On Siegel's problem for E-functions

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Abstract

Siegel defined in 1929 two classes of power series, the E-functions and G-functions, which generalize the Diophantine properties of the exponential and logarithmic functions respectively. He asked whether any E-function can be represented as a polynomial with algebraic coefficients in a finite number of E-functions of the form ${}_pF_q(\lambda z^{q-p+1})$, $q \geq p \geq 1$, with rational parameters. The case of E-functions of differential order less than or equal to 2 was settled in the affirmative by Gorelov in 2004, but Siegel's question is open for higher order. We prove here that if Siegel's question has a positive answer, then the ring $\mathbf{G}$ of values taken by analytic continuations of G-functions at algebraic points must be a subring of the relatively "small" ring $\mathbf{H}$ generated by algebraic numbers, $1/\pi$ and the values of the derivatives of the Gamma function at rational points. Because that inclusion seems unlikely (and contradicts standard conjectures), this points towards a negative answer to Siegel's question in general. As intermediate steps, we first prove that any element of $\mathbf{G}$ is a coefficient of the asymptotic expansion of a suitable E-function, which completes previous results of ours. We then prove (in two steps) that the coefficients of the asymptotic expansion of a hypergeometric E-function with rational parameters are in $\mathbf{H}$. Finally, we prove a similar result for G-functions.

1 Introduction

Siegel [25] introduced in 1929 the notion of E-function as a generalization of the exponential and Bessel functions. We denote by $\overline{\mathbb{Q}} \subset \mathbb{C}$ the field of algebraic numbers.

Definition 1. *A power series* $F(z) = \sum_{n=0}^{\infty} \frac{a_n}{n!} z^n \in \overline{\mathbb{Q}}[[z]]$ *is an E-function if*

(i) $F(z)$ *is solution of a non-zero linear differential equation with coefficients in* $\overline{\mathbb{Q}}(z)$.

(ii) There exists $C > 0$ *such that for any* $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ *and any* $n \geq 0$, $|\sigma(a_n)| \leq C^{n+1}$.

(iii) There exists $D > 0$ *and a sequence of integers* d_n, *with* $1 \leq d_n \leq D^{n+1}$, *such that* $d_n a_m$ *are algebraic integers for all* $m \leq n$.

Siegel's original definition was in fact slightly more general than the above and we shall make some remarks about this in §2.1. Note that (i) implies that the a_n's all lie in a certain number field $\mathbb{K}$, so that in (ii) there are only finitely many Galois conjugates $\sigma(a_n)$ of a_n to consider, with $\sigma \in \mathrm{Gal}(\mathbb{K}/\mathbb{Q})$ (assuming for simplicity that $\mathbb{K}$ is a Galois extension of $\mathbb{Q}$). E-functions are entire, and they form a ring stable under $\frac{\mathrm{d}}{\mathrm{d}z}$ and $\int_0^z$. A power series $\sum_{n=0}^{\infty} a_n z^n \in \overline{\mathbb{Q}}[[z]]$ is said to be a G-function if $\sum_{n=0}^{\infty} \frac{a_n}{n!} z^n$ is an E-function. Examples of G-functions include algebraic functions over $\overline{\mathbb{Q}}(z)$ regular at 0 (this uses Eisenstein's Theorem) and polylogarithms (defined in §2.2).

The generalized hypergeometric series is defined as

$$ {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n}{(1)_n (b_1)_n \cdots (b_q)_n} z^n $$

where $p, q \geq 0$ and $(a)_0 := 1$, $(a)_n := a(a+1)\cdots(a+n-1)$ if $n \geq 1$. The parameters a_j and b_j are in $\mathbb{C}$, with the restriction that $b_j \notin \mathbb{Z}_{\leq 0}$ so that $(b_j)_n \neq 0$ for all $n \geq 0$. We shall also denote it by ${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z]$.

Siegel proved in [25] and [26, §9] that, for any integers $q \geq p \geq 0$, the series

$$ {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z^{q-p+1}\right] \tag{1.1} $$

is an E-function (in the sense of this paper) when $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j. He called them *hypergeometric E-functions*. The simplest examples are $\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!} = {}_1F_1[1; 1; z]$ and the Bessel function $J_0(z) := \sum_{n=0}^{\infty} \frac{(iz/2)^{2n}}{n!^2} = {}_1F_2[1; 1, 1; (iz/2)^2]$. If $a_j \in \mathbb{Z}_{\leq 0}$ for some j, then the series reduces to a polynomial. Any polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form ${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; \lambda z^{q-p+1}]$, with parameters $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $\lambda \in \overline{\mathbb{Q}}$, is an E-function.

The E-functions

$$ L(z) := \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{n}{k} \binom{n+k}{n} \right) \frac{z^n}{n!}, \quad H(z) := \sum_{n=0}^{\infty} \left(\sum_{k=1}^{n} \frac{1}{k} \right) \frac{z^n}{n!} $$

are not of the hypergeometric type (1.1) since the quotient a_{n+1}/a_n of two successive terms is not a rational function of n, but we have

$$ \begin{aligned} L(z) &= e^{(3-2\sqrt{2})z} \cdot {}_1F_1\big[1/2; 1; 4\sqrt{2}z\big], \\ H(z) &= ze^z \cdot {}_2F_2\big[1, 1; 2, 2; -z\big]. \end{aligned} $$

These puzzling identities, amongst others, naturally suggest to study further the role played by hypergeometric series in the theory of E-functions. In fact, Siegel had already stated [25, p. 225] a problem that we reformulate as the following question.

Question 1 (Siegel)**.** *Is it possible to write any E-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of E-functions of the form ${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; \lambda z^{q-p+1}]$, with parameters $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ and $\lambda \in \overline{\mathbb{Q}}$?*

It must be understood that λ, p, q and $q-p$ can take various values in the polynomial. Siegel's original statement is given in §2.1 along with some comments. Gorelov [14, p. 514, Theorem 1] proved that the answer to Siegel's question is positive if the E-function (in Siegel's original sense) satisfies a linear differential equation with coefficients in $\overline{\mathbb{Q}}(z)$ of order ≤ 2. He used some of the pioneering results of André [3] on E-operators. A version of Gorelov's theorem was reproved in [24] for E-functions as in Definition 1 with a method also based on André's results, but somewhat different in the details. It seems difficult to generalize any one of these two approaches when the order is ≥ 3, though Gorelov [15] also obtained further results in the case of E-functions solution of a linear inhomogeneous differential equation of order 2 with coefficients in $\overline{\mathbb{Q}}(z)$, like $H(z)$ above.

In this paper, we adopt another point of view on Siegel's question. To begin with, let us define two subrings of $\mathbb{C}$; the first one was introduced and studied in [9].

Definition 2. $\mathbf{G}$ *denotes the ring of G-values, i.e. the values taken at algebraic points by the analytic continuations of all G-functions.*

$\mathbf{H}$ *denotes the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and the values $\Gamma^{(n)}(r)$, $n \geq 0$, $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$.*

Here, $\Gamma(s) := \int_0^\infty t^{s-1}e^{-t}dt$ is the usual Gamma function for $\Re(s) > 0$, that can be analytically continued to $\mathbb{C} \setminus \mathbb{Z}_{\leq 0}$. We can now state our main result.

Theorem 1. *At least one of the following statements is true:*

(i) $\mathbf{G} \subset \mathbf{H}$*;*

(ii) Siegel's question has a negative answer.

We provide in §2.2 another description of the ring $\mathbf{H}$, and explain there why the inclusion $\mathbf{G} \subset \mathbf{H}$ (and therefore a positive answer to Siegel's question) seems very unlikely; as Y. André, F. Brown and J. Fresán pointed out to us, this inclusion contradicts standard conjectures about (exponential) periods and motivic Galois groups.

The paper is organized as follows. In §2, we comment on Siegel's original formulation of his problem and make some remarks on the ring $\mathbf{H}$. In §3, we prove that any element of $\mathbf{G}$ is a coefficient of the asymptotic expansion of a suitable E-function (Theorem 3). In §4, we prove that the coefficients of the asymptotic expansion of any hypergeometric series ${}_pF_p(z)$ with rational parameters are in $\mathbf{H}$; then we generalize this result to any ${}_pF_q(z^{q-p+1})$ in §5. We complete the proof of Theorem 1 in §6 by comparing the results of the previous sections. Finally, we consider in §7 an analogous problem for G-functions and prove a similar result to Theorem 1. We emphasize that the proof of Theorem 1 uses, in particular, various results obtained in [9] and [10], the proofs of which are crucially based on a deep theorem due to André, Chudnovsky and Katz on the structure of non-zero minimal differential equations satisfied by G-functions; see §7.1 and the references given there.

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2 Comments on Theorem 1

2.1 Siegel's formulation of his problem

In [25] and [26, Chapter II, §9], Siegel proved that the series of the type (1.1) with rational parameters are E-functions, and named them "hypergeometric E-functions". He wrote [26, p. 58]: *Performing the substitution $x \mapsto \lambda x$ for arbitrary algebraic λ and taking any polynomial in x and finitely many hypergeometric E-functions, with algebraic coefficients, we get again an E-function satisfying a homogeneous linear differential equation whose coefficients are rational functions of x. It would be interesting to find out whether all such E-functions can be constructed in the preceding manner.*

Siegel obviously considered E-functions in his sense, which we recall here: in Definition 1, (i) is unchanged but (ii) and (iii) have to be replaced by

(ii') For any $\varepsilon > 0$ and for any $\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$, there exists $N(\varepsilon, \sigma) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon, \sigma)$, $|\sigma(a_n)| \leq n!^{\varepsilon}$.

(iii') There exists a sequence of positive integers d_n such that $d_n a_m$ are algebraic integers for all $m \leq n$ and such that for any $\varepsilon > 0$ there exists $N(\varepsilon) \in \mathbb{N}$ such that for any $n \geq N(\varepsilon)$, $|d_n| \leq n!^{\varepsilon}$.

Again, by (i), there are only finitely many σ to consider for a given E-function. We have chosen to formulate his problem for E-functions in the restricted sense of Definition 1 because the proof of Theorem 1 is based on results which are currently proven only in this sense. However, *a fortiori*, Theorem 1 obviously holds *verbatim* if one considers E-functions in Siegel's sense. Note also that $z = {}_1F_1[0; 1; z] - {}_1F_1[-1; 1; z]$ so that Siegel could have formulated his problem in terms of hypergeometric series only, as we did. Note that the E-function $\frac{1}{z}(e^z - e^{-z}) = \frac{1}{z}{}_1F_1[1; 1; z] - \frac{1}{z}{}_1F_1[1; 1; -z]$ is not a counter-example to Siegel's problem because we also have $\frac{1}{z}(e^z - e^{-z}) = {}_1F_1[1; 2; z] - {}_1F_1[1; 2; -z]$; there is no unicity of the representation of E-functions by polynomials in hypergeometric ones.

Furthermore, the series in (1.1) may be an E-function even if some of its parameters are not rational numbers. For instance, for every $\alpha \in \overline{\mathbb{Q}} \setminus \mathbb{Z}_{\leq 0}$,

$$ {}_1F_1\begin{bmatrix} \alpha+1 \\ \alpha \end{bmatrix}; z\Bigg] = \sum_{n=0}^{\infty} \frac{(\alpha+1)_n}{(1)_n(\alpha)_n} z^n = \sum_{n=0}^{\infty} \frac{\alpha+n}{\alpha} \cdot \frac{z^n}{n!} = \big(1 + \frac{z}{\alpha}\big)e^z $$

is an E-function. Thus, even though Siegel did not consider such examples, the notion of "hypergeometric E-functions" could be interpreted in a broader way than he did in his problem. Galochkin [12] proved the following non-trivial characterization, where E-functions are understood in Siegel's sense.

