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Vector bundles and p-adic Galois representations

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ABSTRACT. Let F be a perfect field of characteristic $p > 0$ complete with respect to a non trivial absolute value. Let E be a non archimedean locally compact field whose residue field is contained in F. To these data, we associate a "complete regular curve" $X = X_{F,E}$ defined over E. If $\overline{F}$ is an algebraic closure of F and $H = \mathrm{Gal}(\overline{F}/F)$, there is an equivalence of categories between continuous finite dimensional E-linear representations of H and semistable vector bundles over X of slope 0. To construct X we first construct the ring B of "rigid analytic functions of the variable π on the punctured unit disk $\{z \in F \mid 0 < |z| < 1\}$".

Let C be the p-adic completion of an algebraic closure $\overline{K}$ of a p-adic field K. A classical construction from p-adic Hodge theory associates to C a field $F = F(C)$ as above and the group G_K acts on the curve $X = X_{F(C),\mathbb{Q}_p}$. We study G_K-equivariant vector bundles over X and classify those which are "de Rham". The two main theorems about p-adic de Rham representations are recovered by considering the special case of semistable vector bundles of slope 0. This paper is a survey. Details and proofs will appear elsewhere.

1. Curves and vector bundles

1.1. General conventions and notations. If R is a commutative ring and M_1, M_2 are R-modules, we denote by $\mathcal{L}_R(M_1, M_2)$ the R-module of R-linear maps $f : M_1 \to M_2$.

If L is a field equipped with a non archimedean absolute value $|\ |$ (or a valuation v), we denote $\mathcal{O}_L = \{x \in L \mid |x| \leq 1\}$ (or $v(x) \geq 0\}$) the corresponding valuation ring, $\mathfrak{m}_L$ the maximal ideal of $\mathcal{O}_L$ and $k_L = \mathcal{O}_L/\mathfrak{m}_L$ the residue field.

As usual, if X is a noetherian scheme, we view a *vector bundle over* X as a locally free coherent $\mathcal{O}_X$-module.

If a group G acts on the left on a noetherian scheme X, an $\mathcal{O}_X$*-representation of* G (resp. a G*-equivariant vector bundle over* X) is a coherent $\mathcal{O}_X$-module (resp. a vector bundle) $\mathcal{F}$ equipped with a semi-linear action of G in the following sense:

- for all $g \in G$, if $g : X \xrightarrow{\sim} X$ is the action of g on X, one is given an isomorphism
$$c_g : g^*\mathcal{F} \xrightarrow{\sim} \mathcal{F},$$

- the following cocyle condition is satisfied

$$c_{g_2} \circ g_2^* c_{g_1} = c_{g_1 g_2}, \quad g_1, g_2 \in G$$

via the identification $g_2^*(g_1^*\mathcal{F}) = (g_1 g_2)^*\mathcal{F}$.

If $X = \mathrm{Spec}(B)$ is affine, an $\mathcal{O}_X$-representation of G is nothing else than a finite type B-module equipped with a semi-linear left action of G.

In this paper, we use freely the formalism of tensor categories (for which we refer to [**DM82**]). For instance, if G is a group acting on a noetherian scheme X, equipped with the tensor product of the underlying $\mathcal{O}_X$-modules, the category $\mathrm{Rep}_{\mathcal{O}_X}(G)$ of $\mathcal{O}_X$-representations of G is an abelian tensor category, though the full sub-category $Bund_X(G)$ of G-equivariant vector bundles is a rigid additive tensor category. If X is a smooth geometrically connected projective curve over a perfect field E, the full subcategory $Bund_X^0(G)$ of G-equivariant vector bundles which are semistable of slope 0 is a tannakian E-linear category.

1.2. Complete regular curves. A *regular curve* X is a separated integral noetherian regular scheme of dimension 1. In other words, X is a separated connected scheme obtained by gluing a finite number of spectra of Dedekind rings.

Let X be a regular curve, $\mathcal{K} = \mathcal{O}_{X,\eta}$ its function field (i.e. the local ring at the generic point η), $|X|$ the set of closed point of X. For any $x \in |X|$, let v_x be the unique discrete valuation of $\mathcal{K}$ such that

$$v_x(\mathcal{K}^*) = \mathbb{Z} \text{ and } \mathcal{O}_{X,x} = \left\{ f \in \mathcal{K} \mid v_x(f) \geq 0 \right\} .$$

The field $\mathcal{K}$, the set of closed points $|X|$ and the collection of valuations $(v_x)_{x\in|X|}$ on $\mathcal{K}$ determine completely the curve X:

i) As a set, the underlying topological space is the disjoint union of $|X|$ and of a set consisting of a single element η.

ii) The non empty open subsets are the complements of the finite subsets of $|X|$. If U is one of them,

$$\Gamma(U, \mathcal{O}_X) = \left\{ f \in \mathcal{K} \mid v_x(f) \geq 0 \text{ for all } x \in U \cap |X| \right\} .$$

If X is a regular curve, the group $\mathrm{Div}(X)$ of *Weil divisors* of X is the free abelian group generated by the $[x]$'s with $x \in |X|$. If $f \in \mathcal{K}^*$, the *divisor of* f is

$$\mathrm{div}(f) = \sum_{x\in|X|} v_x(f)[x] .$$

If X is a regular curve, a coherent $\mathcal{O}_X$-module is a vector bundle if and only if it is torsion free.

A *complete regular curve* is a pair $(X, \deg)$ consisting of a regular curve X and a degree map

$$\deg : |X| \to \mathbb{N}_{>0}$$

such that, for any $f \in \mathcal{K}^*$,

$$\deg(\mathrm{div}(f)) = \sum_{x\in|X|} v_x(f) \deg(x) = 0 . \tag{1}$$

If X is a complete regular curve, then $H^0(X, \mathcal{O}_X)$ is a field. We call it the *field of definition* of X.

REMARK. Equipped with the usual definition of the degree, a smooth projective curve over a field is a complete regular curve. Its function field is finitely generated over its field of definition. It won't be the case for the curves we are going to construct.

Let X be a complete regular curve. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. The *rank of* $\mathcal{F}$ is the dimension of its generic fiber $\mathcal{F}_\eta$ over the function field. If r is the rank of $\mathcal{F}$, choose a vector bundle $\mathcal{E}$ isomorphic to $\mathcal{O}_X^r$ whose generic fiber $\mathcal{E}_\eta$ is equal to $\mathcal{F}_\eta$. For each closed point $x \in |X|$, let $\mathcal{F}'_x$ (resp. $\mathcal{F}''_x$) the kernel (resp. the image) of the natural map $\mathcal{F}_x \to \mathcal{F}_\eta$. We set

$$\lg_x(\mathcal{F}/\mathcal{E}) = \lg_x(\mathcal{F}'_x) + \lg_x(\mathcal{F}''_x/\mathcal{E}_x)$$

where, if M is any $\mathcal{O}_{X,x}$-module of finite length, $\lg_x(M)$ is its length and

$$\lg_x(\mathcal{F}''_x/\mathcal{E}_x) = \lg_x((\mathcal{E}_x + \mathcal{F}''_x)/\mathcal{E}_x) - \lg_x((\mathcal{E}_x + \mathcal{F}''_x)/\mathcal{F}''_x) \ .$$

We have $lg_x(\mathcal{F}/\mathcal{E}) = 0$ for almost all x. We define the degree of $\mathcal{F}$

$$\deg(\mathcal{F}) = \sum_{x \in |X|} \lg_x(\mathcal{F}/\mathcal{E}).\deg(x) \ .$$

