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Deforming semistable Galois representations

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ABSTRACT **Let V be a p-adic representation of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$. One of the ideas of Wiles's proof of FLT is that, if V is the representation associated to a suitable autromorphic form (a modular form in his case) and if V' is another p-adic representation of $Gal(\bar{\mathbf{Q}}/\mathbf{Q})$ "closed enough" to V, then V' is also associated to an automorphic form. In this paper we discuss which kind of local condition at p one should require on V and V' in order to be able to extend this part of Wiles's methods.**

Geometric Galois Representations (refs. 1 and 2; exp. III and VIII). Let $\bar{\mathbf{Q}}$ be a chosen algebraic closure of $\mathbf{Q}$ and $G = Gal(\bar{\mathbf{Q}}/\mathbf{Q})$. For each prime number ℓ, we choose an algebraic closure $\bar{\mathbf{Q}}_\ell$ of $\mathbf{Q}_\ell$ together with an embedding of $\bar{\mathbf{Q}}$ into $\bar{\mathbf{Q}}_\ell$ and we set $G_\ell = Gal(\bar{\mathbf{Q}}_\ell/\mathbf{Q}_\ell) \subset G$. We choose a prime number p and a finite extension E of $\mathbf{Q}_p$.

An *E-representation* of a profinite group J is a finite dimensional E vector space equipped with a linear and continuous action of J.

An *E-representation* V of G is said to be *geometric* if

(*i*) it is unramified outside of a finite set of primes;

(*ii*) it is potentially semistable at p (we will write pst for short).

[The second condition implies that V is de Rham, hence Hodge-Tate, and we can define its *Hodge-Tate numbers* $h^r = h^r(V) = dim_E\,(C_p(r) \otimes_{\mathbf{Q}_p} V)^{G_p}$ where $C_p(r)$ is the usual Tate twist of the p-adic completion of $\bar{\mathbf{Q}}_p$ (one has $\Sigma_{r\in\mathbf{Z}} h^r = d$). It implies also that one can associate to V a representation of the Weil-Deligne group of $\mathbf{Q}_p$, hence a conductor $N_V(p)$, which is a power of p].

Example: If X is a proper and smooth variety over $\mathbf{Q}$ and $m \in \mathbf{N}$, $j \in Z$, then the p-adic representation $H^m_{et}(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(j))$ is geometric.

[Granted the smooth base change theorem, the representation is unramified outside of p and the primes of bad reduction of X. Faltings (3) has proved that the representation is crystalline at p in the good reduction case. It seems that Tsuji (4) has now proved that, in case of semistable reduction, the representation is semistable. The general case can be deduced from Tsuji's result using de Jong's (5) work on alterations].

CONJECTURE (1). *If V is a geometric irreducible E-representation of G, then V comes from algebraic geometry, meaning that there exist X, m, j such that V is isomorphic, as a p-adic representation, to a subquotient of $E \otimes_{\mathbf{Q}_p} H^m_{et}(X_{\bar{\mathbf{Q}}}, \mathbf{Q}_p(j))$.*

Even more should be true. Loosely speaking, say that a geometric irreducible E-representation V of G is *a Hecke representation* if there is a finite $\mathbf{Z}_p$-algebra $\mathcal{H}$, generated by Hecke operators acting on some automorphic representation space, equipped with a continuous homomorphism $\rho : G \to GL_d(\mathcal{H})$, "compatible with the action of the Hecke operators," such that *V comes from $\mathcal{H}$* (i.e., is isomorphic to the one we get from ρ via a map $\mathcal{H} \to E$). Then any geometric Hecke representation of G should come from algebraic geometry and any geometric irreducible representation should be Hecke.

At this moment, this conjecture seems out of reach. Nevertheless, for an irreducible two-dimensional representation of G, to be geometric Hecke means to be a Tate twist of a representation associated to a modular form. Such a representation is known to come from algebraic geometry. Observe that the heart of Wiles's proof of FLT is a theorem (6, th. 0.2) asserting that, if V is a suitable geometric Hecke E-representation of dimension 2, then any geometric E-representation of G which is "close enough" to V is also Hecke.