Theorem (Galochkin). *Let $p, q \geq 1$, $q \geq p$, $a_1, \ldots, a_p, b_1, \ldots, b_q \in (\mathbb{C} \setminus \mathbb{Z}_{\leq 0})^{p+q}$ be such that $a_i \neq b_j$ for all i, j. Then, the series ${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z^{q-p+1}]$ is an E-function if and only if the following two conditions hold:*

(*i*) *The* a_j*'s and* b_j*'s are all in* $\overline{\mathbb{Q}}$*;*

(*ii*) *The* a_j*'s and* b_j*'s which are not rational (if any) can be grouped in* $k \le p$ *pairs* $(a_{j_1}, b_{j_1}), \dots, (a_{j_k}, b_{j_k})$ *such that* $a_{j_\ell} - b_{j_\ell} \in \mathbb{N}$.

It follows that hypergeometric E-functions ${}_pF_q(z^{q-p+1})$ with arbitrary parameters are in fact $\overline{\mathbb{Q}}$-linear combinations of hypergeometric E-functions ${}_{p'}F_{q'}(z^{q'-p'+1})$ (with various values of p' and q') with rational parameters. Hence, there is no loss of generality in considering the latter instead of the former in Siegel's problem.

2.2 The ring H

For $x \in \mathbb{C} \setminus \mathbb{Z}_{\le 0}$, we define the Digamma function

$$\Psi(x) := \frac{\Gamma'(x)}{\Gamma(x)} = -\gamma + \sum_{n=0}^{\infty} \Big(\frac{1}{n+1} - \frac{1}{n+x}\Big),$$

where γ is Euler's constant $\lim_{n\to+\infty}(\sum_{k=1}^n 1/k - \log(n))$, and the Hurwitz zeta function

$$\zeta(s,x) := \frac{(-1)^s}{(s-1)!}\Psi^{(s-1)}(x) = \sum_{n=0}^{\infty} \frac{1}{(n+x)^s}, \quad s \in \mathbb{N},\ s \ge 2.$$

The polylogarithms are defined by

$$\mathrm{Li}_s(z) := \sum_{n=1}^{\infty} \frac{z^n}{n^s}, \quad s \in \mathbb{N}^* = \mathbb{N} \setminus \{0\},$$

where the series converges for $|z| \le 1$ (except at $z = 1$ if $s = 1$). The Beta function is defined as

$$\mathrm{B}(x,y) := \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$$

for $x, y \in \mathbb{C}$ which are not singularities of Beta coming from the poles of Γ at non-positive integers.

In this section, we shall prove the following result.

Proposition 1. *The ring* $\mathbf{H}$ *is generated by* $\overline{\mathbb{Q}}$*,* γ*,* $1/\pi$*,* $\mathrm{Li}_s(e^{2i\pi r})$ ($s \in \mathbb{N}^*$*,* $r \in \mathbb{Q}$*,* $(s, e^{2i\pi r}) \ne (1,1)$)*,* $\log(q)$ ($q \in \mathbb{N}^*$) *and* $\Gamma(r)$ ($r \in \mathbb{Q} \setminus \mathbb{Z}_{\le 0}$)*.*

For any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\le 0}$*,* $\Gamma(r)$ *is a unit of* $\mathbf{H}$*.*

Proof. We first prove that for any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\le 0}$, $\Gamma(r)$ is a unit of $\mathbf{H}$. Indeed, if $r \in \mathbb{N}^*$, then $\Gamma(r) \in \mathbb{N}^*$ and $1/\Gamma(r) \in \mathbb{Q} \subset \mathbf{H}$. If $r \in \mathbb{Q} \setminus \mathbb{Z}$, then by the reflection formula [5, p. 9, Theorem 1.2.1], we have

$$\frac{1}{\Gamma(r)} = \frac{1}{\pi}\sin(\pi r)\Gamma(1-r) \in \mathbf{H}$$

because $1/\pi \in \mathbf{H}$, $\sin(\pi r) \in \overline{\mathbb{Q}} \subset \mathbf{H}$ and $\Gamma(1-r) \in \mathbf{H}$.

From the identity $\Gamma'(x) = \Gamma(x)\Psi(x)$ we obtain that, for any integer $s \geq 1$ and any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$,

$$\Psi^{(s)}(r) = \frac{\Gamma^{(s+1)}(r)}{\Gamma(r)} - \sum_{k=0}^{s-1} \binom{s}{k} \frac{\Gamma^{(s-k)}(r)}{\Gamma(r)} \Psi^{(k)}(r).$$

Since $\Gamma(r)$ is a unit of $\mathbf{H}$, we have $\psi(r) \in \mathbf{H}$ and it follows immediately by induction on s that $\zeta(s,r) = \frac{(-1)^s}{(s-1)!}\Psi^{(s-1)}(r) \in \mathbf{H}$ for any $s \geq 2$ and any $r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. In particular $\gamma = -\Psi(1)$ and the values of the Riemann zeta function $\zeta(s) = \zeta(s,1)$ $(s \geq 2)$ are all in $\mathbf{H}$. Note that γ is not expected to be in $\mathbf{G}$ but that $\zeta(s) = \mathrm{Li}_s(1) \in \mathbf{G}$ for all $s \geq 2$ since polylogarithms are G-functions.

Now the identity $\Gamma'(x) = \Gamma(x)\Psi(x)$ implies by induction that, for any $x \in \mathbb{C} \setminus \mathbb{Z}_{\leq 0}$, we have

$$\Gamma^{(s)}(x) = \Gamma(x) P_s\big(\Psi(x), \zeta(2,x), \ldots, \zeta(s,x)\big) \tag{2.1}$$

for some $P_s \in \mathbb{Q}[X_1, \ldots, X_s]$; for instance $P_0 = 1$, $P_1 = X_1$, and $P_2 = X_1^2 + X_2$. Furthermore, set $p, q \in \mathbb{N}$, $0 < p \leq q$, and $\mu := \exp(2i\pi/q)$. Then,

$$\Psi\Big(\frac{p}{q}\Big) = -\gamma - \log(q) - \sum_{n=1}^{q-1} \mu^{-np}\,\mathrm{Li}_1(\mu^n), \tag{2.2}$$

$$\mathrm{Li}_1(\mu^p) = -\frac{1}{q}\sum_{n=1}^{q} \mu^{np}\,\Psi\Big(\frac{n}{q}\Big), \quad p \neq q \tag{2.3}$$

$$\zeta\Big(s, \frac{p}{q}\Big) = q^{s-1}\sum_{n=1}^{q} \mu^{-np}\,\mathrm{Li}_s(\mu^n), \quad s \geq 2 \tag{2.4}$$

$$\mathrm{Li}_s(\mu^p) = \frac{1}{q^s}\sum_{n=1}^{q} \mu^{np}\,\zeta\Big(s, \frac{n}{q}\Big), \quad s \geq 2. \tag{2.5}$$

We refer to [5, p. 13] for a proof of (2.2), and we observe that (2.4) can be proven in a similar (and even simpler) fashion; then (2.3) follows from applying (2.2) q times, and (2.5) from (2.4) in the same way. From (2.3) and (2.5), we deduce that $\mathrm{Li}_s(\mu^p) \in \mathbf{H}$ for any $s \geq 1$ (with $(s, \mu^p) \neq (1,1)$); then (2.2) implies in turn that $\log(q) \in \mathbf{H}$. The numbers $\log(q)$ and $\mathrm{Li}_s(\mu^p)$ are also in $\mathbf{G}$.

The set of Identities (2.1), (2.2) and (2.4) shows that $\mathbf{H}$ coincides with the ring generated by $\overline{\mathbb{Q}}$, $\gamma = -\Psi(1)$, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$ ($s \in \mathbb{N}^*$, $r \in \mathbb{Q}$, $(s, e^{2i\pi r}) \neq (1,1)$), $\log(q)$ ($q \in \mathbb{N}^*$) and $\Gamma(r)$ ($r \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$). □

Other units of $\mathbf{H}$ can be easily identified, which are also units of $\mathbf{G}$ (see [9, Proposition 1 and the remarks following it]): non-zero algebraic numbers and the values of the Beta function $\mathrm{B}(x,y)$ at rational numbers x, y at which it is defined and non-zero. It follows that $\pi = \Gamma(1/2)^2 = \mathrm{B}(1/2, 1/2)$ and more generally $\Gamma(a/b)^b = (a-1)! \prod_{j=1}^{b-1} \mathrm{B}(a/b, ja/b)$,

$a, b \in \mathbb{N}^*$, are units of $\mathbf{H}$. By the Chowla-Selberg formula [16, Eq. (3)], periods of elliptic curves with complex multiplication by $\mathbb{Q}(i)$ are also units of $\mathbf{H}$.

If Siegel's problem has a positive answer, Theorem 1 yields $\mathbf{G} \subset \mathbf{H}$: any element of $\mathbf{G}$ can be written as a polynomial, with algebraic coefficients, in the numbers γ, $1/\pi$, $\mathrm{Li}_s(e^{2i\pi r})$, $\log(q)$ and $\Gamma(r)$ of Proposition 1. This seems extremely doubtful: we recall that $\mathbf{G}$ contains all the multiple zeta values

$$\zeta(s_1, s_2, \dots, s_n) := \sum_{k_1 > k_2 > \cdots > k_n \geq 1} \frac{1}{k_1^{s_1} k_2^{s_2} \cdots k_n^{s_n}},$$

where the integers s_j are such that $s_1 \geq 2, s_2 \geq 1, \dots, s_n \geq 1$, all values at algebraic points of (multiple) polylogarithms, all elliptic and abelian integrals, etc. For now, we have observed that $\mathbf{G} \cap \mathbf{H}$ contains the ring generated by $\overline{\mathbb{Q}}$, $1/\pi$ and all the values $\mathrm{Li}_s(e^{2i\pi r})$, $\log(q)$ and $\mathrm{B}(x, y)$, and it is in fact possible that both rings are equal.

It is interesting to know what can be deduced from the standard conjectures in the domain, such as the Bombieri-Dwork conjecture "G-functions come from geometry", Grothendieck's period conjecture and its extension to exponential periods by Fresán-Jossen; see [4, Partie III] and [11, p. 201, Conjecture 8.2.5]. In this respect we refer to [9, end of §2.2] for a discussion about the conjectural equality of $\mathbf{G}$ and the algebra generated over periods by $1/\pi$. In a private communication to the authors, Y. André wrote the following argument, which he has autorised us to reproduce here. It shows that $\mathbf{G} \subset \mathbf{H}$ cannot hold under these standard conjectures:

Because of the presence of γ, the inclusion $\mathbf{G} \subset \mathbf{H}$ does not contradict Grothendieck's period conjecture but it certainly contradicts its extension to exponential motives. More precisely, in the description of $\mathbf{H}$ given in Proposition 1, we find γ (a period of an exponential motive E_γ, which is a non-classical extension of Tate motives [11, §12.8]), $\frac{1}{2i\pi}$ (a period of the Tate motive $\mathbb{Q}(1)$), $\mathrm{Li}_s(e^{2i\pi r})$ (periods of a mixed Tate motive over $\mathbb{Z}[1/r]$), $\log(q)$ (a period of a 1-motive over $\mathbb{Q}$), and $\Gamma(r)$ whose suitable powers are periods of Abelian varieties with complex multiplication by $\mathbb{Q}(e^{2i\pi r})$. On the one hand, let M be the Tannakian category of mixed motives over $\overline{\mathbb{Q}}$ generated by all these motives. On the other hand, consider a non CM elliptic curve over $\overline{\mathbb{Q}}$ and E its motive. The periods of E are in $\mathbf{G}$: indeed, it is enough to consider the Gauss hypergeometric solutions centered at $1/2$, and to observe that the periods of the fiber at $1/2$ of the Legendre family can be expressed using values of the Beta function at rational points by the Chowla-Selberg formula, and in particular are algebraic in π and $\Gamma(1/4)$. If $\mathbf{G} \subset \mathbf{H}$, the periods of E are in $\mathbf{H}$. By the exponential period conjecture, E would be in M, which is impossible since the motivic Galois group of M is pro-resoluble, while that of E is GL_2.