Granting to (1), it is independent of the choice of $\mathcal{E}$. The degree may also be defined by:

$$\deg(\mathcal{F}) = \deg(\mathcal{F}_{\text{tor}}) + \deg(\det(\mathcal{F}/\mathcal{F}_{\text{tor}}))$$

where

- $\mathcal{F}_{\text{tor}}$ is the torsion part of $\mathcal{F}$, a finite direct sum of skyscrapers sheaves of finite length $\mathcal{O}_{X,x}$-modules, $x \in |X|$,
- $\deg(\mathcal{F}_{\text{tor}}) = \sum_{x\in|X|} \lg_x(\mathcal{F}_x).\deg(x)$,
- if $\mathcal{L}$ is a line bundle set $\deg(\mathcal{L}) = \deg(\text{div}(s))$ where s is any non-zero meromorphic section of $\mathcal{L}$, $\text{div}(s)$ being the Weil divisor associated to s,
- $\det(\mathcal{F}/\mathcal{F}_{\text{tor}})$ is the line bundle $\bigwedge^{\text{rank}(\mathcal{F})}(\mathcal{F}/\mathcal{F}_{\text{tor}})$.

The point is that, since X is complete, the degree function on line bundles

$$\deg : \text{Div}(X) \longrightarrow \mathbb{Z}$$

factorizes through the group of principal divisors to give a degree function

$$\deg : \text{Div}(X)/\sim\, = \text{Pic}(X) \longrightarrow \mathbb{Z}.$$

If $\mathcal{F}$ is a non-zero coherent $\mathcal{O}_X$-module we define the *slope* of $\mathcal{F}$ as

$$\mu(\mathcal{F}) = \deg(\mathcal{F})/\text{rank}(\mathcal{F}) \in \mathbb{Q} \cup \{+\infty\}$$

(we have $\mu(F) = +\infty$ if and only if $\mathcal{F}$ is torsion).

An $\mathcal{O}_X$-module $\mathcal{F}$ is *semistable* (resp. *stable*) if $\mu(\mathcal{F}') \leq \mu(\mathcal{F})$ (resp. if $\mathcal{F}$ is non-zero and if $\mu(\mathcal{F}') < \mu(\mathcal{F})$) for any proper $\mathcal{O}_X$-submodule $\mathcal{F}'$. A non-zero $\mathcal{O}_X$-module is semistable of slope $+\infty$ if and only if it is a torsion module.

The Harder-Narasimhan theorem holds:

THEOREM 1.1. *Let $\mathcal{F}$ be a non-zero coherent $\mathcal{O}_X$-module. There is a unique filtration*

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \ldots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \mathcal{F}$$

by $\mathcal{O}_X$-submodules with $\mathcal{F}_i/\mathcal{F}_{i-1} \neq 0$, semistable, and

$$\mu(\mathcal{F}_1/\mathcal{F}_0) > \mu(\mathcal{F}_2/\mathcal{F}_1) > \ldots > \mu(\mathcal{F}_m/\mathcal{F}_{m-1}) \ .$$

Moreover, for each $\lambda \in \mathbb{Q} \cup \{+\infty\}$, the full sub-category $Bund_X^{\lambda}$ of the category of coherent $\mathcal{O}_X$-modules whose objects are those which are semistable of slope λ is an abelian E-linear category.

We see that, $\mathcal{F}$ is a vector bundle if and only if $\mu(\mathcal{F}_1/\mathcal{F}_0) \neq +\infty$. In this case, the $\mathcal{F}_i$'s are strict vector subbundles, i.e. the quotients $\mathcal{F}/\mathcal{F}_i$'s are torsion free, hence also vector bundles. If, instead, the torsion sub-module $\mathcal{F}_{\text{tor}}$ is not 0, then $\mathcal{F}_{\text{tor}} = \mathcal{F}_1$.

2. Bounded analytic functions

2.1. The field $\mathcal{E}_{F,E}$. We fix a non archimedean locally compact field E. We denote by p the characteristic of k_E and q the number of elements of k_E. We denote by v_E the valuation of E normalized by $v_E(E^*) = \mathbb{Z}$.

Let F be any perfect field containing k_E. We denote by $\mathcal{E}_{F,E}$ the unique (up to a unique isomorphism) field extension of E, complete with respect to a discrete valuation v extending v_E such that
i) $v(\mathcal{E}_{F,E}^*) = v_E(E^*) = \mathbb{Z}$,
ii) F is the residue field of $\mathcal{E}_{F,E}$.

There is a unique section of the projection $\mathcal{O}_{\mathcal{E}_{F,E}} \to F$ which is multiplicative. We denote it

$$a \mapsto [a] \ .$$

If we choose a uniformizing parameter π of E, any element $f \in \mathcal{E}_{F,E}$ may be written uniquely

$$f = \sum_{n \gg -\infty} [a_n]\pi^n \ \text{ with the } a_n \in F \ ,$$

and $f \in E$ if and only if all the a_n's are in k_E.

We see that, if E is of characteristic p, the map $a \mapsto [a]$ is an homomorphism of rings. If we use it to identify F with a subfield of $\mathcal{E}$, i.e. if we set $[a] = a$ for all $a \in F$, we get

$$E = k_E((\pi)) \ \text{ and } \ \mathcal{E}_{F,E} = F((\pi)) \ .$$

Otherwise, E is a finite extension of $\mathbb{Q}_p$. If $W(F)$ (resp. $W(k_E)$) is the ring of Witt vectors with coefficients in F (resp. k_E), we see that $\mathcal{E}_{F,E}$ can be identified with $E \otimes_{W(k_E)} W(F)$ and that, for all $a \in F$,

$$[a] = 1 \otimes (a, 0, 0, \ldots, 0, \ldots) \ .$$

2.2. Three sub-rings of $\mathcal{E}_{F,E}$. We now fix the perfect field F containing k_E and we assume F to be complete for a given non trivial absolute value $| \ |$. Observe that, as F is perfect, the valuation group is p-divisible, hence the valuation is not discrete.

If there is no risk of confusion, we set $\mathcal{E} = \mathcal{E}_{F,E}$. We still choose a uniformizing parameter π of E. The following subsets of $\mathcal{E}$

$$B^b = B^b_{F,E} = \Big\{ \sum_{n \gg -\infty} [a_n]\pi^n \ \Big| \ \text{there exists } C \text{ such that } |a_n| \leq C, \forall n \ \Big\},$$

$$B^{b,+} = B^{b,+}_{F,E} = \Big\{ \sum_{n\gg -\infty} [a_n]\pi^n \;\Big|\; a_n \in \mathcal{O}_F,\ \forall n \Big\}$$

$$\text{and } A = A_{F,E} = \Big\{ \sum_{n=0}^{+\infty} [a_n]\pi^n \;\Big|\, a_n \in \mathcal{O}_F,\ \forall n \Big\}$$

are $\mathcal{O}_E$-subalgebras of $\mathcal{E}$ and are independent of π. If a is any non-zero element of the maximal ideal $\mathfrak{m}_F$ of $\mathcal{O}_F$, we have

$$B^{b,+} = A[\tfrac{1}{\pi}] \text{ and } B^b = B^{b,+}\big[\tfrac{1}{[a]}\big] \ .$$

When $\mathrm{char}(E) = p$, the ring B^b may be viewed as the ring of rigid analytic functions

$$f : \Delta = \big\{ z \in F \mid 0 < |z| < 1 \big\} \to F$$

which are such that $\pi^n f$ is analytic and bounded on $\big\{ z \in F \mid 0 \leq |z| < 1 \big\}$, for $n \gg 0$.