It seems clear that Wiles's method should apply in more general situations to prove that, starting from a suitable Hecke E-representation of G, any "close enough" geometric representation is again Hecke. The purpose of these notes is to discuss possible generalizations of the notion of "close enough" and the possibility of extending local computations in Galois cohomology which are used in Wiles's theorem. More details should be given elsewhere.

Deformations (7–9). Let $\mathbb{O}_E$ be the ring of integers of E, π a uniformizing parameter and $k = \mathbb{O}_E/\pi\mathbb{O}_E$ the residue field.

Denote by $\mathcal{C}$ the category of local noetherian complete $\mathbb{O}_E$*-algebras* with residue field k (we will simply call the objects of this category $\mathbb{O}_E$-algebras).

Let J be a profinite group and $Rep^f_{\mathbf{Z}_p}(J)$ the category of $\mathbf{Z}_p$-modules of finite length equipped with a linear and continuous action of J. Consider a strictly full subcategory $\mathfrak{D}$ of $Rep^f_{\mathbf{Z}_p}(J)$ stable under subobjects, quotients, and direct sums.

For A in $\mathcal{C}$, an *A-representation* T of J is an A-module of finite type equipped with a linear and continuous action of J. We say that *T lies in D* if all the finite quotients of T viewed as Z_p-representations of J are objects of $\mathfrak{D}$. The A-representations of J lying in $\mathfrak{D}$ form a full subcategory $\mathfrak{D}(A)$ of the category $Rep^{tf}_A(J)$ of A-representations of J.

We say T is *flat* if it is flat ($\Leftrightarrow$ free) as an A-module.

Fix u a (flat !)-k-representation of J lying in $\mathfrak{D}$. For any A in $\mathcal{C}$, let $F(A) = F_{u,J}(A)$ be the set of isomorphism classes of flat A-representations T of J such that $T/\pi T \simeq u$. Set $F_\mathfrak{D}(A) = F_{u,J,\mathfrak{D}}(A)$ = the subset of $F(A)$ corresponding to representations which lie in $\mathfrak{D}$.

PROPOSITION. *If $H^0(J, \mathfrak{gl}(u)) = k$ and $dim_k H^1(J, \mathfrak{gl}(u)) < +\infty$, then F and $F_\mathfrak{D}$ are representable.*

(The ring $R_\mathfrak{D} = R_{u,J,\mathfrak{D}}$ which represents $F_\mathfrak{D}$ is a quotient of the ring $R = R_{u,J}$ representing F.)

Fix also a flat $\mathbb{O}_E$-representation U of J lifting u and lying in $\mathfrak{D}$. Its class defines an element of $F_\mathfrak{D}(\mathbb{O}_E) \subset F(\mathbb{O}_E)$, hence augmentations $\varepsilon_U{:}R \to \mathbb{O}_E$ and $\varepsilon_{U,\mathfrak{D}}{:}R_\mathfrak{D} \to \mathbb{O}_E$.

Set $\mathbb{O}_n = \mathbb{O}_E/\pi^n\mathbb{O}_E$ and $U_n = U/\pi^n U$. If $\mathfrak{p}_U = ker\ \varepsilon_U$ and $\mathfrak{p}_{U,\mathfrak{D}} = ker\ \varepsilon_{U,\mathfrak{D}}$, we have canonical isomorphisms

$$\begin{array}{ccccc} ((\mathfrak{p}_U+\pi^n R)/(\mathfrak{p}_U^2+\pi^n R))^* & \simeq & Ext^1_{\mathbb{O}_n[J]}(U_n, U_n) & \simeq & H^1(J, \mathfrak{gl}(U_n)) \\ \cup & & \cup & & \cup \\ (\mathfrak{p}_{U,\mathfrak{D}}+\pi^n R_\mathfrak{D})/(\mathfrak{p}_{U,\mathfrak{D}}^2+\pi^n R_\mathfrak{D})^* & \simeq & Ext^1_{\mathbb{O}_n,\mathfrak{D}}(U_n, U_n) & =: & H^1_\mathfrak{D}(J, \mathfrak{gl}(U_n)) \end{array}$$

Close Enough to V Representations. We fix a geometric E-representation V of G (morally a "Hecke representation").