We conclude this section with a question of J. Fresán: at which differential order can we expect to find a counter-example to Siegel's problem? Based on the above remarks, it seems unlikely that all the values $\mathrm{Li}_s(\alpha)$ are in $\mathbf{H}$, where the integer $s \geq 1$ and $\alpha \in \overline{\mathbb{Q}}$, $|\alpha| < 1$. From the proof of Theorem 3 below (see Remark 2), we deduce that if $\mathrm{Li}_s(\alpha) \notin \mathbf{H}$, then the E-function

$$\sum_{n=2}^{\infty}\Big(\sum_{k=1}^{n-1}\frac{\alpha^k}{k^s}\Big)\frac{z^n}{n!} \tag{2.6}$$

is such a counter-example. It is of differential order at most $s+2$ because it is in the kernel of the differential operator $P(\theta-2)+zQ(\theta-1)+z^2R(\theta)\in\overline{\mathbb{Q}}[z,\frac{\mathrm{d}}{\mathrm{d}z}]$, where $\theta:=z\frac{\mathrm{d}}{\mathrm{d}z}$ and

$$P(x):=(x+2)(x+1)^{s+1},\quad Q(x):=(x+1)(\alpha x^s-(x+1)^s),\quad R(x):=\alpha x^s.$$

It is thus possible that a counter-example to Siegel's problem already exists at the order 3. However, the function $H(z):=\sum_{n=0}^{\infty}(\sum_{k=1}^{n}\frac{1}{k})\frac{z^n}{n!}$ is an example of order 3 to the problem (see the Introduction) and this shows that one must be careful and not draw hasty conclusions here.

3 Elements of G as coefficients of asymptotic expansions of E-functions

3.1 Definition of asymptotic expansions

As in [10, Definition 5], the asymptotic expansions used throughout this paper are defined as follows, for a function f defined on a sector of the form (3.2).

Definition 3. *Let $\theta\in\mathbb{R}$, and $\Sigma\subset\mathbb{C}$, $S\subset\mathbb{C}$, $T\subset\mathbb{N}$ be finite subsets. Given complex numbers $c_{\rho,\alpha,i,n}$, we write*

$$f(x)\approx\sum_{\rho\in\Sigma}e^{\rho x}\sum_{\alpha\in S}\sum_{i\in T}\sum_{n=0}^{\infty}c_{\rho,\alpha,i,n}x^{-n-\alpha}\log(1/x)^i \tag{3.1}$$

and say that the right-hand side is the asymptotic expansion of $f(x)$ in a large sector bisected by the direction θ if there exist $\varepsilon,R,B,C>0$ and, for any $\rho\in\Sigma$, a function $f_\rho(x)$ holomorphic in the angular sector

$$U=\Big\{x\in\mathbb{C},\ |x|\geq R,\ \theta-\frac{\pi}{2}-\varepsilon\leq\arg(x)\leq\theta+\frac{\pi}{2}+\varepsilon\Big\}, \tag{3.2}$$

such that

$$f(x)=\sum_{\rho\in\Sigma}e^{\rho x}f_\rho(x)$$

and

$$\Big|f_\rho(x)-\sum_{\alpha\in S}\sum_{i\in T}\sum_{n=0}^{N-1}c_{\rho,\alpha,i,n}x^{-n-\alpha}\log(1/x)^i\Big|\leq C^N N!|x|^{B-N} \tag{3.3}$$

for any $x\in U$ and any $N\geq 1$.

This means (see [22, §§2.1 and 2.3]) that for any $\rho \in \Sigma$,

$$\sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} x^{-n-\alpha} \log(1/x)^i \tag{3.4}$$

is 1-summable in the direction θ and its sum is $f_\rho(x)$; notice the parameter B in Eq. (3.3), which accounts for the fact that the last term in the expansion (3.3) does not involve $x^{-(N-1)}$ as in the general setting, but $x^{-(N-1+\alpha)}$ for some $\alpha \in S$.

The sector U is a *large sector*, i.e. a set of the form $\{x \in \mathbb{C},\ |x| \geq R,\ \alpha \leq \arg(x) \leq \beta\}$ with $\beta - \alpha > \pi$. Using a result of Watson (see [22, §2.3]) on asymptotic expansions in large sectors, the sum $f_\rho(x)$ is determined by its asymptotic expansion (3.4). Therefore the expansion on the right-hand side of (3.1) determines $f(x)$ on U. The converse holds too: [10, Lemma 1] asserts that a given function $f(x)$ can have at most one asymptotic expansion in the sense of Definition 3. Of course, to obtain this uniqueness property we assume implicitly (throughout this paper) that Σ, S and T in (3.1) cannot trivially be made smaller, and that for any α there exist ρ and i with $c_{\rho,\alpha,i,0} \neq 0$. Moreover in what follows, S will always be a subset of $\mathbb{Q}$.

3.2 Computing asymptotic expansions of E-functions

In this section, we state [10, Theorem 5] which enables one to determine the asymptotic expansion of an E-function. We refer to [10] for more details.

Let $E(x) = \sum_{n=1}^{\infty} a_n x^n$ be a non-polynomial E-function such that $E(0) = 0$; consider $g(z) = \sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}$. Denote by $\overline{\mathcal{F}} : \mathbb{C}[z, \frac{\mathrm{d}}{\mathrm{d}z}] \to \mathbb{C}[x, \frac{\mathrm{d}}{\mathrm{d}x}]$ the Fourier transform of differential operators, i.e. the morphism of $\mathbb{C}$-algebras defined by $\overline{\mathcal{F}}(z) = \frac{\mathrm{d}}{\mathrm{d}x}$ and $\overline{\mathcal{F}}(\frac{\mathrm{d}}{\mathrm{d}z}) = -x$. Then E is annihilated by an E-operator, which can be written as $\overline{\mathcal{F}}\mathcal{D}$ for some G-operator $\mathcal{D} \in \mathbb{C}[z, \frac{\mathrm{d}}{\mathrm{d}z}]$; and we have $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta \mathcal{D} g = 0$ where δ is the degree in z of $\mathcal{D}$. We denote by Σ the set of all finite singularities of $\mathcal{D}$ and let

$$\mathcal{S} = \mathbb{R} \setminus \{\arg(\rho - \rho'), \rho, \rho' \in \Sigma, \rho \neq \rho'\} \tag{3.5}$$

where all the values modulo 2π of the argument of $\rho - \rho'$ are considered, so that $\mathcal{S} + \pi = \mathcal{S}$.

We fix $\theta \in \mathbb{R}$ with $-\theta \in \mathcal{S}$ (so that the direction θ is not anti-Stokes, i.e. not singular, see for instance [20, p. 79]). For any $\rho \in \Sigma$ we denote by $\Delta_\rho = \rho - e^{-i\theta}\mathbb{R}_+$ the closed half-line of angle $-\theta + \pi \bmod 2\pi$ starting at ρ. Since $-\theta \in \mathcal{S}$, no singularity $\rho' \neq \rho$ of $\mathcal{D}$ lies on Δ_ρ: these half-lines are pairwise disjoint. We shall work in the simply connected cut plane obtained from $\mathbb{C}$ by removing the union of these half-lines. We agree that for $\rho \in \Sigma$ and z in the cut plane, $\arg(z - \rho)$ will be chosen in the open interval $(-\theta - \pi, -\theta + \pi)$. This enables one to define $\log(z - \rho)$ and $(z - \rho)^\alpha$ for any $\alpha \in \mathbb{Q}$.

Now let us fix $\rho \in \Sigma$. Combining theorems of André, Chudnovski and Katz (see [3, §3] or [9, §4.1]), there exist (non necessarily distinct) rational numbers $t_1^\rho, \ldots, t_{J(\rho)}^\rho$, with $J(\rho) \geq 1$, and G-functions $g_{j,k}^\rho$, for $1 \leq j \leq J(\rho)$ and $0 \leq k \leq K(\rho, j)$, such that a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta \mathcal{D}$ around ρ (in the above-mentioned cut plane) is given by the functions

$$f^\rho_{j,k}(z-\rho) = (z-\rho)^{t^\rho_j} \sum_{k'=0}^{k} g^\rho_{j,k-k'}(z-\rho) \frac{\log(z-\rho)^{k'}}{k'!} \tag{3.6}$$

for $1 \le j \le J(\rho)$ and $0 \le k \le K(\rho, j)$. Since $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta \mathcal{D} g = 0$ we can expand g in this basis:

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(\rho,j)} \varpi^\rho_{j,k} f^\rho_{j,k}(z-\rho) \tag{3.7}$$

with connection constants $\varpi^\rho_{j,k}$; Theorem 2 of [9] yields $\varpi^\rho_{j,k} \in \mathbf{G}$.

We denote by $\{u\} \in [0, 1)$ the fractional part of a real number u, and agree that all derivatives of this or related functions taken at integers will be right-derivatives. We let

$$y_{\alpha,i}(z) = \sum_{n=0}^{\infty} \frac{1}{i!} \frac{\mathrm{d}^i}{\mathrm{d}t^i} \Big(\frac{\Gamma(1-\{t\})}{\Gamma(-t-n)} \Big)_{|t=\alpha} z^n \in \mathbb{Q}[[z]] \tag{3.8}$$

for $\alpha \in \mathbb{Q}$ and $i \in \mathbb{N}$. We also denote by $\star$ the Hadamard (coefficientwise) product of formal power series in z, and we consider

$$\eta^\rho_{j,k}(1/x) = \sum_{m=0}^{k} (y_{t^\rho_j,m} \star g^\rho_{j,k-m})(1/x) \in \overline{\mathbb{Q}}[[1/x]] \tag{3.9}$$

for any $1 \le j \le J(\rho)$ and $0 \le k \le K(j, \rho)$. Then [10, Theorem 5] is the following result, where $\widehat{\Gamma} := 1/\Gamma$.

Theorem 2. *In a large sector bisected by the direction θ we have the following asymptotic expansion:*

$$E(x) \approx \sum_{\rho\in\Sigma} e^{\rho x} \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(j,\rho)} \varpi^\rho_{j,k} x^{-t^\rho_j - 1} \sum_{i=0}^{k} \Big(\sum_{\ell=0}^{k-i} \frac{(-1)^\ell}{\ell!} \widehat{\Gamma}^{(\ell)}(1-\{t^\rho_j\}) \eta^\rho_{j,k-\ell-i}(1/x) \Big) \frac{\log(1/x)^i}{i!}. \tag{3.10}$$

This theorem applies to any E-function of the form $z \cdot {}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z^{q-p+1}]$ with rational parameters and $q \ge p \ge 1$. Dividing by z, we then obtain an asymptotic expansion of the hypergeometric E-function ${}_pF_q[a_1, \ldots, a_p; b_1, \ldots, b_q; z^{q-p+1}]$, which is unique because it holds in a large sector bisected by the direction θ.

Remark 1. In the formula (3.10), the term corresponding to a triple (ρ, j, k) is zero whenever $f^\rho_{j,k}(z-\rho)$ is holomorphic at $z = \rho$. This observation will be very useful in the proof of Theorem 4 in §5.2.

3.3 G-values as coefficients of asymptotic expansions

We can now state and prove the main result of this section.

Theorem 3. *For any $\xi \in \mathbf{G}$, there exists an E-function $E(x)$ such that for any $\theta \in [-\pi, \pi)$ outside a finite set, ξ is a coefficient of the asymptotic expansion of $E(x)$ in a large sector bisected by θ.*

Proof. Let $\xi \in \mathbf{G}$; we may assume $\xi \neq 0$. Using [9, Theorem 1], there exists a G-function $h(z)$ holomorphic at $z = 1$ such that $h(1) = \xi$. Let $g(z) = \frac{h(1/z)}{z(z-1)}$. This function has a Taylor expansion at ∞ of the form $\sum_{n=1}^{\infty} \frac{a_n}{z^{n+1}}$, and $E(x) = \sum_{n=1}^{\infty} \frac{a_n}{n!} x^n$ is an E-function. Using the results of [10] recalled in §3.2 we shall compute (partially) its asymptotic expansion at infinity in a large sector bisected by the direction θ, for any $\theta \in [-\pi, \pi)$ outside a finite set; we shall prove that the coefficient of e^x in this expansion is equal to ξ. With this aim in mind, we keep the notation of §3.2, including $\mathcal{D}$ and θ.