2.3. Prime ideals of finite degree. We set $\mathcal{E}_0 = \mathcal{E}_{k_F,E}$.

The projection $\mathcal{O}_F \to k_F$, which we denote as $a \mapsto \tilde{a}$, induces an augmentation map

$$\varepsilon : B^{b,+} \to \mathcal{E}_0 \text{ sending } \sum_{n\gg -\infty} [a_n]\pi^n \text{ to } \sum_{n\gg -\infty} [\tilde{a}_n]\pi^n \ .$$

We have $\varepsilon(A) = \mathcal{O}_{\mathcal{E}_0}$. We say that $\xi \in A$ is *primitive* if $\xi \notin \pi A$ and $\varepsilon(\xi) \neq 0$. The *degree* of a primitive element ξ is

$$\deg(\xi) = v_\pi(\varepsilon(\xi)) \in \mathbb{N} \ .$$

We see that A is a local ring whose invertible elements are exactly the primitive elements of degree 0. A primitive element $\xi \in A$ is *irreducible* if $\deg(\xi) > 0$ and ξ can't be written as the product of two primitive elements of degree > 0. In particular, any primitive element of degree 1 is irreducible.

We say that two primitive irreducible elements ξ and ξ' are *associated* (we write $\xi \sim \xi'$) if there exists η primitive of degree 0 such that $\xi' = \xi\eta$. This is an equivalence relation and we set

$$|Y_{F,E}| = |Y| = \big\{\text{primitive irreducible elements}\big\}/\sim \ .$$

If $y \in |Y|$ is the class of ξ, we set $\deg(y) = \deg(\xi)$.

We say that an ideal $\mathfrak{a}$ of A, $B^{b,+}$ or B^b is *of finite degree* if it is a principal ideal which is generated by a primitive element ξ of A. The *degree* of such an $\mathfrak{a}$ is the degree of ξ.

Proposition 2.3.1. *Let $y \in |Y|$ be the class of a primitive irreducible element ξ. The ideal $\mathfrak{p}_y$ (resp. $\mathfrak{p}_y^{b,+}$, resp. $\mathfrak{p}_y^b$) of A (resp. $B^{b,+}$, resp. B^b) generated by ξ is prime and depends only on y. The map*

$$y \mapsto \mathfrak{p}_y \ (\text{ resp. } y \mapsto \mathfrak{p}_y^{b,+}, \text{ resp. } y \mapsto \mathfrak{p}_y^b \ )$$

induces a bijection between $|Y|$ and the set of prime ideals of finite degree of A (resp. $B^{b,+}$, resp. B^b).

To describe what are the quotients of these rings by a prime ideal of finite degree, it is convenient to introduce the notion of p-perfect field.

2.4. p-perfect fields. A *p-perfect field* is a field L complete with respect to a non trivial non archimedean absolute value $|\ |$ whose residue field k_L is of characteristic p and which is such that the endomorphism $x \mapsto x^p$ of $\mathcal{O}_L/p\mathcal{O}_L$ is surjective.

If L is the fraction field of a complete discrete valuation ring, we see that L is a p-perfect field if and only if k_L is perfect of characteristic p and $\mathfrak{m}_L$ is generated by p.

A *strictly p-perfect field* is a p-perfect field L such that $\mathcal{O}_L$ is not a discrete valuation ring.

Let L be a field complete with respect to a non trivial non archimedean absolute value, with $\mathrm{char}(k_L) = p$ and $\mathcal{O}_L$ not a discrete valuation ring. It is easy to see that

– if a is any element of the maximal ideal $\mathfrak{m}_L$ of $\mathcal{O}_L$ such that $p \in (a)$, then L is strictly p-perfect if and only if the map

$$\mathcal{O}_L/(a) \mapsto \mathcal{O}_L/(a) \ \text{ sending } x \text{ to } x^p$$

is onto,

– if L is of characteristic p, L is strictly p-perfect if and only L is perfect.

Let L be a p-perfect field. We consider the set

$$F(L) = \left\{ x = (x^{(n)})_{n \in \mathbb{N}} \mid x^{(n)} \in L \text{ and } (x^{(n+1)})^p = x^{(n)} \right\} .$$

If $x, y \in F(L)$, we set

$$(x+y)^{(n)} = \lim_{m \mapsto +\infty} (x^{(n+m)} + y^{(n+m)})^{p^m}, \ (xy)^{(n)} = x^{(n)} y^{(n)}$$

(it is easy to see that the limit above exists).

Proposition 2.4.1. *Let L be a p-perfect field. Then $F(L)$ is a perfect field of characteristic p, complete with respect to the absolute value $|\ |$ defined by $|x| = |x^{(0)}|$. Moreover*

i) If $\mathfrak{a} \subset \mathfrak{m}_L$ is a finite type (i.e. principal) ideal of $\mathcal{O}_L$ containing p and if $u \mapsto \tilde{u}$ denote the projection $\mathcal{O}_L \to \mathcal{O}_L/\mathfrak{a}$, the map

$$\mathcal{O}_{F(L)} \to \varprojlim_{n \in \mathbb{N}} \mathcal{O}_L/\mathfrak{a}$$

(with transition maps $v \mapsto v^p$) sending $(x^{(n)})_{n \in \mathbb{N}}$ to $(\widetilde{x^{(n)}})_{n \in \mathbb{N}}$ is an isomorphism of topological rings.

ii) If L contains E as a closed subfield, the map

$$\theta_{L,E} : B^b_{F(L),E} \to L$$

sending $\sum_{n \gg -\infty} [a_n]\pi^n$ to $\sum_{n \gg -\infty} a_n^{(0)} \pi^n$ is a surjective homomorphism of E-algebras (independent of the choice of π). Moreover,

1. *(1) If $\mathcal{O}_L$ is a discrete valuation ring, $F(L)$ is the residue field of L equipped with the trivial valuation and $\theta_{L,E}$ is an isomorphism.*
2. *(2) If L is strictly p-perfect, we have $|F(L)| = |L|$ and the kernel of $\theta_{L,E}$ is a prime ideal of $B^b_{F(L),E}$ of degree 1. We have*

$$\theta_{L,E}(B^{b,+}_{F(L),E}) = L \ \text{ and } \ \theta_{L,E}(A_{F(L),E}) = \mathcal{O}_L .$$

REMARKS. (1) If L is of characteristic p, the map $x \mapsto x^{(0)}$ is a canonical isomorphism of the field $F(L)$ onto the residue field of L if L is not strictly p-perfect and onto L otherwise. Then, all the results are obvious. If L is strictly p-perfect and if λ is the unique element of $F(L)$ such that $\lambda^{(0)} = \pi$, then $\pi - [\lambda]$ is a generator of $\ker \theta_{L,E}$.

(2) If L is strictly perfect of characteristic 0, it's not always true that there exists $\lambda \in F(L)$ such that $\pi - [\lambda]$ is a generator of $\ker \theta_{L,E}$ (which is equivalent to saying that $\lambda^{(0)} = \pi$). This is true if F is algebraically closed, but such a λ is not unique !

All the ideals of degree 1 are obtained by this construction: Let $\mathcal{L}$ be the set of isomorphism classes of pairs (L, ι) where L is a p-perfect field containing E as a closed subfield and $\iota : F(L) \to F$ is an isomorphism of topological fields. If (L, ι) is such a pair, let $\theta_L : B^b \to L$ be the homomorphism deduced from $\theta_{L,E} : B^b_{F(L),E} \to L$ by *transport de structure*.