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We choose a G-stable $\mathcal{O}_E$-lattice U of V and assume $u = U/\pi U$ absolutely irreducible (hence V is a fortiori absolutely irreducible).

We fix also a finite set of primes S containing p and a full subcategory $\mathcal{D}_p$ of $Rep^f_{\mathbf{Z}_p}(G_p)$, stable under subobjects, quotients, and direct sums.

For any E-representation W of G_p, we say *W lies in $\mathcal{D}_p$* if a G_p-stable lattice lies in $\mathcal{D}_p$.

We say an E-representation of G is *of type* $(S, \mathcal{D}_p)$ if it is unramified outside of S and lies in $\mathcal{D}_p$.

Now we assume V is of type $(S, \mathcal{D}_p)$. We say an E-representation V' of G is $(S, \mathcal{D}_p)$*-close to V* if:

(i) given a G-stable lattice U' of V', then $U'/\pi U' \simeq u$;

(ii) V' is of type $(S, \mathcal{D}_p)$.

Then, if $\mathbf{Q}_S$ denote the maximal Galois extension of $\mathbf{Q}$ contained in $\bar{\mathbf{Q}}$ unramified outside of S, deformation theory applies with $J = G_S = Gal(\mathbf{Q}_S/\mathbf{Q})$ and $\mathcal{D}$ the full subcategory of $Rep^f_{\mathbf{Z}_p}(G_S)$ whose objects are T's which, viewed as representations of G_p, are in $\mathcal{D}_p$. But if we want the definition of $(S, \mathcal{D}_p)$-close to V to be good for our purpose, it is crucial that the category $\mathcal{D}_p$ is *semistable*, i.e., is such that any E-representation of G_p lying in $\mathcal{D}_p$ is pst.

We would like also to be able to say something about the conductor of an E-representation of G_p lying in $\mathcal{D}_p$. Since $H^1_{\mathcal{D}}(J, \mathfrak{gl}(U_n))$ is the kernel of the natural map

$$H^1(G_S, \mathfrak{gl}(U_n)) \rightarrow H^1_{\mathcal{D}_p}(G_p, \mathfrak{gl}(U_n)),$$

it is better also if we are able to compute $H^1_{\mathcal{D}_p}(G_p, \mathfrak{gl}(U_n))$.

In the rest of these notes, we will discuss some examples of such semistable categories $\mathcal{D}_p$'s.

Examples of Semi-Stable $\mathcal{D}_p$'s.

Example 1: The category $\mathcal{D}_p^{cr}$ (application of (10); cr, crystalline).

For any $\mathcal{O}_E$-algebra A, consider the category $MF(A)$ whose objects are A-module M of finite type equipped with

(i) a decreasing filtration (indexed by $\mathbf{Z}$),

$$\ldots Fil^i\, M \supset Fil^{i+1}\, M \supset \ldots,$$

by sub-A-modules, direct summands as $\mathbf{Z}_p$-modules, with $Fil^i\, M = M$ for $i \ll 0$ and $= 0$ for $i \gg 0$;

(ii) for all $i \in \mathbf{Z}$, an A-linear map $\phi^i : Fil^i\, M \rightarrow M$, such that $\phi^i\,|_{Fil^{i+1}\, M} = p\phi^{i+1}$ and $M = \Sigma Im\ \phi^i$.

With an obvious definition of the morphisms, $MF(A)$ is an A-linear abelian category.

For $a \leq b \in \mathbf{Z}$, we define $MF^{[a,b]}(A)$ to be the full subcategory of those M, such that $Fil^a\, M = M$ and $Fil^{b+1}\, M = 0$. If $a < b$, we define also $MF^{]a,b]}(A)$ as the full subcategory of $MF^{[a,b]}(A)$ whose objects are those M with no nonzero subobjects L with $Fil^{a+1}\, L = 0$.