We let $\rho = 1$ (eventhough we still write ρ for better readability), and consider a basis of local solutions of $(\frac{\mathrm{d}}{\mathrm{d}z})^\delta \mathcal{D}$ around ρ with functions $f_{j,k}^\rho$ and $g_{j,k}^\rho$ as in §3.2. By Frobenius' method, upon shifting t_j^ρ by an integer we may assume that $g_{j,0}^\rho(0) \neq 0$. Moreover, upon performing $\overline{\mathbb{Q}}$-linear combinations of the basis elements and a permutation of the indices, we may assume that $t_1^\rho < \ldots < t_{J(\rho)}^\rho$ so that the solutions $f_{j,k}^\rho$ have pairwise distinct asymptotic behaviours at 0, namely $f_{j,k}^\rho(s) \sim \frac{g_{j,0}^\rho(0)}{k!} s^{t_j^\rho} \log(s)^k$. At last, dividing each $f_{j,k}^\rho$ with $g_{j,0}^\rho(0)$ we may assume that $g_{j,0}^\rho(0) = 1$ for any j.

Now consider the expansion

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(\rho,j)} \varpi_{j,k}^\rho f_{j,k}^\rho (z - \rho). \tag{3.11}$$

Let $T = \{(j,k),\ \varpi_{j,k}^\rho \neq 0\}$. Since g is not identically zero, T is not empty. Let $j_0 \in \{1, \ldots, J(\rho)\}$ be the minimal value such that $(j_0, k) \in T$ for some k, and let k_0 be the maximal value such that $(j_0, k_0) \in T$. Then on the right-hand side of Eq. (3.11), the leading term as $z \to \rho$ is given by $(j,k) = (j_0, k_0)$, so that

$$g(z) \sim \frac{\varpi_{j_0,k_0}^\rho}{k_0!} (z-\rho)^{t_{j_0}^\rho} \log(z-\rho)^{k_0} \tag{3.12}$$

since $g_{j_0,0}^\rho(0) = 1$. Now recall that $g(z) = \frac{h(1/z)}{z(z-1)}$ with $h(1) = \xi \neq 0$ and $\rho = 1$; therefore $g(z) \sim \frac{\xi}{z-1}$. Comparing this with Eq. (3.12) yields $t_{j_0}^\rho = -1$, $k_0 = 0$, and $\varpi_{j_0,0}^\rho = \xi$.

Let us consider the asymptotic expansion given by Theorem 2, and especially the coefficient of e^x that we denote by α. This coefficient comes from the multiple sum in Eq. (3.10). In this sum, we have $\varpi_{j,k}^\rho = 0$ for any $j < j_0$ and any k (by definition of j_0), so that these terms do not contribute to the value of α. For any $j > j_0$ we have $t_j^\rho > t_{j_0}^\rho = -1$ so that $-t_j^\rho - 1 < 0$ and the corresponding terms do not contribute either. Therefore the value of

α is given only by the terms corresponding to $j = j_0$ (with $t^\rho_{j_0} = -1$):

$$\alpha = \sum_{k=0}^{K(\rho,j_0)} \varpi^\rho_{j_0,k} \sum_{\ell=0}^{k} \frac{(-1)^\ell}{\ell!} \widehat{\Gamma}^{(\ell)}(1) \eta^\rho_{j_0,k-\ell}(0).$$

Now recall that by definition, $k_0 = 0$ is the maximal value of k such that $\varpi^\rho_{j_0,k} \neq 0$. Therefore the previous sum has (at most) one non-zero term: the one corresponding to $k = 0$. Since $\widehat{\Gamma}(1) = 1$ and $\varpi^\rho_{j_0,0} = \xi$ we have $\alpha = \xi \eta^\rho_{j_0,0}(0) = \xi y_{-1,0}(0) g^\rho_{j_0,0}(0) = \xi$ using Eqs. (3.8) and (3.9). This concludes the proof that the coefficient of e^x in the asymptotic expansion of $E(x)$ is equal to ξ. $\square$

Remark 2. Let us follow the proof of Theorem 3 with $\xi = \mathrm{Li}_s(\alpha)$, $\alpha \in \overline{\mathbb{Q}}^\star$, $|\alpha| < 1$. We may choose $h(z) = \mathrm{Li}_s(\alpha z)$ so that

$$g(z) = \frac{\mathrm{Li}_s(\alpha/z)}{z(z-1)} = \sum_{n=2}^{\infty} \Big(\sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \Big) z^{-n-1} \text{ as } z \to \infty, \text{ and } E(x) = \sum_{n=2}^{\infty} \Big(\sum_{k=1}^{n-1} \frac{\alpha^k}{k^s} \Big) \frac{x^n}{n!}.$$

Therefore ξ is a coefficient of an asymptotic expansion of the function (2.6).

4 Asymptotic expansion of the hypergeometric series ${}_pF_p(z)$

In this section, we prove the following result (recall that asymptotic expansions have been defined in §3.1). It will be generalized in §5 but it is important to prove it first.

Proposition 2. *Let $\theta \in (-\pi, \pi) \setminus \{0\}$, and $f(z) := {}_pF_p[a_1, \ldots, a_p; b_1, \ldots, b_p; z]$ be a hypergeometric function with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then $f(z)$ has an asymptotic expansion*

$$f(z) \approx \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} z^{-n-\alpha} \log(1/z)^i$$

in a large sector bisected by θ, with $\Sigma \subset \{0, 1\}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ both finite, and coefficients $c_{\rho,\alpha,i,n}$ in $\mathbf{H}$.

Proof. If one of the a_j's is in $\mathbb{Z}_{\leq 0}$, the hypergeometric series is in $\mathbb{C}[z]$ and the conclusion clearly holds with $c_{\rho,\alpha,i,n}$ in $\overline{\mathbb{Q}}$. From now on, as for the b_j's, we assume that none of the a_j's is in $\mathbb{Z}_{\leq 0}$. In the beginning of the proof we do not need to assume that $a_j, b_j \in \mathbb{Q}$.

Let

$$R(s) = R(\underline{a}, \underline{b}; s) := \frac{\prod_{j=1}^p \Gamma(a_j + s)}{\prod_{j=1}^p \Gamma(b_j + s)} \Gamma(-s).$$

The poles of $R(s)$ are located at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \ldots, p$, and at $\mathbb{Z}_{\geq 0}$. We define the series

$$L_p(\underline{a}, \underline{b}; z) := \sum_{j=1}^{p} \sum_{k=0}^{\infty} \mathrm{Residue}\big(R(s) z^s, s = -a_j - k\big).$$

Set $\nu := \sum_{j=1}^{p} a_j - \sum_{j=1}^{p} b_j$,

$$C_k(\underline{a}, \underline{b}) = \sum_{k_1 \geq 0, \ldots, k_p \geq 0, N_p = k} \frac{(1-a_p)_{k_p} \prod_{j=1}^{p-1}(a_{j+1} + b_{j+1} - a_j)_{k_j} \prod_{j=1}^{p}(B_j + N_{j-1})_{k_j}}{\prod_{j=1}^{p} k_j!},$$

where for every j, $B_j = \sum_{m=1}^{j} b_m$ and $N_j = \sum_{m=1}^{j} k_m$. We define the formal series

$$K_p(\underline{a}, \underline{b}; z) := e^z \sum_{k=0}^{\infty} C_k(\underline{a}, \underline{b}) z^{\nu - k} \in e^z z^{\nu} \mathbb{Q}[\underline{a}, \underline{b}][[1/z]].$$

Combining [21, p. 212] and [17, p. 283, Theorem] (partially reproved in [27, p. 113, Theorem 4.1, Eq. (4.4)]), as $z \to \infty$ in the sector $-\frac{3\pi}{2} < \arg(z) < \frac{\pi}{2}$, we have the asymptotic expansion

$${}_pF_p\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}; z\right] \approx \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \Big(L_p(\underline{a}, \underline{b}; e^{i\pi} z) + K_p(\underline{a}, \underline{b}; z) \Big),$$

while if $z \to \infty$ in the sector $-\frac{\pi}{2} < \arg(z) < \frac{3\pi}{2}$, we have

$${}_pF_p\left[\begin{matrix} a_1, \ldots, a_p \\ b_1, \ldots, b_p \end{matrix}; z\right] \approx \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} \Big(L_p(\underline{a}, \underline{b}; e^{-i\pi} z) + K_p(\underline{a}, \underline{b}; z) \Big).$$

These two expansions satisfy Definition 3 above: they hold in a *large sector* bisected by any $\theta \in (-\pi, 0)$, respectively any $\theta \in (0, \pi)$, and the holonomic formal series $L_p(\underline{a}, \underline{b}; e^{\pm i\pi} z)$ and $e^{-z} K_p(\underline{a}, \underline{b}; z)$ are both 1-summable in the direction θ because of the growth of their coefficients.

These asymptotic expansions are refined versions of Barnes and Wright's fundamental works [6, 28] and are consequences of the general expansion of the Meijer G-function [21, Chapter V]. Note that the Meijer G-function is not related to Siegel's G-functions, though by specialization of its parameters the former provides examples of the latter. In the next two subsections, we provide more explicit expressions for the function $L_p(\underline{a}, \underline{b}; z)$ under the assumption that the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ (whereas up to now this assumption was not used), in order to prove that all coefficients of its asymptotic expansion belong to $\mathbf{H}$.

4.1 R has simple poles

If the a_j's are pairwise distinct modulo $\mathbb{Z}$, then the poles of $R(s)$ are simple, and we have

$$L_p(\underline{a}, \underline{b}; z) = \sum_{j=1}^{p} \sum_{k=0}^{\infty} (-1)^k \frac{\Gamma(a_j + k) \prod_{i=1, i \neq j}^{p} \Gamma(a_i - a_j - k)}{k! \prod_{i=1}^{p} \Gamma(b_i - a_j - k)} z^{-a_j - k}.$$

When the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} L_p(\underline{a}, \underline{b}; z)$ is thus equal to a finite sum

$$\sum_j z^{-a_j} f_j(z)$$

with $f_j(z) \in \mathbf{H}[[1/z]]$. Note that the element $1/\pi \in \mathbf{H}$ appears through the use of the reflection formula $\frac{1}{\Gamma(s)} = \frac{1}{\pi} \sin(\pi s) \Gamma(1-s)$ for rational values of s.

4.2 R has multiple poles

We assume that the a_j's and b_j's are in $\mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Up to reordering the a_j's, we can group them in ℓ groups as follows: for $m = 0, \dots, \ell - 1$, we have

$$a_{j_m+1}, a_{j_m+2}, \dots, a_{j_{m+1}} \text{ equal mod } \mathbb{Z}, \quad a_{j_m+1} \text{ the smallest one in the group,}$$

where the a_{j_m} are pairwise distinct mod $\mathbb{Z}$ for $m = 1, \dots, \ell$, and $0 = j_0 < j_1 < j_2 < \dots < j_\ell = p$.