PROPOSITION 2.4.2. *The map $\mathcal{L} \to \{$ideals of degree $1\}$ sending the class of (L, ι) to the kernel of θ_L is bijective.*

2.5. Algebraic extensions of strictly p-perfect fields.

PROPOSITION 2.5.1. *Let L_0 be a strictly p-perfect field containing E as a closed subfield, $F_0 = F(L_0)$ and $\mathfrak{m}$ the kernel of the map $\theta_{L_0,E} : B^b_{F_0,E} \to L_0$.*

i) If L is a finite extension of L_0, then L is strictly p-perfect and $F(L)$ is a finite extension of $F(L_0)$ of the same degree.

ii) If F is a finite extension of F_0, the ideal $B^b_{F,E}\mathfrak{m}$ of $B^b_{F,E}$ is maximal and the quotient of $B^b_{F,E}$ by this ideal is a finite extension of L_0 of the same degree.

The functor $L \to F(L)$ is an equivalence of categories between finite extensions of L_0 and finite extensions of F_0. The functor $F \mapsto B^b_{F,E}/B^b_{F,E}\mathfrak{m}$ is a quasi-inverse.

REMARK. This equivalence extends in an obvious way to étale algebras. Hence, we see that the small étale site of L_0 can be identified with the small étale site of F_0.

2.6. Finite divisors. We can now give a complete description of the prime ideals of finite degree.

PROPOSITION 2.6.1. *If F is algebraically closed, a primitive element is irreducible if and only if it is of degree 1.*

PROPOSITION 2.6.2. *Let $y \in |Y|$, $d = \deg(y)$, $\xi = \sum_{n=0}^{+\infty}[c_n]\pi^n$ a primitive element lifting y, $L_y = B^b/\mathfrak{p}^b_y$ and $\theta_y : B^b \to L_y$ the projection. We set $||y|| = |c_0|^{1/d}$. Then:*

i) The ideals $\mathfrak{p}^b_y$ and $\mathfrak{p}^{b,+}_y$ are maximal and

$$B^{b,+}/\mathfrak{p}^{b,+}_y = L_y \ .$$

ii) There is a unique absolute value $|\ |_y$ on the field L_y such that $|\theta_y([a])|_y = |a|$ for all $a \in F$. Equipped with this absolute value, L_y is a p-perfect field containing E as a closed subfield. Moreover $|\pi|_y = ||y||$.

iii) The map $F \to F(L_y)$ sending a to $(\theta_y([a^{p^{-n}}]))_{n\in\mathbb{N}}$ is a continuous homomorphism of topological fields identifying $F(L_y)$ with a finite extension of F of degree d.

v) The ring $A/\mathfrak{p}_y$ is a $\mathcal{O}_E$-subalgebra of $\mathcal{O}_{L_y}$ whose fraction field is L_y.

We define the group $\mathrm{Div}_f(Y)$ of *finite divisors of* Y as the free abelian group with basis the $[y]$'s for $y \in |Y|$. Hence any finite divisor may be written uniquely

$$D = \sum_{y\in|Y|} n_y[y] \quad \text{with the } n_y \in \mathbb{Z}, \text{ almost all } 0 \ .$$

The degree of such a D is $\sum_{y\in|Y|} n_y \deg(y)$.

We denote $\mathrm{Div}_f^+(Y)$ the monoïd of *finite effective divisors*, i.e. of divisors $D = \sum n_y[y]$ with $n_y \geq 0$ for all y. From the previous proposition, one deduces:

Corollary 2.6.1. *The map from* $\mathrm{Div}_f^+(Y)$ *to the multiplicative monoïd of ideals of finite degree of* A *(resp.* $B^{b,+}$*, resp.* B^b*) sending* $\sum_{y\in|Y|} n_y[y]$ *onto* $\prod_{y\in|Y|}(\mathfrak{p}_y)^{n_y}$ *(resp.* $\prod_{y\in|Y|}(\mathfrak{p}_y^{b,+})^{n_y}$*, resp.* $\prod_{y\in|Y|}(\mathfrak{p}_y^{b})^{n_y}$*) is an isomorphism of monoïds.*

3. The rings of rigid analytic functions

3.1. Norms and completions. For $f = \sum_{n\gg-\infty}[a_n]\pi^n \in B^b$, and $0 < \rho < 1$, we define

$$|f|_\rho = \max_{n\in\mathbb{Z}} |a_n|\rho^n \ .$$

We also set

$$|f|_0 = q^{-r} \text{ if } r \text{ is the smallest integer such that } a_r \neq 0, \text{ and } |f|_1 = \sup_{n\in\mathbb{Z}} |a_n|.$$

For any $\rho \in [0,1]$, the map $f \mapsto |f|_\rho$ is a *multiplicative norm* on B^b, i.e. we have

$$|f+g|_\rho \leq \max\{|f|_\rho, |g|_\rho\} \ , \ |fg|_\rho = |f|_\rho |g|_\rho \ \text{ and } \ |f|_\rho = 0 \iff f = 0 \ .$$

For any non empty interval $I \subset [0,1]$, we denote

$$B_I = B_{F,E,I}$$

the completion of B^b for the family of the $|\ |_\rho$'s for $\rho \in I$ [1].

Proposition 3.1.1. *Let* $I \subset [0,1]$ *be a non empty interval. For any* $\rho \in I$, $|\ |_\rho$ *is a norm on* B_I *(i.e., if* $b \in B_I$ *is* $\neq 0$*, then* $|b|_\rho \neq 0$*). Moreover:*

i) If $J \subset I$ *is an interval, the induced map*

$$B_I \to B_J$$

is a continuous injective map.

ii) If $I = [\rho_1, \rho_2]$ *is a non empty closed interval contained in* $[0,1[$*, then* B_I *is a Banach* E*-algebra: if we set*

$$A^b_{F,E,I} = A^b_I = \left\{ f \in B^{b,+} \mid |f|_{\rho_1} \leq 1 \text{ and } |f|_{\rho_2} \leq 1 \right\} \ ,$$

then $B_I = A_I[1/\pi]$ *where* $A_I = A_{F,E,I}$ *is the* π*-adic completion of* A^b_I*.*

[1] Say that a sequence $(f_n)_{n\in\mathbb{N}}$ is a Cauchy sequence over the interval I if for any $\rho \in I$ and any $\epsilon > 0$, there exists N such that $|f_m - f_n|_\rho < \epsilon$ if m and n are $\geq N$. Say that two Cauchy sequences $(f_n)_{n\in\mathbb{N}}$ and $(g_n)_{n\in\mathbb{N}}$ are equivalent if, for any $\rho \in I$ and any $\epsilon > 0$, there exists N such that $|f_n - g_n|_\rho < \epsilon$ if $n \geq N$. An element of $B_{F,E,I}$ may be viewed as an equivalence class of Cauchy sequences over I.

iii) If $I \subset [0,1[$ is not restricted to $[0]=\{0\}$, then B_I is a Fréchet-E-algebra (inverse limit of Banach E-algebras): If $\mathcal{I}_I$ is the set of closed intervals contained in I, the map

$$B_I \to \varprojlim_{J\in\mathcal{I}_I} B_J$$

is a homeomorphism of topological rings.

iv) We have $B_{[0,1]} = B^b$ and $B_{[0]} = \mathcal{E}$.

In what follow, if $J \subset I$, we use the injective map $B_I \to B_J$ to identify B_I with a subring of B_J.