As full subcategories of $MF(A)$, $MF^{[a,b]}(A)$ and $MF^{]a,b]}(A)$ are stable under taking subobjects, quotients, direct sums, and **extensions.**

If $\bar{\mathbf{Z}}_p$ denote the p-adic completion of the normalization of $\mathbf{Z}_p$ in $\bar{\mathbf{Q}}_p$, the ring

$$A_{cris} = \varprojlim H^0((Spec(\bar{\mathbf{Z}}_p/p)/W_n)_{crys}, struct.sheaf)$$

is equipped with an action of G_p and a morphism of Frobenius $\phi : A_{cris} \rightarrow A_{cris}$. There is a canonical map $A_{cris} \rightarrow \bar{\mathbf{Z}}_p$ whose kernel is a divided power ideal J. Moreover, for $0 \leq i \leq p - 1$, $\phi(J^{[i]}) \subset p^i A_{cris}$. Hence, because A_{cris} has no p-torsion, we can define for such an i, $\phi^i : J^{[i]} \rightarrow A_{cris}$ as being the restriction of ϕ to $J^{[i]}$ divided out by p^i.

For M in $MF^{[-(p-1),0]}(A)$, we then can define $Fil^o(A_{cris} \otimes M)$ as the sub-A-module of $A_{cris} \otimes_{\mathbf{Z}_p} M$, which is the sum of the images of the $Fil^i A_{cris} \otimes Fil^{-i} M$, for $0 \leq i \leq p - 1$. We can define $\phi^o : Fil^0(A_{cris} \otimes M) \rightarrow A_{cris} \otimes M$ as being $\phi^i \otimes \phi^{-i}$ on $Fil^i\, A_{cris} \otimes Fil^{-i}\, M$. If we set

$$\underline{U}(M) = (Fil^O(A_{cris} \otimes_{\mathbf{Z}_p} M))_{\phi^0=1},$$

this is an A-module of finite type equipped with a linear and continuous action of G_p. We get in this way an A-linear functor

$$\underline{U} : MF^{[-(p-1),0]}(A) \rightarrow Rep^{ft}_A(G_p)$$

which is exact and faithful. Moreover, the restriction of $\underline{U}$ to $MF^{]-(p-1),0]}(A)$ is fully faithful. We call $\mathcal{D}^{cr}_p(A)$ the essential image.

PROPOSITION. *Let V' be an E-representation of G_p. Then V' lies in $\mathcal{D}^{cr}_p$ if and only if the three following conditions are satisfied*:

(i) *V' is crystalline (i.e., V' is pst with conductor $N_{V'}(p) = 1$);*

(ii) *$h^r(V') = 0$ if $r > 0$ or $r < -p + 1$;*

(iii) *V' has no nonzero subobject V'' with $V''(-p + 1)$ unramified.*

Moreover (11), *if X is a proper and smooth variety over $\mathbf{Q}_p$ with good reduction and if $r,n \in \mathrm{N}$ with $0 \leq r \leq p - 2$, $H^r_{et}(X_{\bar{\mathbf{Q}}_p}, \mathbf{Z}/p^n\mathbf{Z})$ is an object of $\mathcal{D}^{cr}_p(\mathbf{Z}_p)$.*

Remarks: (i) Define $\mathcal{D}^{ff}_p$ as the full subcategory of $Rep^f_{\mathbf{Z}_p}(G_p)$, whose objects are representations which are isomorphic to the general fiber of a finite and flat group scheme over $\mathbf{Z}_p$. If $p \neq 2$, $\mathcal{D}^{ff}_p$ is a full subcategory **stable under extensions** of $\mathcal{D}^{cr}_p$ (this is the essential image of $MF^{[-1,0]}(\mathbf{Z}_p)$).

(ii) Deformations in $\mathcal{D}^{cr}_p$ don't change Hodge type: if V,V' are E-representations of G_p, lying in $\mathcal{D}^{cr}_p$ and if one can find lattices U of V and U' of V' such that $U/\pi U \simeq U'/\pi U'$, then $h^r(V) = h^r(V')$ for all $r \in \mathbf{Z}$ (if $U/\pi U = \underline{U}(M)$, $h^r(V) = dim_k gr^{-r}M$).

Computation of $H^1_{\mathcal{D}^{cr}_p}$. This can be translated in terms of the category $MF(\mathcal{O}_E) \supset MF^{]-p+1,0]}(\mathcal{O}_E)$.