Then, for every $j \in \{j_m + 1, \dots, j_{m+1}\}$, we have

$$\Gamma(a_j + s) = (a_{j_m+1} + s)_{a_j - a_{j_m+1}} \Gamma(a_{j_m+1} + s).$$

Set $d_m := j_m - j_{m-1} \geq 1$, $c_m := a_{j_{m-1}+1}$ and

$$P(s) := \prod_{m=0}^{\ell-1} \left(\prod_{j=j_m+1}^{j_{m+1}} (a_{j_m+1} + s)_{a_j - a_{j_m+1}} \right) \in \mathbb{Q}[s].$$

Hence,

$$R(s) = P(s)\Gamma(-s) \frac{\prod_{m=1}^{\ell} \Gamma(c_m + s)^{d_m}}{\prod_{m=1}^{p} \Gamma(b_m + s)}.$$

To compute the residue of $R(s)z^{-s}$ at $s = -c_n - k$ for given $n \in \{1, \dots, \ell\}$ and $k \in \mathbb{Z}_{\geq 0}$, we write

$$\Gamma(c_n + s) = \frac{\Gamma(c_n + s + k + 1)}{(c_n + s)_k (c_n + s + k)}$$

and define

$$\Phi_{c_n,k}(s) := z^{-s} P(s)\Gamma(-s) \frac{\prod_{m=1, m\neq n}^{\ell} \Gamma(c_m + s)^{d_m}}{\prod_{m=1}^{p} \Gamma(b_m + s)} \cdot \frac{\Gamma(c_n + s + k + 1)^{d_n}}{(c_n + s)_k^{d_n}}$$

which is holomorphic at $s = -c_n - k$. We thus deduce from

$$R(s)z^{-s} = \frac{\Phi_{c_n,k}(s)}{(c_n + s + k)^{d_n}}$$

that

$$\text{Residue}\big(R(s)z^{-s}, s = -c_n - k\big) = \frac{1}{(d_n - 1)!} \Phi_{c_n,k}^{(d_n-1)}(-c_n - k).$$

It follows that $\frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p} \Gamma(a_j)} L_p(\underline{a}, \underline{b}; z)$ is equal to a finite sum

$$\sum_{j,t} z^{-a_j} \log(1/z)^t f_{j,t}(z)$$

with $f_{j,t}(z) \in \mathbf{H}[[1/z]]$. (The powers of $\log(1/z)$ come from the differentiation of z^{-s} with respect to s.) This concludes the proof of Proposition 2. $\square$

5 Asymptotic expansion of the hypergeometric E-function ${}_pF_q(z^{q-p+1})$

The goal of this section is to prove the following result, which generalizes Proposition 2. We have chosen to present and prove this proposition in an independent part because the case $p = q$ is one of the steps in the proof of Proposition 3 below, towards the proof of the general case $q \geq p \geq 1$. Moreover Proposition 2 is slightly more precise than Theorem 4 in the sense that the non-explicited "finite set" below is $\{0\}$ in Proposition 2; however, this precision is not important for us.

Theorem 4. *Let $\theta \in (-\pi, \pi)$, and $F(z) := {}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z^{q-p+1}]$ be a hypergeometric E-function with $q \geq p \geq 1$ and parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Assume that θ does not belong to some finite set that depends only on F. Then $F(z)$ has an asymptotic expansion*

$$F(z) \approx \sum_{\rho \in \Sigma} e^{\rho z} \sum_{\alpha \in S} \sum_{i \in T} \sum_{n=0}^{\infty} c_{\rho,\alpha,i,n} z^{-n-\alpha} \log(1/z)^i \tag{5.1}$$

in a large sector bisected by θ, with $\Sigma \subset \{0, 1\}$, $S \subset \mathbb{Q}$ and $T \subset \mathbb{N}$ both finite, and coefficients $c_{\rho,\alpha,i,n}$ in $\mathbf{H}$.

Remark 3. There exist well-known results in the literature from which follows the existence of an asymptotic expansion of a given ${}_pF_q[a_1, \dots, a_p; b_1, \dots, b_q; z]$ in any sector of opening $< 2\pi$. See the references [6, 21, 28] already cited in §4. However, when z is changed to z^{q-p+1}, these expansions hold only in sectors of opening $< 2\pi/(q-p+1)$, so that when $q > p$, these openings are $< \pi$. Unicity in a large sector, which is crucial for us, is thus not guaranteed anymore. The existence and unicity of the expansion (5.1) is a consequence of Theorem 2 applied to $z \cdot {}_pF_q(z^{q-p+1})$ because such a series is an E-function when its parameters are rational and it vanishes at $z = 0$. We don't know explicit expressions for the coefficients $c_{\rho,\alpha,i,n}$ such as those obtained in §4 when $p = q$, but the result is enough for our purposes.

5.1 Connections constants of the G-function $(1/z) \cdot {}_{p+1}F_p(1/z)$

We use the same notation as in §3.3 and §4. Using unicity in large sectors of the asymptotic expansion of a ${}_{p+1}F_{p+1}(z)$ series with rational parameters, we shall study the connection constants of its Laplace transform, which is a $(1/z) \cdot {}_{p+1}F_p(1/z)$ series.

More precisely, consider the hypergeometric E-function

$$f(z) := {}_{p+1}F_{p+1} \begin{bmatrix} a_1, \dots, a_{p+1} \\ 1, b_1, \dots, b_p \end{bmatrix}; z \Bigg]$$

where $p \geq 0$ and $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. The Laplace transform of $f(z)$ is the G-function

$$g(z) := \int_0^\infty f(t) e^{-zt} dt = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n n!}{(1)_n (1)_n (b_1)_n \cdots (b_p)_n} \frac{1}{z^{n+1}} = \frac{1}{z} \cdot {}_{p+1}F_p \begin{bmatrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{bmatrix}; \frac{1}{z} \Bigg]. \tag{5.2}$$

The finite singularities of the differential equation satisfied by $g(z)$ are 0 and 1. Locally around $\rho \in \{0,1\}$, this equation has a basis of solutions $(f^{\rho}_{j,k}(z-\rho))_{j,k}$ of the form (3.6) to which we connect to $g(z)$ as in (3.7):

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(\rho,j)} \varpi^{\rho}_{j,k} f^{\rho}_{j,k}(z-\rho).$$

As observed in the proof of Theorem 3, we may assume in the expansions given by (3.6) and (3.7) that for any ρ the rational numbers $t^{\rho}_1, \ldots, t^{\rho}_{J(\rho)}$ are pairwise distinct, and that $g^{\rho}_{j,0}(0) \neq 0$ for any ρ and any j. Then we have the following result.

Proposition 3. *For any ρ, any j and any k such that $f^{\rho}_{j,k}(z-\rho)$ is not holomorphic at $z=\rho$, we have $\varpi^{\rho}_{j,k} \in \mathbf{H}$.*

We believe that $\varpi^{\rho}_{j,k} \in \mathbf{H}$ also when $f^{\rho}_{j,k}(z-\rho)$ is holomorphic at $z=\rho$, but the proof of this result would require a new idea. However, the case where $f^{\rho}_{j,k}(z-\rho)$ is holomorphic at $z=\rho$ is useless for proving Theorem 4 (see Remark 1 after Theorem 2).

Proof. We first recall that $\widehat{\Gamma}(x) := 1/\Gamma(x)$. We shall in fact prove Proposition 3 for any G-function $g(z)$ which is the Laplace transform of an E-function $f(x)$ with an asymptotic expansion of the form (3.1) in a large sector bisected by any θ (except finitely many values mod 2π), with coefficients $c_{\rho,\alpha,i,n} \in \mathbf{H}$. Then Proposition 2 shows that the function $g(z)$ defined in (5.2) has this property.

Let ρ, j_0 and k_0 be fixed. We shall prove that $\varpi^{\rho}_{j_0,k_0} \in \mathbf{H}$; to begin with, we consider the case where $t^{\rho}_{j_0} \notin \mathbb{N}$. By induction we may assume that for any j such that $t^{\rho}_{j_0} - t^{\rho}_j$ is a positive integer and for any k, we have $\varpi^{\rho}_{j,k} \in \mathbf{H}$. The initial step of this inductive proof corresponds to the case where there is no such j, so that this assumption holds trivially. Let us denote by κ the coefficient of $e^{\rho x} x^{-t^{\rho}_{j_0}-1} \log(1/x)^{k_0}$ in the asymptotic expansion of $f(x)$ in a large sector bisected by a given θ; by assumption we have $\kappa \in \mathbf{H}$. Now this asymptotic expansion is unique, and (except for finitely many values of θ mod 2π) the following expression of κ follows from Theorem 2:

$$\kappa = \sum_{\substack{1 \le j \le J(\rho) \\ t^{\rho}_{j_0} - t^{\rho}_j \in \mathbb{N}}} \sum_{k=k_0}^{K(j,\rho)} \varpi^{\rho}_{j,k} \sum_{\ell=0}^{k-k_0} \frac{(-1)^{\ell}}{\ell! k_0!} \widehat{\Gamma}^{(\ell)}(1-\{t^{\rho}_{j_0}\}) \; [x^{t^{\rho}_{j_0}-t^{\rho}_j}](\eta^{\rho}_{j,k-k_0-\ell}(x))$$

where $[x^n](\eta^{\rho}_{j,i}(x))$ is the coefficient of x^n in the power series $\eta^{\rho}_{j,i}(x) \in \overline{\mathbb{Q}}[[x]]$. It follows from the reflection formula $\widehat{\Gamma}(1-x) = \frac{1}{\pi}\sin(\pi x)\Gamma(x)$ that $\widehat{\Gamma}^{(\ell)}(1-\{t^{\rho}_{j_0}\}) \in \mathbf{H}$ for any $\ell \in \mathbb{N}$, since $t^{\rho}_{j_0} \in \mathbb{Q}$. We have assumed that $\varpi^{\rho}_{j,k} \in \mathbf{H}$ for any pair (j,k) such that $t^{\rho}_{j_0} - t^{\rho}_j$ is a positive integer, so that in the expression of κ the total contribution of these terms belongs to $\mathbf{H}$. We recall that for any $j \neq j_0$ we have assumed that $t_j \neq t_{j_0}$, so that all terms with $j \neq j_0$ are included here. Proceeding by decreasing induction on k_0 we may also assume that $\varpi^{\rho}_{j_0,k} \in \mathbf{H}$ for any k such that $k_0+1 \le k \le K(j_0,\rho)$, or that there is no such k. Then the term corresponding to each such k in the expression of κ belongs also to $\mathbf{H}$. Since $\kappa \in \mathbf{H}$ we deduce that the only remaining term, namely the one with $(j,k) = (j_0, k_0)$, belongs to $\mathbf{H}$: we have $\varpi^{\rho}_{j_0,k_0}\lambda \in \mathbf{H}$ with $\lambda = \frac{1}{k_0!}\,\widehat{\Gamma}(1 - \{t^{\rho}_{j_0}\})\ \eta^{\rho}_{j_0,0}(0)$. Now with the notation of §3.2 we have $\eta^{\rho}_{j_0,0}(0) = y_{t^{\rho}_{j_0},0}(0) g^{\rho}_{j_0,0}(0) = \frac{\Gamma(1-\{t^{\rho}_{j_0}\})}{\Gamma(-t^{\rho}_{j_0})} g^{\rho}_{j_0,0}(0)$ so that $\lambda \neq 0$ and $\frac{1}{\lambda} = \frac{k_0!\Gamma(-t^{\rho}_{j_0})}{g^{\rho}_{j_0,0}(0)} \in \mathbf{H}$. Therefore we have proved that $\varpi^{\rho}_{j_0,k_0} = \frac{1}{\lambda}\cdot \varpi^{\rho}_{j_0,k_0}\lambda$ belongs to $\mathbf{H}$.