If $I \subset [0,1[$ contains 0 then B_I can be identified with a subring of $\mathcal{E}$:

$$B_I = \Big\{ \sum_{n\gg-\infty} [a_n]\pi^n \in \mathcal{E} \mid \forall\, \rho \in I,\ |a_n|\rho^n \to 0 \text{ for } n \to +\infty \Big\}\ .$$

If $I \subset [0,1[$ contains 0, we set

$$B^+_{F,E,I} = B^+_I = \big\{ b \in B_I \mid |b|_0 \le 1 \big\} = B_I \cap \mathcal{O}_{\mathcal{E}}\ .$$

Similarly if $I \subset [0,1]$ contains 1, we set

$$B^+_I = \big\{ b \in B_I \mid |b|_1 \le 1 \big\}\ .$$

We have

$$B^+_{[0,1]} = B^{b,+} \text{ and } A = B^{b,+} \cap \mathcal{O}_{\mathcal{E}} = \big\{ b \in B^b = B_{[0,1]} \mid |b|_0 \le 1 \text{ and } |b|_1 \le 1 \big\}$$

We also write

$$B^+_{F,E} = B^+ = B^+_{]0,1]} \text{ and } B_{F,E} = B = B_{]0,1[}\ .$$

If $\mathrm{char}(E)=p$ and if $I \subset]0,1[$ the ring B_I can be identified with the ring of rigid analytic functions

$$f : \big\{ z \in F \text{ with } |z| \in I \big\} \to F\ .$$

In particular $B := B_{]0,1[}$ is the ring of rigid analytic functions on the punctured open unit disk.

Similarly, if $\mathrm{char}(E)=p$ and if $0 \in I \subset [0,1[$, then B^+_I may be identified with the ring of analytic functions

$$f : \big\{ z \in F \text{ with } |z| \in I \big\} \to F\ ,$$

though B_I is the ring of meromorphic rigid analytic functions in the same range, with no pole away from 0.

Remark. Let $I \subset]0,1[$. Let $(a_n)_{n\in\mathbb{Z}}$ be elements in F such that, for all $\rho \in I$, we have $|a_n|\rho^n \to 0$ whenever $n \to +\infty$ and also when $n \to -\infty$. Then the series

$$\sum_{n\in\mathbb{Z}} [a_n]\pi^n$$

converges (in both directions) to an element of B_I. If $\mathrm{char}(E)=p$, any element of B_I may be written uniquely like that. If $\mathrm{char}(E)=0$, we don't know if it is always possible and, when it is possible, we don't know if this writing is unique (but it seems unlikely in general).

3.2. Newton polygons. Let v the valuation of F normalized by $|a| = q^{-v(a)}$ for all $a \in F$. Let $I \subset [0,1]$ be an interval containing 0. The map

$$(a_n)_{n\in\mathbb{Z}} \longmapsto \sum_{n\in\mathbb{Z}} [a_n]\pi^n$$

is a bijection between the set of sequences $(a_n)_{n\in\mathbb{Z}}$ of elements of F such that

i) $a_n = 0$ for $n \ll 0$,

ii) for all $\rho \in I$, $a_n \rho^n \to 0$ for $n \to +\infty$

and B_I. If $f = \sum_{n\gg -\infty} [a_n]\pi^n \in B_I$ is non-zero, the *Newton polygon of* f is the convex hull $\mathrm{Newt}(f)$ of the points of the real plane of coordinates $(n, v(a_n))$ for $n \in \mathbb{Z}$. If $J \subset I$ is an interval, $\mathrm{Newt}_J(f)$ is the sub-polygon of $\mathrm{Newt}(f)$ obtained by deleting all segments whose slopes s are such that $q^s \notin I$.

PROPOSITION 3.2.1. *Let $I \subset [0,1]$ be an interval and let $\overline{I}$ be the smallest interval containing I and 0. Then $B_{\overline{I}}$ is a dense subring of B_I. If $f \in B_I$ and if $(f_n)_{n\in\mathbb{N}}$ is a sequence of elements of $B_{\overline{I}}$ converging to f, then the sequence $(\mathrm{Newt}_I(f_n))_{n\in\mathbb{N}}$ has a limit, i.e., for any closed interval $J \subset I$, the sequence of the $\mathrm{Newt}_J(f_n)$ is stationary. This limit is independent of the choice of the sequence $(f_n)_{n\in\mathbb{N}}$.*

We call this limit $\mathrm{Newt}_I(f)$.

3.3. Divisors. For any interval $I \subset [0,1]$ different from $\emptyset, \{1\}$, we set

$$|Y_I| = \{ y \in |Y| \mid ||y|| \in I \} ,$$

and we define *the group* $\mathrm{Div}(Y_I)$ *of divisors of* Y_I[2]:

i) If I is closed and $I \subset [0,1[$, we set

$$\mathrm{Div}(Y_I) = \{ \sum_{y\in|Y_I|} n_y[y] \mid n_y = 0 \text{ for almost all } y \} .$$

ii) If $I \subset [0,1[$ is not closed and if $\mathcal{J}_I$ denote the set of closed ideals $J \subset I$, we set

$$\mathrm{Div}(Y_I) = \{ \sum_{y\in|Y_I|} n_y[y] \mid \ \forall J \in \mathcal{J}_I, n_y = 0 \text{ for almost all } y \text{ with } ||y|| \in J \} .$$

iii) If $1 \in I$, we define I' as the complement of 1 in I, we choose $\rho_0 \in I'$ and we set

$$\mathrm{Div}(Y_I) = \{ \sum_{y\in|Y_I|} n_y[y] \in \mathrm{Div}(Y_{I'}) \mid \sum_{||y||\geq\rho_0} n_y \log(||y||) > -\infty \}$$

(independent of the choice of ρ_0).

For any I, we denote by $\mathrm{Div}^+(Y_I)$ the monoïd of *effective divisors* i.e. of divisors $D = \sum n_y[y] \in \mathrm{Div}(Y_I)$ such that $n_y \geq 0$ for all y.

[2]See the remark 3.4.1 below for a geometric interpretation of these constructions.

3.4. Closed ideals. For any $y \in |Y|$, we choose a primitive element ξ_y representing y.

PROPOSITION 3.4.1. *Let $I \subset [0,1]$ be a non empty interval and $y \in |Y|$. If $||y|| \notin I$, then ξ_y is invertible in B_I. If $||y|| \in I$ and if $L_y = B^b/(\xi_y)$, the projection of B^b to L_y extends by continuity to a surjective homomorphism of E-algebras*

$$\theta_y : B_I \to L_y$$

whose kernel is the maximal ideal generated by ξ_y.

The map

$$y \mapsto \mathfrak{m}_{I,y} = \text{ideal of } B_I \text{ generated by } \xi_y$$

is an injective map from $|Y_I|$ to the set of maximal ideals of B_I.

THEOREM 3.1. *Let $I \subset [0,1]$ an interval different from $\emptyset, \{1\}$. For any $y \in |Y_I|$, we have $\cap_{n\in\mathbb{N}}(\mathfrak{m}_{I,y})^n = 0$. Let $f \in B_I$ a non-zero element. For any $y \in |Y_I|$, let $v_y(f)$ be the biggest integer n such that $f \in (\mathfrak{m}_y)^n$. Then*

$$\text{div}(f) = \sum_{y\in|Y_I|} v_y(f)[y] \in \text{Div}^+(Y_I) \ .$$

Moreover, for any $\rho = q^{-r} \in I$ with $r > 0$, the length $\mu_\rho(f)$ of the projection on the horizontal axis of the segment of $\text{Newt}_I(f)$ *of slope $-r$ is finite and*

$$\sum_{||y||=\rho} v_y(f)\deg(y) = \mu_\rho(f) \ .$$

COROLLARY 3.4.1. *Let $I \subset [0,1]$ an interval different from $\emptyset, \{1\}$. Then:*

i) Any non-zero closed prime ideal of B_I is maximal and principal.

ii) The map $|Y_I| \to$ {closed maximal ideals of B_I} sending y to $\mathfrak{m}_{I,y}$ is a bijection.

iii) If $I \subset [0,1[$ and is closed, any ideal of B_I is closed and B_I is a principal domain.