In $MF(\mathcal{O}_E)$, define $H^i_{MF}(\mathbf{Q}_p, M)$ as being the i^{th} derived functor of the functor $Hom_{MF(\mathcal{O}_E)}(\mathcal{O}_E, -)$. These groups are the cohomology of the complex

$$Fil^0\, M \xrightarrow{1-\phi} M \rightarrow 0 \rightarrow 0 \rightarrow \ldots$$

If we set $t_M = M/Fil^0 M$, this implies $lg_{\mathcal{O}_E} H^1_{\mathcal{D}^{cr}_p}(\mathbf{Q}_p, M) = lg_{\mathcal{O}_E} H^0 + lg_{\mathcal{O}_E} t_M$.

Hence, if U is a G_p-stable lattice of an E-representation V of G_p lying in $\mathcal{D}^{cr}_p$, and if, for any $i \in \mathbf{Z}$, $h_r = h_r(V)$, with obvious notations, we get $H^1_{\mathcal{D}^{cr}_p}(\mathbf{Q}_p, \mathfrak{gl}(U_n)) = Ext^1_{MF^{]-p+1,0]}(A)}(M_n, M_n) = Ext^1_{MF(A)}(M_n, M_n) = H^1_{MF}(\mathbf{Q}_p, End_{\mathcal{O}_E}(M_n))$ and $lg_{\mathcal{O}_E} H^1_{\mathcal{D}^{cr}_p}(\mathbf{Q}_p, \mathfrak{gl}(U_n)) = lg_{\mathcal{O}_E}\ H^0(\mathbf{Q}_p, \mathfrak{gl}(U_n)) + nh$, where $h = \Sigma_{i<j} h_i h_j$ [this generalizes a result of Ramakrishna (9)].

A Special Case. Of special interest is the case where $H^0(\mathbf{Q}_p, \mathfrak{gl}(u)) = k$, which is equivalent to the representability of the functor $F_{u,G_p,\mathcal{D}^{cr}}$. In this case, $H^1_{\mathcal{D}^{cr}_p}(\mathbf{Q}_p, \mathfrak{gl}(U_n)) \simeq (\mathcal{O}_n)^{h+1}$ and $H^1_{\mathcal{D}^{cr}_p}(\mathbf{Q}_p, \mathfrak{sl}(U_n)) \simeq (\mathcal{O}_n)^h$. Moreover, because there is no H^2, the deformation problem is smooth, hence $R_{u,G_p,\mathcal{D}^{cr}_p} \simeq \mathcal{O}_E[[X_0, X_1, X_2, \ldots, X_h]]$.

Example 2: $\mathcal{D}^{na}_p$ (the naive generalization of $\mathcal{D}^{cr}_p$ to the semistable case).

For any $\mathcal{O}_E$-algebra A, we can define the category $MFN(A)$ whose objects consist of a pair (M, N) with M object of $MF(A)$ and $N : M \rightarrow M$ such that

(i) $N(Fil^i M) \subset Fil^{i-1} M$,

(ii) $N\phi^i = \phi^{i-1} N$.

With an obvious definition of the morphisms, this is an abelian A-linear category and $MF(A)$ can be identified to the full subcategory of $MFN(A)$ consisting of M's with $N = 0$.

We have an obvious definition of the category $MFN^{]-p+1,0]}(A)$. There is a natural way to extend $\underline{U}$ to a functor

$$\underline{U} : MFN^{]-p+1,0]}(A) \rightarrow Rep^f_{Z_p}(G_p)$$

again exact and fully faithful. We call $\mathcal{D}^{cr}_p(A)$ the essential image.

There is again a simple characterization of the category $\mathcal{D}^{na}_p(E)$ of E-representations of G_p lying in $\mathcal{D}^{na}_p$ as a suitable full subcategory of the category of semistable representations with crystalline semisimplification. Moreover:

If $p \neq 2$, the category of semistable V values with $h^r(V) = 0$ if $r \notin \{0, -1\}$ is a full subcategory **stable under extensions** of $\mathcal{D}_p^{na}(E)$.

