. Of course U depends on γ, but this is not important in the proof. There exists a family $E_0(X, Y), \ldots, E_N(X, Y)$ of bi-homogeneous polynomials of bi-degree (a, b) which represents the addition law on $U \times U$; we let $X = (X_0, \ldots, X_N)$, $Y = (Y_0, \ldots, Y_N)$ and $E = (E_0, \ldots, E_N)$.

For any $y \in U$, after choosing a system $(y_0, \ldots, y_N) \in \mathbb{C}^{N+1}$ of projective coordinates of y in $\mathbb{P}^N$ we may consider for any $P \in \mathcal{R}(G)_D$ the polynomial

$$t_y P(X) = P(E(X, y)) \in \mathcal{R}(G)_{aD}.$$

The linear map $t_y : \mathcal{R}(G)_D \to \mathcal{R}(G)_{aD}$ represents the translation by y. Moreover if P vanishes to order at least T (resp. does not vanish) at a given point z and if $z - y \in U$ then $t_y P$ vanishes to order at least T (resp. does not vanish) at the point $z - y$. Of course the map t_y depends on D, E and on the choice of $(y_0, \ldots, y_N)$, but we omit this dependence in the notation t_y.

In the proof we shall use repeatedly the following fact: since H_i may take only finitely many values (see Lemma 4.8), a constant that depends on H_i can actually be chosen in terms of $G, \gamma_1, \ldots, \gamma_l$. Given a constant N (depending on $\gamma_1, \ldots, \gamma_l$), we may also assume that D is a multiple of N. Indeed $\mathcal{B}_{G,\Gamma(\underline{S}),T,D}$ is a non-increasing function of D when the subset $\Gamma(\underline{S})$ and the order of vanishing T are held constant, so it is enough to prove Theorem 2.1 for a slightly smaller value of D (resp. to prove Theorem 2.2 for a slightly larger value of D).

We first prove Theorem 2.2. Let $\gamma \in \Gamma((1 - \varepsilon)\underline{S})$ and assume that $P \in \mathcal{R}(G)_D$ vanishes to order at least T at any point of $\Gamma(\underline{S})$. Consider $Q = t_\gamma P \in \mathcal{R}(G)_{aD}$: then Q vanishes to order at least T at any point of $\Gamma(\varepsilon \underline{S})$. We let $\Omega_1 = \Gamma(\frac{\varepsilon}{\dim H_i} \underline{S}) \cap H_i$ and denote by $Q_1 \in \mathcal{R}(H_i)_{aD}$ the restriction of Q to H_i. Then Q_1 vanishes to order at least T at any point of $\Omega_1[\dim H_i]$. Moreover, for any connected algebraic subgroup $H \subsetneq H_i$ Lemma 4.10 yields

$$\mathrm{Card}(\Omega_1 \bmod H) > c_{17} \mathfrak{S}_{i-1}^{\dim(H_i/H)}$$

where c_{17} depends only on $\gamma_1, \ldots, \gamma_l$. Since $D < c_5^{-1} \mathfrak{S}_{i-1} T$ this implies (provided that c_5 is large enough) that

$$\mathrm{Card}(\Omega_1 \bmod H) T^{\dim(H_i/H)} > c_7 D^{\dim(H_i/H)}$$

where c_7 is the constant in Proposition 3.1 applied in the algebraic group H_i. This Proposition yields $Q_1 = 0 \in \mathcal{R}(H_i)_{aD}$ so that P vanishes identically on $\gamma + (H_i \cap U)$.

Now the zero element of G belongs to $H_i \cap U$ (because we have assumed $\Gamma \subset U$), so that $H_i \cap U$ is non-empty. Since U is an open subset of G, we obtain that $H_i \cap U$ is Zariski dense in H_i. This density does not change by translation, so that P vanishes identically on $\gamma + H_i$ because it vanishes on $\gamma + (H_i \cap U)$. This concludes the proof of Theorem 2.2.