PROPOSITION 3.4.2. *Let $I \subset [0,1[$ a non empty interval. For any non-zero closed ideal $\mathfrak{a}$ of B_I and any $y \in |Y_I|$, let $v_y(\mathfrak{a})$ the biggest integer $n \leq 0$ such that $\mathfrak{a} \subset (\mathfrak{m}_{I,y})^n$. Then*

$$\text{div}(\mathfrak{a}) = \sum_{y\in|Y_I|} v_y(\mathfrak{a})[y] \in \text{Div}^+(Y_I) \ .$$

The map

$$\{\textit{non-zero closed ideals of } B_I\} \to \text{Div}^+(Y_I) \ ,$$

so defined, is an isomorphism of monoïds.

REMARK 3.4.1. Let $I \subset [0,1[$ an interval different from $\emptyset$, $\{1\}$.

– If I is closed, we see that $\text{Div}(Y_I)$ is nothing but the group of divisors of the regular curve $Y_I = \text{Spec}(B_I)$ and that $|Y_I|$ may be identified to the set of closed points of Y_I.

– Otherwise, we may consider the inductive system of regular curves

$$Y_I = (Y_J = \text{Spec} B_J)_{J\in\mathcal{I}_I} \ .$$

If $J_1 \subset J_2$ belong to $\mathcal{I}_I$, we have morphisms of abelian groups

$$\text{Div}(Y_{J_1}) \to \text{Div}(Y_{J_2}) \ \text{ and } \ \text{Div}(Y_{J_2}) \to \text{Div}(Y_{J_1})$$

induced by the fact that, if $\mathfrak{a}$ is a non-zero ideal of B_{J_1} then $\mathfrak{a} \cap B_{J_2}$ is a non-zero ideal of B_{J_2}, though, if $\mathfrak{b}$ is a non zero ideal of B_{J_2}, then $B_{J_1}\mathfrak{b}$ is a non zero ideal of B_{J_1}. We see that $\mathrm{Div}(Y_I)$ is the inverse limit of the $\mathrm{Div}(Y_J)$ for $J \in \mathcal{I}_I$. The direct limit of these groups consists of the subgroup

$$\mathrm{Div}_f(Y_I) = \Big\{ \sum_{y \in |Y_I|} n_y[y] \in \mathrm{Div}(Y_I) \Big|\ n_y = 0 \text{ for almost all } y \in |Y_I| \Big\} \ .$$

3.5. Factorization. From the above proposition, we see that the analogue in this context of the classical question "does there exist an analytic function which has a given set of zeros with fixed multiplicities " becomes the question:

"Let $D \in \mathrm{Div}^+(Y_I)$. Does there exist $f \in B_I$ such that $\mathrm{div}(f) = D$?"

The answer to this question is "yes for any D" if and only if any closed ideal is principal.

The answer to this question is obviously "yes" if $I \subset [0, 1[$ is closed. This is also "yes" if $I =]0, \rho]$ for some $\rho \in]0, 1[$ (see cor. 3.5.1 below). But it is "no" in general.

Recall that one says that the field F is *spherically complete* if the intersection of any decreasing sequence of non empty balls contained in F is non empty.

For instance, if k is an algebraically closed field of characteristic p,

i) the completion of an algebraic closure of the field $k((u))$ is not spherically complete,

ii) If G is a divisible totally ordered abelian group (e.g. $G = \mathbb{Q}$ or $\mathbb{R}$), we may consider the subset F of all formal series of the form

$$f = \sum_{g \in G} a_g g \quad \text{with } a_g \in k \ ,$$

such that the support of f

$$\mathrm{supp}(f) = \{g \in G \mid a_g \neq 0\}$$

is a well ordered subset of G. Then, with the obvious addition, multiplication and absolute value, F is an algebraically closed field which is spherically complete [**Poo93**].

PROPOSITION 3.5.1. *Let $I \subset [0, 1[$ be a non closed interval. Then:*

i) If F is not spherically complete, there are closed ideals of B_I which are not principal.

ii) If F is spherically complete and char$(E) = p$, any closed ideal of B_I is principal.

It is likely that (ii) remains true whenever $\mathrm{char}(E) = 0$.

Without any assumption on F, if I is an interval whose closure contains 0, any divisor

$$\sum_{y \in |Y_I|} n_y[y]$$

such that $n_y = 0$ if $||y|| \geq \rho$ for $\rho \in I$ big enough, is the divisor of a function. More precisely, for any $y \in |Y_I|$ we denote by d_y the degree of y and we choose a *π-primitive element* ξ (i.e. an element $\xi_y \in A$ such that $|\xi_y - \pi^{d_y}|_1 < 1$) representing y (one can show that such an element always exists). Then:

Proposition 3.5.2. *Let $\overline{I}\subset[0,1]$ an interval containing* 0*, not reduced to* $\{0\}$*, and* I *the complement of* $\{0\}$ *in* $\overline{I}$*. Let*

$$D=\sum_{y\in|Y_I|}n_y[y]\in \mathrm{Div}^+(Y_I)\ .$$

i) For any $\rho\in I$, the infinite product

$$f_{\leq\rho}=\prod_{||y||\leq\rho}\frac{\xi_y}{\pi^{d_y}}$$

converges in $B^+_{]0,1]}\subset B_I$ *and* $\mathrm{div}(f_{\leq\rho})=\sum_{||y||\leq\rho}n_y[y]$.

ii) If there exists $f\in B_I$ *such that* $\mathrm{div}(f)=D$ *then* $f=f_{\leq\rho}f_{>\rho}$ *for some* $f_{>\rho}\in B_{\overline{I}}$ *and* $\mathrm{div}(f_{>\rho})=\sum_{||y||>\rho}n_y[y]$.

In particular, if $I=]0,1[$, $f_{>\rho}\in B^b_{[0,1[}$*. In this case,* $f\in B_{]0,1]}$ *(resp* $B^+_{]0,1]}$*) if and only if* $f_{>\rho}\in B^b$ *(resp.* $B^{b,+}$*).*

Corollary 3.5.1. *i) If* $I=]0,\rho]$ *for some* $\rho\in]0,1[$*, any closed ideal of* B_I *is principal.*

ii) An ideal of $B_{]0,1[}$ *or of* $B_{]0,1]}$ *is closed if and only if it is an intersection of principal ideals.*

3.6. Units. The ring A is a local ring. Therefore, if $\mathfrak{m}_A$ is its maximal ideal, the multiplicative group A^* of invertible elements of A is the complement of $\mathfrak{m}_A$ in A. With obvious notations, we have also

$$A^*=[\mathcal{O}_F^*]\times\mathcal{U}_F\quad\text{with}\quad\mathcal{U}_F=\big\{1+\sum_{n=1}^{\infty}[a_n]\pi^n\mid a_n\in\mathcal{O}_F\big\}\ .$$