For $0 \leq r < p-1$, let $\mathcal{D}_p^{ord,r}$ the full subcategory of $Rep^f_{\mathbf{Z}_p}(G_p)$ of T's such that there is a filtration (necessarily unique)

$$0 = F_{r+1}T \subset F_rT \subset \ldots F_1T \subset F_0T = T$$

such that $gr_iT(-i)$ is unramified for all i; then $\mathcal{D}_p^{ord,r}$ is a full subcategory of $\mathcal{D}_p^{na}$ **stable under extensions**.

Again, in $\mathcal{D}_p^{na}$, deformations don't change Hodge type. The conductor may change.

Computation of $H^1_{\mathcal{D}_p^{na}}(\mathbf{Q}_p, \mathfrak{gl}(U_n))$. As before, this can be translated in terms of the category $MFN(\mathcal{O}_E) \supset MFN^{]-p+1,0]}(\mathcal{O}_E)$: if we define $H^i_{MFN}(\mathbf{Q}_p, M)$ as the i^{th}-derived functor, in the category $MFN(\mathcal{O}_E)$, of the functor $Hom_{MFN(\mathcal{O}_E)}(\mathcal{O}_E, -)$, these groups are the cohomology of the complex

$$Fil^0M \to Fil^{-1}M \oplus M \to M \to 0 \to 0 \to \ldots$$

(with $x \mapsto (Nx, (1-\phi^0)x)$ and $(y, z) \mapsto (1-\phi^{-1})y - Nz$). Again, in this case, $H^1_{\mathcal{D}_p^{na}}(\mathbf{Q}_p, \mathfrak{gl}(U_n)) = Ext^1_{MFN^{]-p+1,0]}(A)}(M_n, M_n) = Ext^1_{MFN(A)}(M_n, M_n) = H^1_{MFN}(\mathbf{Q}_p, End_{\mathcal{O}_E}(M_n))$. But,

(*i*) the formula for the length is more complicated, and

(*ii*) the (local) deformation problem is not always smooth.

Example 3: $\mathcal{D}_p^{st}$ [the good generalization of $\mathcal{D}_p^{cr}$ to the semistable case, theory due to Breuil (12)].

Let $S = \mathbf{Z}_p\langle u\rangle$ be the divided power polynomial algebra in one variable u with coefficients in $\mathbf{Z}_p$. If $v = u - p$, we have also $S = \mathbf{Z}_p\langle v\rangle$. Define:

(*a*) Fil^iS as the ideal of S generated by the $v^m/m!$, for $m \geq i$;

(*b*) ϕ as the unique $\mathbf{Z}_p$-endomorphism such that $\phi(u) = u^p$;

(*c*) N as the unique $\mathbf{Z}_p$-derivation from S to S such that $N(u) = -u$.

For $r \leq p-1$, ϕ^r: $Fil^rS \to S$ is defined by $\phi^r(x) = p^{-r}\phi(x)$.

If $r \leq p-2$, let $'\mathcal{M}_0^r$ be the category whose objects consist of:

(*i*) an S-module $\mathcal{M}$,

(*ii*) a sub-S-module $Fil^r\mathcal{M}$ of $\mathcal{M}$ containing $Fil^rS.\mathcal{M}$,

(*iii*) a linear map ϕ^r: $Fil^r\mathcal{M} \to \mathcal{M}$, such that $\phi^r(sx) = \phi^r(s).\phi(x)$ (where ϕ: $\mathcal{M} \to \mathcal{M}$ is defined by $\phi(x) = \phi^r(v^rx)/\phi^r(v^r)$), with an obvious definition of the morphisms. We consider the full subcategory $\mathcal{M}_0^r$ of $'\mathcal{M}_0^r$ whose objects satisfy

(*i*) as an S-module $\mathcal{M} \simeq \oplus_{1\leq i\leq d}S/p^{n_d}S$ for suitable integers d and $(n_i)_{1\leq i\leq d}$;

(*ii*) as an S-module $\mathcal{M}$ is generated by the image of ϕ^r.