To conclude the proof, let us prove that $\varpi^{\rho}_{j_0,k_0} \in \mathbf{H}$ in the case where $t^{\rho}_{j_0} \in \mathbb{N}$; assuming that $f^{\rho}_{j_0,k_0}(z-\rho)$ is not holomorphic at $z = \rho$, we have $k_0 \geq 1$ in this case. We denote by κ the coefficient of $e^{\rho x} x^{-t^{\rho}_{j_0}-1}\log(1/x)^{k_0-1}$ in the asymptotic expansion of $f(x)$ in a large sector; note that the exponent of $\log(1/x)$ has changed with respect to the first case. Using Theorem 2 we obtain:

$$\kappa = \sum_{\substack{1\leq j\leq J(\rho)\\ t^{\rho}_{j_0}-t^{\rho}_{j}\in\mathbb{N}}} \sum_{k=k_0-1}^{K(j,\rho)} \varpi^{\rho}_{j,k} \sum_{\ell=0}^{k-k_0+1} \frac{(-1)^{\ell}}{\ell!(k_0-1)!}\,\widehat{\Gamma}^{(\ell)}(1-\{t^{\rho}_{j_0}\})\ [x^{t^{\rho}_{j_0}-t^{\rho}_{j}}](\eta^{\rho}_{j,k-k_0+1-\ell}(x)).$$

As above we may assume that for any j such that $t^{\rho}_{j_0} - t^{\rho}_{j}$ is a positive integer and for any k, we have $\varpi^{\rho}_{j,k} \in \mathbf{H}$. By decreasing induction on k we may also assume, in the same way, that $\varpi^{\rho}_{j_0,k} \in \mathbf{H}$ for any k such that $k_0+1 \leq k \leq K(j_0,\rho)$, or that there is no such k. Then as above, all terms with $j \neq j_0$ or $k \leq k_0+1$ belong to $\mathbf{H}$. Since $\kappa \in \mathbf{H}$ we deduce that

$$\sum_{k=k_0-1}^{k_0} \varpi^{\rho}_{j_0,k} \sum_{\ell=0}^{k-k_0+1} \frac{(-1)^{\ell}}{\ell!(k_0-1)!}\,\widehat{\Gamma}^{(\ell)}(1-\{t^{\rho}_{j_0}\})\ \eta^{\rho}_{j_0,k-k_0+1-\ell}(0) \in \mathbf{H}.$$

Now with the notation of §3.2, $t^{\rho}_{j_0} \in \mathbb{N}$ implies that $y_{t^{\rho}_{j_0},0}(x)$ is identically zero (as noticed in [10, p. 44]), so that $\eta^{\rho}_{j_0,0}(x)$ is identically zero too. Therefore the terms with $\ell = k - k_0 + 1$ vanish in the above sum. The only term that remains is the one with $k = k_0$ and $\ell = 0$, so that we have

$$\varpi^{\rho}_{j_0,k_0}\frac{1}{(k_0-1)!}\,\widehat{\Gamma}(1-\{t^{\rho}_{j_0}\})\ \eta^{\rho}_{j_0,1}(0) \in \mathbf{H}.$$

Letting $\lambda = \frac{1}{(k_0-1)!}\,\widehat{\Gamma}(1-\{t^{\rho}_{j_0}\})\ \eta^{\rho}_{j_0,1}(0)$, it is sufficient to prove that $\lambda \neq 0$ and $\frac{1}{\lambda} \in \mathbf{H}$. We have $(k_0-1)!\Gamma(1-\{t^{\rho}_{j_0}\}) \in \mathbf{H}$, and $\eta^{\rho}_{j_0,1}(0) = y_{t^{\rho}_{j_0},1}(0) g^{\rho}_{j_0,0}(0)$ using the fact that $y_{t^{\rho}_{j_0},0}(0) = 0$ since $t^{\rho}_{j_0} \in \mathbb{N}$. We have assumed that $g^{\rho}_{j_0,0}(0)$ is a non-zero algebraic integer; at last we have (see [10, Eq. (4.6)] for more details):

$$y_{t^{\rho}_{j_0},1}(0) = \frac{\mathrm{d}}{\mathrm{d}t}\Big(\frac{\Gamma(1-\{t\})}{\Gamma(-t)}\Big)_{|t=t^{\rho}_{j_0}} = \frac{\mathrm{d}}{\mathrm{d}t}\Big((-t)_{t^{\rho}_{j_0}+1}\Big)_{|t=t^{\rho}_{j_0}} = (-1)^{t^{\rho}_{j_0}+1}(t^{\rho}_{j_0})! \in \overline{\mathbb{Q}}^{\star}.$$

This concludes the proof that $\lambda \neq 0$ and $\frac{1}{\lambda} \in \mathbf{H}$, and that of Proposition 3. □

5.2 Proof of Theorem 4

Proof. We shall now transfer the result obtained in §5.1 to ${}_pF_q(z^{q-p+1})$ series.

Consider the hypergeometric E-function

$$F(z) := {}_pF_q\left[\begin{matrix} a_1, \dots, a_p \\ b_1, \dots, b_q \end{matrix}; z^r\right]$$

where $q \geq p \geq 1$, $r := q - p + 1 \geq 1$, and $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Consider also the hypergeometric G-function

$$g(z) := \frac{1}{z} \cdot {}_{q+1}F_q\left[\begin{matrix} a_1, \dots, a_p, 1/r, 2/r, \dots, r/r \\ b_1, \dots, b_q \end{matrix}; \frac{1}{z}\right].$$

The Laplace transform of $F(z)$ is

$$G(z) := \int_0^\infty F(t)e^{-zt}dt = \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_p)_n (rn)!}{(1)_n (b_1)_n \cdots (b_q)_n} \frac{1}{z^{rn+1}} = z^{r-1} g\Big(\frac{z^r}{r^r}\Big).$$

The finite singularities of the (essentially hypergeometric) differential equation satisfied by $g(z)$ are 0 and 1. As in the previous sections (see §3.2), we connect $g(z)$ on a local basis $(f_{j,k}^\rho(z-\rho))_{j,k}$ around $z = \rho \in \{0, 1\}$:

$$g(z) = \sum_{j=1}^{J(\rho)} \sum_{k=0}^{K(j,\rho)} \varpi_{j,k}^\rho f_{j,k}^\rho(z-\rho)$$

where

$$f_{j,k}^\rho(z-\rho) = (z-\rho)^{t_j^\rho} \sum_{\ell=0}^{k} g_{j,k-\ell}^\rho(z-\rho) \frac{\log(z-\rho)^\ell}{\ell!}$$

with $t_j^\rho \in \mathbb{Q}$ and $g_{j,k-\ell}^\rho(z-\rho) \in \overline{\mathbb{Q}}[[z-\rho]]$ holomorphic at $z = \rho$; moreover, the $g_{j,k-\ell}^\rho$ are G-functions. Here, θ is chosen such that $\log(z)$ is defined with $-\pi - \theta < \arg(z) < \pi - \theta$ and $\log(z-1)$ is defined with $-\pi - \theta < \arg(z-1) < \pi - \theta$. For later use, we also impose that θ is such that the following property holds: the half-lines $L_0 := -e^{-i\theta}\mathbb{R}_+$ and $L_k := re^{2i\pi k/r} - e^{-i\theta}\mathbb{R}_+$ ($k = 1, 2, \dots, r$) are such that $L_j \cap L_k = \emptyset$ for $j \neq k$.

We may assume that for any ρ the rational numbers $t_1^\rho, \dots, t_{J(\rho)}^\rho$ are pairwise distinct, and that $g_{j,0}^\rho(0) \neq 0$ for any ρ and any j. Then Proposition 3 proved in §5.1 shows that $\varpi_{j,k}^\rho \in \mathbf{H}$ for any j, k, ρ, except maybe when $f_{j,k}^\rho(z-\rho)$ is holomorphic at $z = \rho$ (i.e., when $k = 0$ and $t_j^\rho \in \mathbb{N}$).

The set of finite singularities of the differential equation $\mathcal{E}$ satisfied by $G(z)$ is

$$\{0, e^{2i\pi/r}r, e^{4i\pi/r}r, \dots, e^{2ri\pi/r}r\}.$$

The function $G(z)$ can be continued to the simply connected cut plane $\mathcal{L} := \mathbb{C} \setminus \cup_{k=0}^r L_k$. Since the minimal differential equations satisfied by $G(z)$ and $g(z)$ have the same order, $(z^{r-1} f_{j,k}^0(z^r/r^r))_{j,k}$ is a basis of solutions of $\mathcal{E}$ at $z = 0$ while $(z^{r-1} f_{j,k}^1(z^r/r^r - 1))_{j,k}$ is a basis of solutions of $\mathcal{E}$ at any one of the points $z = re^{2i\ell\pi/r}$, $\ell = 1, \dots, r$. These bases are not necessarily of the form (3.6) but this is not essential for our purpose.

Let us first consider the connection of $G(z)$ with the basis of solutions at $z = 0$. Below, $\log(z)$ is defined by $-\pi - \theta < \arg(z) < \pi - \theta$, and we have $\log(z^r/r^r) = r\log(z) - r\log(r)$ for all $z \in \mathbb{C}$ such that $c_1(r) < \arg(z) < c_2(r)$, for some well chosen constants $c_1(r) < c_2(r)$. Note that $r\log(r) \in \mathbf{H}$. We then have, for all $z \in \mathcal{L}$ such that $c_1(r) < \arg(z) < c_2(r)$,

$$
\begin{aligned}
G(z) &= \sum_{j=1}^{J(0)} \sum_{k=0}^{K(j,0)} \varpi_{j,k}^0 z^{r-1} f_{j,k}^0(z^r/r^r) \\
&= \sum_{j=1}^{J(0)} \sum_{k=0}^{K(j,0)} \varpi_{j,k}^0 r^{-rt_j^0} z^{rt_j^0+r-1} \sum_{\ell=0}^{k} g_{j,k-\ell}^0(z^r/r^r) \frac{\log(z^r/r^r)^\ell}{\ell!} \\
&= \sum_{j=1}^{J(0)} \sum_{k=0}^{K(j,0)} \varpi_{j,k}^0 r^{-rt_j^0} z^{rt_j^0+r-1} \sum_{\ell=0}^{k} g_{j,k-\ell}^0(z^r/r^r) S_\ell(\log(z)),
\end{aligned}
$$

where $S_\ell[x] \in \mathbf{H}[x]$ is of degree ℓ. Note that $r^{-rt_j^0} \in \overline{\mathbb{Q}}$ and $g_{j,k-\ell}^0(z^r/r^r) \in \overline{\mathbb{Q}}[[z]]$. Recall that, by Proposition 3, we have $\varpi_{j,k}^0 \in \mathbf{H}$ except maybe when $k = 0$ and $t_j^0 \in \mathbb{N}$. We observe that in this special case, $\varpi_{j,0}^0$ appears in the formula above as the coefficient of a function holomorphic at 0, because $t_j^0 \in \mathbb{N}$ implies $rt_j^0 + r - 1 \in \mathbb{N}$ and $g_{j,0}^0(z^r/r^r)$ is holomorphic at 0. Hence, for all $z \in \mathcal{L}$ such that $c_1(r) < \arg(z) < c_2(r)$, we have

$$
G(z) = \sum_{j=1}^{J(0)} \sum_{k=0}^{K(j,0)} \Omega_{j,k}^0 z^{rt_j^0+r-1} \sum_{\ell=0}^{k} G_{j,k-\ell}^0(z) \frac{\log(z)^\ell}{\ell!} \tag{5.3}
$$

with $\Omega_{j,k}^0 \in \mathbb{C}$, $G_{j,k-\ell}^0(z) \in \mathbf{H}[[z]]$ are holomorphic at $z = 0$, and $\Omega_{j,k}^0 \in \mathbf{H}$ for any pair (j,k) such that $k \geq 1$ or $rt_j^0 + r - 1 \notin \mathbb{N}$. Now, Eq. (5.3) extends to $\mathcal{L}$ by analytic contination, i.e. the assumption $c_1(r) < \arg(z) < c_2(r)$ can be dropped. For our application, it is enough to know that the functions $G_{j,k-\ell}^0(z)$ are in $\mathbf{H}[[z]]$ and not necessarily G-functions. Moreover, we need no information on the nature of the constants $\Omega_{j,0}^0$ for j such that $rt_j^0 + r - 1 \in \mathbb{N}$ because they are factors of terms in (5.3) that are holomorphic at 0, and therefore do not contribute to the asymptotic expansion of $F(z)$. Note that it may happen that some $t_{j_0} \notin \mathbb{N}$ is such that $rt_{j_0}^0 + r - 1 \in \mathbb{N}$, in which case we know that $\Omega_{j_0,0}^0$ is indeed in $\mathbf{H}$ but this information will not be useful to complete the proof of Theorem 4.