We have also

$$(B^{b,+})^*=\pi^{\mathbb{Z}}\times A^*=\pi^{\mathbb{Z}}\times[\mathcal{O}_F^*]\times\mathcal{U}_F\quad\text{and}\quad(B^b)^*=\pi^{\mathbb{Z}}\times[F^*]\times\mathcal{U}_F\ .$$

If f is an invertible element of $B_{]0,1[}$ we must have $\mathrm{div}(f)=0$, which implies that $f\in B^b$. Therefore,

$$(B_{]0,1[})^*=(B_{]0,1]})^*=(B^b)^*\quad\text{and}\quad(B^+)^*=(B^{b,+})^*\ .$$

4. The curve X in the case where F is algebraically closed

4.1. Construction of the curve. The *Frobenius automorphism* φ on B^b is the unique E-automorphism which is continuous for $|\ |_0$ and induces $x\mapsto x^q$ on F. It satisfies

$$\varphi\Big(\sum_{n\gg-\infty}[a_n]\pi^n\Big)=\sum_{n\gg-\infty}[a_n^q]\pi^n\ .$$

For any $f\in B^b$ and any $\rho\in[0,1]$, we have $|\varphi(f)|_{\rho^q}=(|f|_\rho)^q$. This implies that φ extends by continuity to an automorphism (still denoted φ) of $B=B_{]0,1[}$.

We consider the graded E-algebra

$$P_\pi=P_{F,E,\pi}=\bigoplus_{d\in\mathbb{N}}P_{\pi,d}\quad\text{with}\quad P_{\pi,d}=P_{F,E,\pi,d}=\big\{b\in B\mid\varphi(b)=\pi^d b\big\}\ .$$

The natural map $P_\pi\to B$ is injective and we use it to identify P_π with a subring of B. We have $P_\pi\subset B^+$.

We define the scheme

$$X = X_{F,E} = \text{Proj } P_\pi \ .$$

One can show that X is independent of the choice of π: If π' is another uniformizing parameter of E and if $X' = \text{Proj } P_{\pi'}$, the function field of X' (viewed as a subfield of the fraction field of B) is the function field $\mathcal{K}$ of X and the set of closed points of X' (viewed as a subset of the set of normalized discrete valuations on $\mathcal{K}$) is the set of closed points of X.

On the other hand, the line bundles

$$\mathcal{O}_X(d)_\pi = \widetilde{\bigoplus_{n\in\mathbb{Z}} P_{\pi,n+d}}$$

(with the convention that $P_{\pi,m} = 0$ for $m < 0$) depend on the choice of π.

We have

$$P_{\pi,0} = \big\{u \in B \mid \varphi(u) = u\big\} = E \ .$$

4.2. The Lubin-Tate formal group. Set

$$\ell_\pi(X) = \sum_{n=0}^{+\infty} \frac{X^{q^n}}{\pi^n} \in E[[X]]$$

and $\Phi_\pi(X,Y) \in E[[X,Y]]$ the unique formal power series $\equiv X + Y \ (\bmod (X,Y)^2)$ such that

$$\ell_\pi(\Phi_\pi(X,Y)) = \ell_\pi(X) + \ell_\pi(Y) \ .$$

Then, $\Phi_\pi(X,Y) \in \mathcal{O}_E[[X,Y]]$ and defines a one parameter formal group law over $\mathcal{O}_E$ which is a Lubin-Tate formal group over $\mathcal{O}_E$ associated to the uniformizing parameter π ([**LT65**], [**Ser67**], §3).

For any linearly topologized complete $\mathcal{O}_E$-algebra Λ, we may consider the topological $\mathcal{O}_E$-module $\Phi_\pi(\Lambda)$: The underlying topological space is the topological space underlying the ideal of elements of Λ which are topologically nilpotent, with the addition $(x,y) \mapsto \Phi_\pi(x,y)$ and the multiplication by $\alpha \in \mathcal{O}_E$ given by $x \mapsto f_{\pi,\alpha}(x)$ where $f_{\pi,\alpha}(X) \in \mathcal{O}_E[[X]]$ is the unique power series $\equiv \alpha X \ (\bmod X^2)$ such that $\ell_\pi(f_\alpha(X)) = \alpha\ell_\pi(X)$.

Let C be an algebraically closed field containing E, complete for an absolute value extending the given absolute value on E. We may consider the Tate module

$$T_C(\Phi_\pi) = \mathcal{L}_{\mathcal{O}_E}(E/\mathcal{O}_E, \Phi_\pi(\mathcal{O}_C)) \ .$$

This is a free-$\mathcal{O}_E$-module of rank one. If we denote by $\Phi_\pi(\mathcal{O}_{\overline{E}})$ the inductive limit (or the union) of the $\Phi_\pi(\mathcal{O}_{E'})$, for E' varying through the finite extensions of E contained in C, we have also $T_C(\Phi) = \mathcal{L}_{\mathcal{O}_E}(E/\mathcal{O}_E, \Phi_\pi(\mathcal{O}_{\overline{E}}))$.

If $V_C(\Phi_\pi)$ is the one dimensional E-vector space $E \otimes_{\mathcal{O}_E} T_C(\Phi_\pi)$, we have a short exact sequence

$$0 \to V_C(\Phi_\pi) \to \mathcal{L}_{\mathcal{O}_E}(E, \Phi_\pi(\mathcal{O}_C)) \to C \to 0 \tag{1}$$

where the map $\mathcal{L}_{\mathcal{O}_E}(E, \Phi_\pi(\mathcal{O}_C)) \to C$ is $f \mapsto \ell_\pi(f(1))$.

The perfectness of $\mathcal{O}_F$ implies that multiplication by π on the $\mathcal{O}_E$-module $\Phi_\pi(\mathcal{O}_F)$ is bijective, so $\Phi_\pi(\mathcal{O}_F)$ is an E-vector space. We see that $\Phi_\pi(\mathcal{O}_F)$ depends only on the special fiber Φ_{π,k_E} of Φ_π (a formal $\mathcal{O}_E$-module over the residue field k_E of $\mathcal{O}_E$).

PROPOSITION 4.2.1. *For any x in the maximal ideal $\mathfrak{m}_F$ of $\mathcal{O}_F$, the series $\sum_{n\in\mathbb{Z}} \pi^{-n}[x^{q^n}]$ converges in B and its sum $L_\pi(x)$ belongs to $P_{\pi,1}$. The map*

$$L_\pi : \Phi_\pi(\mathcal{O}_F) \to P_{\pi,1}$$

so defined is an isomorphism of topological E-vector spaces.

REMARK. *This construction can be generalized: For $d \in \mathbb{N}$, one may interpret P_d as being "the sections over $\mathcal{O}_F$ of an E-sheaf $S^d_{E,\pi}$ for the syntomic topology over k_E".*

In the rest of the section 4, we assume F *algebraically closed.*

The automorphism φ generates a torsion free cyclic group $\varphi^{\mathbb{Z}}$ of automorphisms of B. This group acts also on $|Y|$ and on $\mathrm{Div}(Y) = \mathrm{Div}(Y_{]0,1[})$. If λ, λ' are non-zero elements of $\mathfrak{m}_F$ such that $\pi - [\lambda]$ and $\pi - [\lambda']$ have the same image in $|Y|$, this implies that $|\lambda| = |\lambda'|$. If $\pi - [\lambda]$ is a lifting of $y \in |Y|$ and $n \in \mathbb{Z}$ then $\pi - [\lambda^{q^n}]$ is a lifting of $\varphi^n(y)$, so if $y \in |Y|$ then the $\varphi^n(y)$'s for $n \in \mathbb{Z}$ are all distinct.