We now prove Theorem 2.1. We argue by decreasing induction on i. Letting $\mathfrak{S}_r = 1$ this result is meaningful for $i = r$, and trivially true since $H_r = G$. From now on, we let $i \in \{0, \dots, r-1\}$ and assume that Theorem 2.1 holds for $i+1$.

Assume there exists $\gamma \in \mathcal{B}_{G,\Gamma(\underline{S}),T,D}$ with $\gamma \notin \Gamma(\underline{S}) + H_i$. Since Theorem 2.1 holds for $i+1$ and $\mathfrak{S}_{i+1} \leq \mathfrak{S}_i$, we have $\gamma \in \Gamma(\underline{S}) + H_{i+1}$. Let $\beta \in \Gamma(\underline{S})$ and $h \in H_{i+1}$ be such that $\gamma = \beta + h$. Consider $\Omega_2 = (-\beta + \Gamma(\underline{S})) \cap H_{i+1}$, and notice that $h \notin \Omega_2 + H_i$ since $\gamma \notin \Gamma(\underline{S}) + H_i$.

Now H_{i+1}/H_i is a commutative algebraic group, so we can choose (arbitrarily) a projective embedding $H_{i+1}/H_i \hookrightarrow \mathbb{P}^{M_i}$. With respect to this embedding (and that of H_{i+1} in $\mathbb{P}^N$), the projection $H_{i+1} \to H_{i+1}/H_i$ is given, on an open subset of H_{i+1} which contains $\Omega_2 \cup \{h\}$, by homogeneous polynomials $R_{i,0}, \dots, R_{i,M_i}$ of the same degree, say a_i. It is possible to ensure that a_i depends only on the embeddings of H_{i+1}/H_i and H_{i+1}, and not on Ω_2 or h. We put $R_i = (R_{i,0}, \dots, R_{i,M_i})$.

Let $\hbar = h \bmod H_i$ and $\overline{\Omega}_2 = \Omega_2 \bmod H_i$, so that $\hbar \notin \overline{\Omega}_2$. Then $(\overline{\Omega}_2 \backslash \{\hbar\})\{\dim(H_{i+1}/H_i)\}$ is a subset of $\Gamma(2n\underline{S}) \cap H_{i+1} \bmod H_i$ since $\dim(H_{i+1}/H_i) \leq \dim G = n$, so that Lemma 4.9 yields

$$\operatorname{Card}\,((\overline{\Omega}_2 \backslash \{\hbar\})\{\dim(H_{i+1}/H_i)\} \cap \overline{H}) < c_{13}\mathfrak{S}_i^{\dim \overline{H}}$$

for any connected algebraic subgroup H of G such that $H_i \subsetneq H \subset H_{i+1}$, where $\overline{H} = H/H_i$. Since $D > c_4 \mathfrak{S}_i T$, Proposition 3.3 applies in the algebraic group H_{i+1}/H_i if c_4 is sufficiently large: it provides $P_1 \in \mathcal{R}(H_{i+1}/H_i)_{D/2aa_i}$ which vanishes to order at least T at any point of $\overline{\Omega}_2$ and does not vanish at $\hbar$ (because $\hbar \notin \overline{\Omega}_2$). Then $P_1 \circ R_i \in \mathcal{R}(H_{i+1})_{D/2a}$ vanishes to order at least T at any point of $\Omega_2 = (-\beta + \Gamma(\underline{S})) \cap H_{i+1}$, and does not vanish at the point h. Choose $P_2 \in \mathcal{R}(G)_{D/2a}$ such that $P_1 \circ R_i$ is the restriction of P_2 to H_{i+1} so that the same vanishing and non-vanishing properties hold for P_2. Then $P_3 = t_{-\beta} P_2 \in \mathcal{R}(G)_{D/2}$ vanishes to order at least T at any point of $\Gamma(\underline{S}) \cap (\beta + H_{i+1})$, and does not vanish at the point $\beta + h = \gamma$ (because U contains Γ and γ).