Let us now consider the connection of $G(z)$ with the basis of solutions at $z = e^{2\ell i\pi/r}r$. For simplicity, we shall assume that $\ell = 0$ but the general case can be obtained similarily. Below, $\log(z - r)$ is defined with $-\pi - \theta < \arg(z - r) < \pi - \theta$. We have $\log(z^r/r^r - 1) - \log(z - r) = Q(z - r) \in \mathbf{H}[[z - r]]$ for $z \in \mathbb{C}$ such that $|z - r| < \kappa$ and $d_1(r) < \arg(z) < d_2(r)$, for some well chosen constants κ and $d_1(r) < d_2(r)$. (The series $Q(z - r)$ may change if the angular sector $d_1(r) < \arg(z) < d_2(r)$ is changed to another one.) Hence, for $z \in \mathcal{L}$

such that $|z-r|<\kappa$ and $d_1(r)<\arg(z)<d_2(r)$, we have

$$\begin{aligned}
G(z) &= \sum_{j=1}^{J(1)}\sum_{k=0}^{K(j,1)} \varpi_{j,k}^1 z^{r-1} f_{j,k}^1(z^r/r^r) \\
&= \sum_{j=1}^{J(1)}\sum_{k=0}^{K(j,1)} \varpi_{j,k}^1 (z^r/r^r-1)^{t_j^1} z^{r-1} \sum_{\ell=0}^{k} g_{j,k-\ell}^1(z^r/r^r-1)\frac{\log(z^r/r^r-1)^\ell}{\ell!} \\
&= \sum_{j=1}^{J(1)}\sum_{k=0}^{K(j,1)} \varpi_{j,k}^1 (z-r)^{t_j^1} P_j(z-r) \sum_{\ell=0}^{k} \frac{1}{\ell!} g_{j,k-\ell}^1(z^r/r^r-1)\big(\log(z-r)+Q(z-r)\big)^\ell
\end{aligned}$$

where $P_j(z-r)\in\overline{\mathbb{Q}}[[z-r]]$. As above, Proposition 3 yields $\varpi_{j,k}^1\in\mathbf{H}$ for any j, k, except maybe when $k=0$ and $t_j^1\in\mathbb{N}$. Hence, for all $z\in\mathcal{L}$ such that $|z-r|<\kappa$ and $d_1(r)<\arg(z)<d_2(r)$, we have

$$G(z) = \sum_{j=1}^{J(1)}\sum_{k=0}^{K(j,1)} \Omega_{j,k}^r\cdot(z-r)^{t_j^1}\sum_{\ell=0}^{k} G_{j,k-\ell}^r(z-r)\frac{\log(z-r)^\ell}{\ell!} \tag{5.4}$$

with $\Omega_{j,k}^r\in\mathbb{C}$, $G_{j,k-\ell}^r(z-r)\in\mathbf{H}[[z-r]]$ are holomorphic at $z=r$, and $\Omega_{j,k}^r\in\mathbf{H}$ for any j, k such that $k\geq 1$ or $t_j^1\notin\mathbb{N}$. Now, Eq. (5.4) extends to $\mathcal{L}$ by analytic continuation, i.e. the conditions that $|z-r|<\kappa$ and $d_1(r)<\arg(z)<d_2(r)$ can be dropped. For our application, it is enough to know that $G_{j,k-\ell}^r(z)$ are in $\mathbf{H}[[z]]$ and not necessarily G-functions. Moreover, we need no information on the nature of the constants $\Omega_{j,0}^r$ for j such that $t_j^1\in\mathbb{N}$ because they are factors of terms in (5.3) that are holomorphic at $z=r$, and therefore do not contribute to the asymptotic expansion of $F(z)$ (using Remark 1 stated after Theorem 2). Eq. (5.4) can be generalized to the other singularities $e^{2i\ell\pi/r}r$ with obvious changes.

Now, we use the analytic continuation to $\mathcal{L}$ of Eqs. (5.3), (5.4) and analogues at $z=e^{2\ell i\pi/r}r$ for other values of ℓ in the formula of Theorem 2, and we deduce Theorem 4. □

6 Application to Siegel's problem: proof of Theorem 1

Proof. Assume that Siegel's question has an affirmative answer, and let $\xi\in\mathbf{G}$. Theorem 3 provides an E-function $E(z)$ and a finite set $S\subset(-\pi,\pi)$ such that for any $\theta\in(-\pi,\pi)\setminus S$, ξ is a coefficient of the asymptotic expansion of $E(z)$ in a large sector bisected by θ. Now an affirmative answer to Siegel's question yields n hypergeometric series $f_1,\ldots,f_n$ of the type ${}_pF_q(z^{q-p+1})$ (for various values of $q-p$, p and q such that $q\geq p\geq 1$) with rational parameters, n algebraic numbers $\lambda_1,\ldots,\lambda_n$, and a polynomial $P\in\overline{\mathbb{Q}}[X_1,\ldots,X_n]$, such that $E(z)=P(f_1(\lambda_1 z),\ldots,f_n(\lambda_n z))$. Choose $\theta\in(-\pi,\pi)$ outside a suitable finite set. Then Theorem 4 implies that for any i, the asymptotic expansion of $f_i(\lambda_i z)$ in a large sector bisected by θ has coefficients in $\mathbf{H}$. The same holds for $E(z)=P(f_1(\lambda_1 z),\ldots,f_n(\lambda_n z))$

because $\mathbf{H}$ is a $\overline{\mathbb{Q}}$-algebra. Since such an asymptotic expansion is unique (see §3.1), the coefficient ξ belongs to $\mathbf{H}$. This concludes the proof of Theorem 1. $\square$

7 A Siegel type problem for G-functions

We recall that $\sum_{n=0}^{\infty} a_n z^n$ is a G-function if $\sum_{n=0}^{\infty} a_n z^n/n!$ is an E-function (in the sense of this paper). G-functions form a ring stable under $\frac{\mathrm{d}}{\mathrm{d}z}$ and $\int_0^z$; they are not entire (unless they are polynomials), they have a finite number of singularities and they can be analytically continued in a cut plane with cuts at these singularities. Moreover, given any algebraic function $\alpha(z)$ over $\overline{\mathbb{Q}}(z)$ holomorphic at 0 and any G-function $f(z)$, the functions $\alpha(z)$ and $f(z\alpha(z))$ are G-functions.

For any integer $p \geq 0$, the hypergeometric series

$$ {}_{p+1}F_p\left[\begin{matrix} a_1, \dots, a_{p+1} \\ b_1, \dots, b_p \end{matrix}; z\right] := \sum_{n=0}^{\infty} \frac{(a_1)_n \cdots (a_{p+1})_n}{(1)_n (b_1)_n \cdots (b_p)_n} z^n \tag{7.1} $$

is a G-function when $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$ for all j; Galochkin's classification can be adapted to describe all the hypergeometric G-functions of type ${}_{p+1}F_p$. The simplest examples are ${}_1F_0[a; \cdot; z] = (1-z)^{-a}$ ($a \in \mathbb{Q}$) and ${}_2F_1[1, 1; 2; z] = -\log(1-z)/z$. If $a_j \in \mathbb{Z}_{\leq 0}$ for some j, then the series reduces to a polynomial. Any polynomial with coefficients in $\overline{\mathbb{Q}}$ of functions of the form $\mu(z)\, {}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; \lambda(z)]$ is a G-function, where $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, and $\mu(z), \lambda(z)$ are algebraic over $\overline{\mathbb{Q}}(z)$ and holomorphic at $z = 0$, with $\lambda(0) = 0$.

In the spirit of Siegel's problem for E-functions, it is natural to ask the following question.

Question 2. *Is it possible to write any G-function as a polynomial with coefficients in $\overline{\mathbb{Q}}$ of G-functions of the form $\mu(z) \cdot {}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; \lambda(z)]$, with $p \geq 0$, $a_j \in \mathbb{Q}$, $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$, $\mu(z), \lambda(z)$ algebraic over $\overline{\mathbb{Q}}(z)$ and holomorphic at $z = 0$, and $\lambda(0) = 0$?*

We prove in this section a result similar to that for E-functions (recall that the inclusion $\mathbf{G} \subset \mathbf{H}$ is very unlikely: see §2.2).

Theorem 5. *At least one of the following statements is true:*

(*i*) $\mathbf{G} \subset \mathbf{H}$*;*

(*ii*) *Question 2 has a negative answer under the further assumption that the algebraic functions λ have a common singularity in $\overline{\mathbb{Q}}^* \cup \{\infty\}$ at which they all tend to ∞.*

Our method seems inoperant if this further assumption on the λ's is dropped. Question 2 is related to a conjecture of Dwork [8, p. 784] concerning the classification of certain operators in $\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$ of order 2, which was disproved by Krammer [19]; later on, Bouw-Möller [7] gave other counter-examples of a different nature. Dwork's conjecture said that a globally nilpotent operator of order 2 either has a basis of algebraic solutions over $\overline{\mathbb{Q}}(z)$ or is an algebraic pullback of the hypergeometric equation for the ${}_2F_1$ with rational parameters. We will not define here globally nilpotent operators (see the references), but they are conjectured to coincide with G-operators in $\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$ (which in particular include operators in $\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]\setminus\{0\}$ which are minimal for some non-zero G-function). It is known that operators "coming from geometry" are G-operators and globally nilpotent, by results of André [2, p. 111] and Katz [18, Theorem 10.0] respectively. The Krammer and Bouw-Möller operators "come from geometry" and in [7, §9], the authors even produced explicit G-functions solutions of their operators which are neither algebraic functions nor algebraic pullbacks of a ${}_2F_1$ with rational parameters. However, this does not rule out the possibility that these G-functions could be polynomials in more variables in ${}_{p+1}F_p$ hypergeometric functions with various values of $p \geq 1$. It is known that G-operators of order 1 are exactly those operators in $\overline{\mathbb{Q}}(z)[\frac{d}{dz}]$ of order 1 with a solution of the form $\prod_{j=1}^m (1-\lambda_j z)^{s_j} (= \prod_{j=1}^m {}_1F_0[-s_j; \cdot; \lambda_j z])$, for some $m \geq 1$, $\lambda_j \in \overline{\mathbb{Q}}$ and $s_j \in \mathbb{Q}$; this is already a non-trivial result based on the André-Chudnovsky-Katz Theorem. It is possible that a classification of inhomogeneous linear differential equations of order 1 with coefficients in $\mathbb{Q}(z)$ and admitting a G-function for solution could be obtained as well but this is currently unknown. (The analogous classification problem for E-functions was solved by Gorelov in [13], and reproved by a different method as a consequence of a more general result in [23]; in both cases, the proof is not easy.) After Dwork's unsuccessful attempt, a classification of G-operators of order 2 –or more simply of operators of order 2 minimal for some non-zero G-function– seems elusive, in clear contrast with Gorelov's classification of E-functions of order ≤ 2.

Finally, if there exist $\alpha \in \overline{\mathbb{Q}}$, $|\alpha| < 1$, and $s \in \mathbb{N}$ such that $\mathrm{Li}_s(\alpha) \notin \mathbf{H}$, then the proof given below shows that $\mathrm{Li}_s(\frac{\alpha z}{z-\alpha})$ provides a counter-example, of differential order $s+1$, to Question 2 with the restriction in Theorem 5.

7.1 G-values as connection constants of G-functions

Given a non-zero G-function $f(z)$, let L denote a non-zero operator in $\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]$ such that $Lf(z) = 0$ and of minimal order for f. By standard results of André, Chudnovsky and Katz recalled in [3, §3] or [9, §4.1] (and already used in §3.2), L is fuchsian with rational exponents and, at any $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, L admits a $\mathbb{C}$-basis of solutions of the form

$$F(z-\alpha) := \sum_{e\in E}\sum_{k\in K}(z-\alpha)^e \log(z-\alpha)^k f_{e,k}(z-\alpha)$$

where $E \subset \mathbb{Q}$ and $K \subset \mathbb{N}$ are finite sets, and the $f_{e,k}(z)$ are G-functions; if $\alpha = \infty$, $z-\alpha$ has to be replaced by $1/z$. We call such a basis an ACK basis of L at α. The determination of $\log(z-\alpha)$ is fixed but somewhat irrelevant to our purpose; monodromy around the singularities and ∞ of any solution of L produces further coefficients in $\overline{\mathbb{Q}}[\pi] \subset \mathbf{G}$ for the local expansions of this solution at these points.

Given an element $F(z-\beta)$ of an ACK basis of L at some point $\beta \in \overline{\mathbb{Q}} \cup \{\infty\}$ and an ACK basis $F_1(z-\alpha), \ldots, F_\mu(z-\alpha)$ of L at $\alpha \in \overline{\mathbb{Q}} \cup \{\infty\}$, we can connect locally around

α an analytic continuation of $F(z-\beta)$ (in a suitable cut plane) to this ACK basis:

$$F(z-\beta)=\sum_{j=1}^{\mu}\omega_j F_j(z-\alpha), \tag{7.2}$$

where, following a general terminology, the complex numbers $\omega_1,\dots,\omega_\mu$ are called *connection constants*. In [9, Theorem 2], we proved that $\omega_1,\dots,\omega_\mu$ are in fact in $\mathbf{G}$. We prove here a converse result.