Finally, define $\mathcal{M}^r$ as the category whose objects are objects $\mathcal{M}$ of $\mathcal{M}_0^r$ equipped with a linear endomorphism

$$N : \mathcal{M} \to \mathcal{M}$$

satisfying

(*i*) $N(sx) = N(s).x + s.N(x)$ for $s \in S, x \in \mathcal{M}$,

(*ii*) $v.N(Fil^r\mathcal{M}) \subset Fil^r\mathcal{M}$,

(*iii*) if $x \in Fil^r\mathcal{M}$, $\phi^1(v).N(\phi^r(x)) = \phi^r(v.N(x))$.

This turns out to be an abelian $\mathbf{Z}_p$-linear category and we call $MFB^{[-r,o]}(\mathbf{Z}_p)$ the opposite category.

For A an $\mathcal{O}_E$-algebra, one can define in a natural way the category $MFB^{[-r,o]}(A)$ (for instance, if A is artinian, an object of this category is just an object of $MFB^{[-r,o]}(\mathbf{Z}_p)$ equipped with an homomorphism of A into the ring of the endomorphisms of this object).

Breuil defines natural "inclusions":

$$MFB^{[-r-1,o]}(A) \subset MFB^{[-r,o]}(A) \text{ (if } r+1 \leq p-2),$$

$$MF^{[-r,o]}(A) \subset MFN^{[-r,o]}(A) \subset MFB^{[-r,o]}(A).$$

Moreover, the simple objects of $MF^{[-r,o]}(k)$, $MFN^{[-r,o]}(k)$, and $MFB^{[-r,o]}(k)$ are the same. Breuil extends $\underline{U}$ to $MFB^{[-r,o]}(A)$ and proves that this functor is again exact and fully faithful. We call $\mathcal{D}_p^{st,r}(A)$ the essential image.

Let V be an E-representation of G_p. Breuil proves that, if V lies in $\mathcal{D}_p^{st,r}$ then V is semistable and $h^m(V) = 0$ if $m>0$ or $m < -r$. Conversely, it seems likely that if V satisfies these two conditions, V lies in $\mathcal{D}_p^{st,r}$. This is true if $r = 1$, and it has been proven by Breuil if $E = \mathbf{Q}_p$ and V is of dimension 2. More importantly, Breuil proved also

PROPOSITION (13). *Let X be a proper and smooth variety over* $\mathbf{Q}_p$. *Assume X as semistable reduction and let* r, n $\in \mathbf{N}$ *with* $0 \leq$ r $\leq$ p-2; *then* $H^r_{et}(X_{\bar{\mathbf{Q}}_p}, \mathbf{Z}/p^n\mathbf{Z})$ *is an object of* $\mathcal{D}_p^{st,r}(\mathbf{Z}_p)$.

When working with $\mathcal{D}_p^{st,r}$, *deformation may change the Hodge type* (the conductor also). The computation of $H^1_{\mathcal{D}_p^{st,r}}(\mathbf{Q}_p, \mathfrak{gl}(U_n))$ still reduces to a computation in $MFB^{[-r,o]}(\mathcal{O}_E)$ (or equivalently in $\mathcal{M}^r$). This computation becomes difficult in general but can be done in specific examples.

Final Remarks. Let L be a finite Galois extension of $\mathbf{Q}_p$ contained in $\bar{\mathbf{Q}}_p$, $\mathcal{O}_L$ the ring of integers and $e_L = e_L/\mathcal{D}_p$.

(*a*) Call $\mathcal{D}_p^{ff,L}$, the full subcategory of $Rep^f_{\mathbf{Z}_p}(G_p)$ whose objects are representations which, when restricted to $Gal(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$, extends to a finite and flat group scheme over $\mathcal{O}_L$. If $e_L \leq p-1$, an E-representation V lies in $\mathcal{D}_p$ if and only if it becomes crystalline over L and $h^m(V) = 0$ for $m \notin \{0, -1\}$). If $e_L < p-1$, Conrad (14) defines an equivalence between $\mathcal{D}_p^{ff,L}$ and a nice category of filtered modules equipped with a Frobenius and an action of $Gal(L/\mathbf{Q}_p)$. Using it, one can get the same kind of results as we described for $\mathcal{D}_p^{cr}$. For $e_L = p-1$, the same thing holds if we require that the representation of $Gal(\bar{\mathbf{Q}}_p/\mathbf{Q}_p)$ extends to a connected finite and flat group scheme over $\mathcal{O}_L$.