On the other hand, we can choose an embedding of the commutative algebraic group G/H_{i+1} in projective space $\mathbb{P}^{M'_i}$. On an open subset of G which contains $\Gamma \cup \{\gamma\}$ the projection $G \to G/H_{i+1}$ is given as above by a family $R'_i = (R'_{i,0}, \dots, R'_{i,M'_i})$ of homogeneous polynomials of the same degree a'_i and this degree can be made independent from γ.

Now let $\overline{\Omega}_3 = \overline{\Gamma(\underline{S})} \backslash \{\overline{\beta}\}$, where $\overline{\Gamma(\underline{S})}$ and $\overline{\beta}$ are the images of $\Gamma(\underline{S})$ and β in G/H_{i+1}. Then $\overline{\Omega}_3\{\dim(G/H_{i+1})\} \subset \Gamma(2n\underline{S}) \bmod H_{i+1}$ since $\dim(G/H_{i+1}) \leq n$, so that Lemma 4.9 (with $i+1$ instead of i) yields

$$\operatorname{Card}(\overline{\Omega}_3\{\dim(G/H_{i+1})\} \cap \overline{H}) < c_{13}\mathfrak{S}_{i+1}^{\dim \overline{H}}$$

for any connected algebraic subgroup H such that $H_{i+1} \subsetneq H \subset G$, with $\overline{H} = H/H_{i+1}$. Now we have $D > c_4\mathfrak{S}_i T \geq c_4\mathfrak{S}_{i+1}T$; if c_4 is sufficiently large then Proposition 3.3 (applied in G/H_{i+1}) provides $Q_1 \in \mathcal{R}(G/H_{i+1})_{D/2a'_i}$ which vanishes to order at least T at any point of $\overline{\Omega}_3 = \overline{\Gamma(\underline{S})}\backslash\{\overline{\beta}\}$, and does not vanish at the point $\overline{\beta}$. Then $Q_2 = Q_1 \circ R'_i \in \mathcal{R}(G)_{D/2}$ vanishes to order at least T at any point of $\Gamma(\underline{S})\backslash(\beta + H_{i+1})$, and does not vanish at $\gamma = \beta + h$.

We consider now $P = P_3 Q_2 \in \mathcal{R}(G)_D$. We have $P(\gamma) \neq 0$ because $P_3(\gamma) \neq 0$ and $Q_2(\gamma) \neq 0$. For $\gamma' \in \Gamma(\underline{S})$, if $\gamma' \in \beta + H_{i+1}$ then P_3 vanishes to order at least T at γ'; otherwise Q_2 does. Therefore P vanishes to order at least T at any point of $\Gamma(\underline{S})$. Since $P(\gamma) \neq 0$ and $\gamma \in \mathcal{B}_{G,\Gamma(\underline{S}),T,D}$, this is a contradiction.

6 Possible generalizations

It would be interesting to generalize Theorems 2.1 and 2.2 in at least two directions.

The first one would be to replace $\Gamma(\underline{S})$ with a fixed finite set Ω. A first step would be to find constants c_4 and c_5 in Theorems 2.1 and 2.2 which do not depend on $\gamma_1, \dots, \gamma_l$. Interpolation and multiplicity estimates are known in this setting (see Sect. 3.1), but the proof leaves no hope to obtain this result unless new ideas are used. For instance, the chain of subgroups $(H_i)_{0 \leq i \leq r}$ constructed in Sect. 4 does not depend on the torsion part of Γ: it is trivial as soon as Γ has rank 0. For an analogous reason, in Masser's interpolation estimate [5] (and in the first author's generalization [3]), the constant depends also on $\gamma_1, \dots, \gamma_l$. The case of an arbitrary finite set Ω was dealt with in [4] using a more geometric approach in terms of Seshadri constants.

The second way to generalize Theorems 2.1 and 2.2 would be to consider vanishing along analytic subgroups of G. The only problem is that the interpolation estimate of [4] (stated above as Proposition 3.2) is not known in this setting. The only available interpolation estimate is the one proved by the first author [3], but it is not sufficiently precise to deduce the corresponding generalization of Proposition 3.3 (even with $\Omega = \Gamma(\underline{S})$).