Theorem 6. *Let $\xi\in\mathbf{G}\setminus\{0\}$ and $\alpha\in\overline{\mathbb{Q}}^*\cup\{\infty\}$. There exist a non-zero G-function $F(z)$ solution of a differential operator $L\in\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]\setminus\{0\}$, of minimal order for F, and an ACK basis $F_1(z-\alpha),\dots,F_\mu(z-\alpha)$ of L at α such that the analytic continuation of $F(z)$ in a suitable cut plane is given by*

$$\sum_{j=1}^{\mu}\omega_j F_j(z-\alpha),$$

where $\omega_1=F(\alpha)=\xi$.

Proof. Let $\xi\in\mathbf{G}\setminus\{0\}$. We first assume that $\alpha\neq\infty$. By [9, Theorem 1], there exists a non-zero G-function $G(z)$ of radius of convergence $>|\alpha|$ such that $G(\alpha)=\xi$. Let $L\in\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]\setminus\{0\}$ be of minimal order for G. Let $F_1(z-\alpha),\dots,F_\mu(z-\alpha)$ be an ACK basis of L at α. Up to relabeling the basis, we can assume without loss of generality that there exists $\lambda\le\mu$ such that

$$G(z)=\sum_{j=1}^{\lambda}\omega_j F_j(z-\alpha) \tag{7.3}$$

in a cut plane locally around $z=\alpha$, where the $\omega_j\in\mathbf{G}$ are all non-zero. Up to performing $\overline{\mathbb{Q}}$-linear combinations of the F_j's, we can assume without loss of generality that for all $j\ge 2$, $F_j(z-\alpha)=o(F_1(z-\alpha))$ locally around $z=\alpha$. (Doing so, $(F_1(z-\alpha),\dots,F_\mu(z-\alpha))$ remains an ACK basis of L at α.) Hence, $G(z)\sim\omega_1F_1(z-\alpha)$ as $z\to\alpha$ in the cut plane. Because $G(\alpha)=\xi\neq 0$ and $\omega_1\neq 0$, this implies that $\kappa:=F_1(0)$ is non-zero; therefore $\kappa\in\overline{\mathbb{Q}}^*$ (see [9, Lemma 5] with $\mathbb{A}=\overline{\mathbb{Q}}$). Upon replacing F_1 with $\kappa^{-1}F_1$, we may assume that $\kappa=1$; then we have $\xi=G(\alpha)=\omega_1$.

To work around ∞, we fix $\alpha\in\overline{\mathbb{Q}}^*$ and keep the same notations; we consider now $H(z):=G(\frac{\alpha z}{z-\alpha})$ and $H_j(\frac{1}{z}):=F_j(\frac{\alpha z}{z-\alpha}-\alpha)$. Then H is a non-zero G-function solution of a differential operator $M\in\overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}]\setminus\{0\}$ minimal for H (trivially deduced from L). Now, $H_1(\frac{1}{z}),\dots,H_\mu(\frac{1}{z})$ is an ACK basis of M at ∞ and, by (7.3), we have

$$H(z)=\sum_{j=1}^{\lambda}\omega_j H_j\Big(\frac{1}{z}\Big)$$

still with $\omega_1=\xi$ and $H(z)\to\xi$ as $z\to\infty$ in a suitable cut plane. ∎

7.2 Analytic continuation of the hypergeometric series ${}_{p+1}F_p(z)$

In this section, we prove the following result.

Theorem 7. *Let $f(z) = {}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; z]$ be a hypergeometric series with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Then, the analytic continuation of $f(z)$ to the domain defined by $|\arg(-z)| < \pi$ and $|z| > 1$ is given by*

$$\sum_{j=1}^{p+1} \sum_{\ell} z^{-a_j} \log(1/z)^{\ell} f_{j,\ell}(1/z)$$

where the sum over the integer ℓ is finite and each $f_{j,\ell}(z) \in \mathbf{H}[[z]]$ converges for $|z| < 1$.

Proof. Let

$$R(s) = R(\underline{a}, \underline{b}; s) := \frac{\prod_{j=1}^{p+1} \Gamma(a_j + s)}{\prod_{j=1}^{p} \Gamma(b_j + s)} \Gamma(-s).$$

The poles of $R(s)$ are located at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$, $j = 1, \dots, p+1$, and at $\mathbb{Z}_{\geq 0}$. We define the series

$$M_p(\underline{a}, \underline{b}; z) := \sum_{j=1}^{p+1} \sum_{k=0}^{\infty} \text{Residue}\big(R(s)(-z)^s, s = -a_j - k\big),$$

which converges for any z such that $|\arg(-z)| < \pi$ and $|z| > 1$.

Then the analytic continuation of f to the domain defined by $|\arg(-z)| < \pi$ and $|z| > 1$ is given by

$$f(z) = \frac{\prod_{j=1}^{p} \Gamma(b_j)}{\prod_{j=1}^{p+1} \Gamma(a_j)} M_p(\underline{a}, \underline{b}; z).$$

See the discussion on the Meijer G-function, of which $M_p(\underline{a}, \underline{b}; z)$ is a special case, in [21, §5.3.1], and in particular Eqs. (5) and (17) there. The same method as in §4.1 and §4.2 shows that

$$M_p(\underline{a}, \underline{b}; z) = \sum_{j=1}^{p+1} \sum_{\ell} z^{-a_j} \log(1/z)^{\ell} f_{j,\ell}(1/z)$$

where the sum over the integer $\ell \geq 0$ is finite and each $f_{j,\ell}(z) \in \mathbf{H}[[z]]$ converges for $|z| < 1$. This completes the proof. □

Note that when the a_j's are pairwise distinct mod $\mathbb{Z}$, the poles of $R(s)$ at $-a_j - k$, $k \in \mathbb{Z}_{\geq 0}$ are all distinct and we simply have

$$f(z) = \sum_{j=1}^{p+1} (-z)^{-a_j} \frac{\prod_{k=1, k\neq j}^{p+1} \Gamma(a_k - a_j)}{\prod_{k=1, k\neq j}^{p+1} \Gamma(a_k)} \times \frac{\prod_{k=1}^{p} \Gamma(b_k)}{\prod_{k=1}^{p} \Gamma(b_k - a_j)} {}_{p+1}F_p \left[\begin{matrix} a_j, 1 - b_1 + a_j, \dots, 1 - b_p + a_j \\ 1 - a_1 + a_j, \dots * \dots, 1 - a_{p+1} + a_j \end{matrix}; -\frac{1}{z} \right]$$

where $*$ means that the term $1 - a_j + a_j$ is omitted.

7.3 Proof of Theorem 5

Proof. We start with some general considerations. Let $F(z)$ be a G-function and $L \in \overline{\mathbb{Q}}(z)[\frac{\mathrm{d}}{\mathrm{d}z}] \setminus \{0\}$ be a minimal operator annihilating F. Given a cut plane and $\alpha \in \overline{\mathbb{Q}}^* \cup \{\infty\}$, the local behaviour around α of the analytic continuation of F is described by an ACK basis of L at α. In particular, if $|z|$ is large enough, the analytic continuation of F is of the form

$$\sum_{e\in E}\sum_{k\in K}\sum_{n\geq 0} c_{e,k,n} z^{-e-n}\log(1/z)^k \tag{7.4}$$

where $c_{e,k,n} \in \mathbf{G}$, $E \subset \mathbb{Q}$ and $K \subset \mathbb{N}$ are finite sets (recall that the connection constants ω_j in Eq. (7.2) belong to $\mathbf{G}$, by [9, Theorem 2]). Monodromy around ∞ and the singularities of F produces further coefficients in $\overline{\mathbb{Q}}[\pi] \subset \mathbf{G}$ for the local expansions of F at these points. Hence any analytic continuation of F is in fact of the form (7.4) at ∞.

Let now $f(z) = {}_{p+1}F_p[a_1, \dots, a_{p+1}; b_1, \dots, b_p; z]$ be a hypergeometric series with parameters $a_j \in \mathbb{Q}$ and $b_j \in \mathbb{Q} \setminus \mathbb{Z}_{\leq 0}$. Let $\mu(z), \lambda(z) \in \overline{\overline{\mathbb{Q}}(z)}$ be holomorphic at $z = 0$, with $\lambda(0) = 0$ and $\lambda(z) \to \infty$ as $z \to \infty$. A more precise result than (7.4) can be obtained for the G-function $g(z) := \mu(z)f(\lambda(z))$. We first recall that the analytic continuations of $\mu(z)$ and $\lambda(z)$ in suitable cut planes admit convergent Puiseux expansions at ∞ of the form

$$\sum_{n\geq -m} a_n z^{-n/d} \in \overline{\mathbb{Q}}[[1/z^{1/d}]]$$

for some integers $m \geq 0$ and $d \geq 1$. Moreover, there exists $n \leq -1$ such that $a_n \neq 0$ for λ because $\lambda(z) \to \infty$ as $z \to \infty$. Using Theorem 7, we then deduce that $g(z)$ admits an analytic continuation at ∞ of the form (7.4) with $c_{e,k,n} \in \mathbf{H}$. Again, because $\overline{\mathbb{Q}}[\pi] \subset \mathbf{H}$, any analytic continuation of g is of the form (7.4) at ∞ with $c_{e,k,n} \in \mathbf{H}$.

For $j = 1, \dots, N$, consider $g_j(z) := \mu_j(z)f_j(\lambda_j(z))$ where $f_j(z)$ is a ${}_{p+1}F_p$ hypergeometric series with rational parameters and $\mu_j(z), \lambda_j(z) \in \overline{\overline{\mathbb{Q}}(z)}$ are holomorphic at $z = 0$, with $\lambda_j(0) = 0$ and $\lambda_j(z) \to \infty$ as $z \to \infty$. For any polynomial $P \in \overline{\mathbb{Q}}[X_1, \dots, X_N]$, it follows from the above discussion that any analytic continuation of the G-function $P(g_1(z), \dots, g_N(z))$ is also of the form (7.4) at ∞ with $c_{e,k,n} \in \mathbf{H}$.

Let us now assume that Question 2 has a positive answer when all the λ's tend to ∞ as $z \to \infty$. Given $\xi \in \mathbf{G} \setminus \{0\}$, consider the non-zero G-function $F(z)$ given by Theorem 6 for $\alpha = \infty$: in a suitable cut plane, its analytic continuation is of the form (7.4) with $c_{0,0,0} = \xi$. On the other hand, we have

$$F(z) = P\big(g_1(z), \dots, g_N(z)\big)$$

in a neighborhood of $z = 0$, where the polynomial P and the g_j's are as above. The properties of this specific analytic continuation of $F(z)$ and of those of the right-hand side imply that $\xi \in \mathbf{H}$. Hence $\mathbf{G} \subset \mathbf{H}$ in this case.

If the λ's all tend to ∞ as $z \to \beta \in \overline{\mathbb{Q}}^*$, then the above argument can be adapted using Puiseux expansions of the μ's and λ's of the form $\sum_{n\geq -m} a_n(z-\beta)^{n/d} \in \overline{\mathbb{Q}}[[(z-\beta)^{1/d}]]$, a G-function F given by Theorem 6 with $\alpha = \beta$ and an ACK basis at β. □

Remark 4. We emphasize the fact that in the above proof we need to make the λ's all tend to ∞ in a neighborhood of a common point in $\overline{\mathbb{Q}}^* \cup \{0\}$ because we can then compare various expansions at infinity of the form (7.4) with the help of Theorem 7. It is indeed not known if a version of Theorem 7 holds at a finite point α instead of ∞. More precisely $f(z)$ can still be locally written as

$$\sum_{j=1}^{p+1} \sum_{\ell} (z-\alpha)^{a_j} \log(z-\alpha)^{\ell} f_{j,\ell}(z-\alpha)$$

with $f_{j,\ell}(z-\alpha) \in \mathbb{C}[[z-\alpha]]$, but it is not known if the $f_{j,\ell}(z-\alpha)$ are in $\mathbf{H}[[z]]$.

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