(*b*) More generally, Breuil's construction should extend to E-representations becoming semistable over L with $h^m(V) = 0$ if $m > 0$ or $< -(p-1)/e_L$ ($\leq -(p-1)/e_L$ with a "grain de sel").

(*c*) Let $Rep\mathbf{Q}_p(G_p)^r_{cris,L}$ (resp. $Rep\mathbf{Q}_p(G_p)^r_{st,L}$) be the category of $\mathbf{Q}_p$-representations V of G_p becoming crystalline over L (resp. semistable) with $h^m(V) = 0$ if $m > 0$ or $m < -r$. Let $\mathcal{D}_p^{cris,r,L}$ (resp. $\mathcal{D}_p^{st,r,L}$) be the full subcategory of $Rep^f_{\mathbf{Z}_p}(G_p)$ consisting of T's for which one can find an object V of $Rep\mathbf{Q}_p(G_p)^r_{cris,L}$ (resp. $Rep\mathbf{Q}_p(G_p)^r_{st,L}$) G_p-stable lattices $U' \subset U$ of V such that $T \simeq U/U'$. I feel unhappy not being able to prove the following:

Conjecture. $C_p^{cris,r,L}$ *(resp.* $C_p^{st,r,L}$*): Let V be a* $\mathbf{Q}_p$-*representation of V lying in* $\mathcal{D}_p^{cris,r,L}$ (resp. $\mathcal{D}_p^{st,r,L}$). *Then V an object of* $Rep_{\mathbf{Q}_p}(G_p)^r_{cris,L}$ *(resp.* $Rep_{\mathbf{Q}_p}(G_p)^r_{st,L}$*).*

The only cases I know $C_p^{cris,r,L}$ are $r = 0$, $r = 1$, and $e_L \leq p-1$, $r \leq p-1$, and $e_L = 1$. The only cases I know $C_p^{st,r,L}$ are $r = 0$, $r = 1$, and $e_L \leq p-1$. Of course, each time we know the answer is yes, this implies that the category is semistable.

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1. Fontaine, J.-M. & Mazur, B. (1995) *Geometric Galois Representations*, Hong Kong Conference on Elliptic Curves and Modular Forms (International Press, Boston), pp. 41–78.
2. Périodes p-adiques, Astérisque 223 (1994) *Soc. Math. de France.*
3. Faltings, G. (1989) in *Algebraic Analysis, Geometry and Number Theory* (The Johns Hopkins University Press, Baltimore, MD), pp. 25–80.
4. Tsuji, T. *On Syntomic Cohomology of Higher Degree of a Semi-Stable Family*, preprint.
5. de Jong, A. J. (1995) *Smoothness, Semi-Stability, and Alterations*, preprint.
6. Wiles, A. (1995) *Ann. Math.* **141,** 443–551.
7. Schlessinger, M. (1968) *Trans. A. M. S.* **130,** 208–222.
8. Mazur, B. (1989) in *Galois Groups Over Q* (M.S.R.I. Publications, Springer), pp. 385–437.

9. Ramakrishna, R. (1993) *Comp. Math.* **87,** 269–286.
10. Fontaine, J.-M. & Laffaille, G. (1982) *Construction de Représentations p-adiques* Ann. Scient. E. N. S. **15,** 547–608.
11. Fontaine, J.-M. & Messing, W. (1987) *p-adic Periods and p-adic étale Cohomology, Contemporary Mathematics* **67,** 179–207.
12. Breuil, C. (1995) *Construction de Représentations p-adiques Semi-Stables*, Université de Paris-Sud, preprint.
13. Breuil, C. (1996) *Cohomologie étale de Torsion et Cohomologie Cristalline en Réduction Semi-Stable*, Université de Paris-Sud, preprint.
14. Conrad, B. (1996) *Finite Honda Systems and Supersingular Elliptic Curves*, Ph.D. (Princeton University, Princeton).