Let us introduce this setting more precisely, and mention how the other tools used in this paper generalize to it. Let $T_0(G)$ denote the tangent space to G at 0, seen as the space of translation-invariant vector fields on G. Let W be a subspace of $T_0(G)$, of dimension $d \geq 0$, and $\underline{\partial} = (\partial_1, \dots, \partial_d)$ be a basis of W. For a family $\underline{T} = (T_1, \dots, T_d)$ of d positive real numbers, we let $\mathbb{N}^d_{\underline{T}}$ be the set of all $\sigma = (\sigma_1, \dots, \sigma_d)$ such that $0 \leq \sigma_j < T_j$ for any $j \in \{1, \dots, d\}$.

We denote by Op_W the set of all polynomials in $\partial_1, \dots, \partial_d$, i.e. the space of differential operators along W, and by $\mathrm{Op}_{\underline{\partial},\underline{T}}$ the subspace of Op_W spanned by the monomials $\partial^\sigma = \partial_1^{\sigma_1} \dots \partial_d^{\sigma_d}$ for $\sigma \in \mathbb{N}^d_{\underline{T}}$. We assume (without loss of generality: see [6], p. 492) that $\Gamma \subset \{X_0 \neq 0\} \subset \mathbb{P}^N$, and we say that a polynomial $P \in \mathcal{R}(G)_D$ vanishes up to order $\underline{T}$ along W at a point $\gamma \in \Gamma$ if $\partial^\sigma(P/X_0^D)(\gamma) = 0$ for any $\sigma \in \mathbb{N}^d_{\underline{T}}$. If $T_1 = \dots = T_d = T$ and $W = T_0(G)$, this means that P vanishes to order at least T at γ.

With this notation, one would replace everywhere "vanishing to order at least T" with "vanishing up to order $\underline{T}$ along W", and $T^{\dim H}$ with $\dim(\mathrm{Op}_{W \cap T_0(H)} \cap \mathrm{Op}_{\underline{\partial},\underline{T}})$.

The corresponding multiplicity estimate (Proposition 3.1) has been proved by Philippon [10]. For any $j \in \{0, \dots, d\}$ let $W_j = \mathrm{Span}(\partial_1, \dots, \partial_j)$. Then $\dim(W_j \cap T_0(H))$ plays a role analogous to the one of $\mathrm{rk}(\Gamma_j \cap H)$. Given $S_1 \geq \cdots \geq S_l \geq 1$ and $T_1 \geq \cdots \geq T_d \geq 1$, let us define

$$\varphi_{\underline{S},\underline{T}}(K) = \sum_{j=1}^{l} \mathrm{rk}\left(\frac{\Gamma_j \cap K}{\Gamma_{j-1} \cap K}\right) \log S_j + \sum_{j=1}^{d} \dim\left(\frac{W_j \cap T_0(K)}{W_{j-1} \cap T_0(K)}\right) \log T_j$$

for any connected algebraic subgroup K of G. Then in Sect. 4.1 it is possible to replace $\varphi_{\underline{S}}$ with $\varphi_{\underline{S},\underline{T}}$. The chain of subgroups constructed in this way depends on $\gamma_1, \dots, \gamma_l$, $\partial_1, \dots, \partial_d$, $S_1, \dots, S_l$, $T_1, \dots, T_d$. In Lemmas 4.9 and 4.10, the left hand side of the inequalities has to be multiplied by $\dim\left(\frac{\mathrm{Op}_{W \cap T_0(H')} \cap \mathrm{Op}_{\partial,\underline{T}}}{\mathrm{Op}_{W \cap T_0(H'')} \cap \mathrm{Op}_{\partial,\underline{T}}}\right)$ with $\{H', H''\} = \{H, H_i\}$. Moreover in the definition of $\mathfrak{S}_i$, $\varphi_{\underline{S}}$ should also be replaced with $\varphi_{\underline{S},\underline{T}}$ (see Eq. (17)). It seems reasonable to conjecture that Theorems 2.1 and 2.2 hold in this setting, provided $\mathfrak{S}_i T$ is replaced with this new value of $\mathfrak{S}_i$.

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