This implies that it is possible to choose for each $y \in |Y|$ an element $\lambda_y \in \mathfrak{m}_F$ such that $\pi - [\lambda_y]$ is a lifting of y and, for all y,

$$\lambda_{\varphi(y)} = (\lambda_y)^q \ .$$

We make such a choice once and for all. If $y \in |Y|$, the field

$$L_y = B^b/(\pi - [\lambda_y]) = B^+/(\pi - [\lambda_y]) = B/(\pi - [\lambda_y])$$

is algebraically closed. The multiplicative map $\mathcal{O}_F \to \mathcal{O}_{L_y}$ sending a to $\theta_y([a])$ induces, by passing to the quotients, an isomorphism of rings

$$\mathcal{O}_F/\lambda_y\mathcal{O}_F \to \mathcal{O}_{L_y}/\pi\mathcal{O}_{L_y} \ .$$

Moreover, φ induces a canonical isomorphism $L_y \to L_{\varphi(y)}$.

For any linearly topologized complete $\mathcal{O}_E$-algebra Λ, we denote $\mathcal{V}_{E,\pi}(\Lambda)$ the E-vector space $\mathcal{L}_{\mathcal{O}_E}(E, \Phi_\pi(\Lambda))$.

PROPOSITION 4.2.2. *Let $y \in |Y|$. The natural maps*

$$\mathcal{V}_{E,\pi}(\mathcal{O}_C) \to \mathcal{V}_{E,\pi}(\mathcal{O}_{L_y}/\pi\mathcal{O}_{L_y}) \leftarrow \mathcal{V}_{E,\pi}(\mathcal{O}_F/\lambda_y\mathcal{O}_F) \leftarrow \mathcal{V}_{E,\pi}(\mathcal{O}_F) \to \Phi_\pi(\mathcal{O}_F) \to P_{\pi,1}$$

are all isomorphisms.

ii) We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \to & V_C(\Phi_\pi) & \to & \mathcal{V}_{E,\pi}(\mathcal{O}_C) & \to & C & \to & 0 \\ & & \downarrow & & \downarrow & & \| & & \\ 0 & \to & P_{\pi,1} \cap \ker\theta_y & \to & P_{\pi,1} & \to & C & \to & 0 \end{array}$$

where the lines are exact and the vertical arrows are isomorphisms.

REMARK. *There is an explicit way to construct a generator t of $P_{\pi,1} \cap \ker\theta_y$:* From the fact that F is algebraically closed, one deduces easily that one can find $t_+ \in A$ not divisible by π such that

$$\varphi(t_+) = (\pi - [\lambda_y])t_+ \ .$$

On the other hand the infinite product

$$t_- = \prod_{n=0}^{+\infty} \left(1 - \frac{[\lambda_y^{q^n}]}{\pi}\right)$$

converges in B^+. We may take $t = t_- t_+$.

4.3. Divisors of X. Let $\mathrm{Div}(Y)_{\varphi=1}$ the subgroup of $\mathrm{Div}(Y)$ consisting of the divisors D such that $\varphi(D) = D$ and $\mathrm{Div}^+(Y)_{\varphi=1}$ the submonoïd of $\mathrm{Div}^+(Y)$ consisting of effective divisors such that $\varphi(D) = D$.

If $D = \sum_{y \in |Y|} n_y[y] \in \mathrm{Div}(Y)$ we have $\varphi(D) = \sum_{y \in |Y|} n_y[\varphi(y)]$, therefore $D \in \mathrm{Div}(X)$ if and only if $n_y = n_{\varphi(y)}$ for all y.

Choose $\rho \in]0, 1[$. As $]\rho^q, \rho] \subset [\rho^q, \rho]$, we have $n_y = 0$ for almost all y such that $\rho^q < ||y|| \leq \rho$. On the other hand, for any $y \in |Y|$, there is a unique $n \in \mathbb{Z}$ such that $\rho^q < ||\varphi^n(y)|| \leq \rho$. Therefore:

Proposition 4.3.1. *For any $y \in Y$, set* $\delta(y) = \sum_{n \in \mathbb{Z}} [\varphi^n(y)]) \in \mathrm{Div}(Y)_{\varphi=1}$ *and*

$$\Delta = \left\{ D \in \mathrm{Div}(Y) \mid \text{ there exists } y \in |Y| \text{ such that } D = \delta(y) \right\} .$$

Then $\mathrm{Div}(Y)_{\varphi=1}$ *(resp.* $\mathrm{Div}^+(Y)_{\varphi=1}$*) is a free abelian group (resp. monoïd) and the elements of Δ form a basis.*

Proposition 4.3.2. *i) Let $y \in |Y|$ and t a generator of $E_y = P_{\pi,1} \cap \mathfrak{m}_y$. Then*

$$\mathrm{div}(t) = \delta(y) .$$

ii) Let $d \in \mathbb{N}_{>0}$ and $u \in P_{\pi,d}$ non zero. There exists $t_1, t_2, \dots, t_d \in P_{\pi,1}$ such that

$$u = t_1 t_2 \dots t_d .$$

Moreover, if $t'_1, t'_2, \dots, t'_d \in P_{\pi,1}$ are such that $u = t'_1 t'_2 \dots t'_d$, there exists $\sigma \in \mathfrak{S}_d$ and $\lambda_1, \lambda_2, \dots, \lambda_d \in E^*$ such that $t'_i = \lambda_i t_{\sigma(i)}$ for all i.

This proposition is an easy consequence of what we already know: (i) is formal. To prove (ii), we observe that the ideal generated by u is fixed by φ^n for all $n \in \mathbb{Z}$, hence $\mathrm{div}(u) \in \mathrm{div}^+(Y)_{\varphi=1}$. Therefore we can write

$$\mathrm{div}(u) = D_1 + D_2 + \dots + D_r$$

with $D_i \in \Delta$. If $D_i = \delta(y_i)$, if $\mathfrak{m}_i$ is the maximal ideal of B corresponding to y_i and if t_i is a generator of $P_{F,1} \cap \mathfrak{m}_i$, then we must have

$$u = \lambda t_1 t_2 \dots t_r$$

with $\lambda \in B^*$. Therefore, we must have $r = d$ and $\varphi(\lambda) = \lambda$, hence $\lambda \in E^*$. The assertion follows.

An easy consequence of this proposition is the following result:

Theorem 4.1. *Let $|X|$ be the set of closed points of X and set* $\deg(x) = 1$ *for all $x \in |X|$. Then X is a complete curve whose field of definition is E. Moreover:*

i) Let $D \in \Delta$, $t \in P_{\pi,1}$ non-zero such that $\mathrm{div}(t) = D$, *$y \in |Y|$ such that $D = \delta(y)$ and $L_D = L_y$. Then*

a) the homogeneous ideal of P_π generated by t defines a closed point x_D of X whose local ring is a discrete valuation ring and residue field is L_D, *b) the complement of x_D in X is an affine scheme which is the spectrum of a principal domain.*

ii) The map $D \mapsto x_F$ is a bijection $\Delta \to |X|$ inducing canonical isomorphisms

$$\mathrm{Div}(Y)_{\varphi=1} \to \mathrm{Div}(X) \text{ and } \mathrm{Div}^+(Y)_{\varphi=1} \to \mathrm{Div}^+(X) \ .$$

4.4. Vector bundles. For each $d \in \mathbb{Z}$, $\mathcal{O}_X(d)_\pi$ is a line bundle of degree d. Proposition 4.3.2 implies trivially:

PROPOSITION 4.4.1. *We have*

$$Pic^0(X) = 0 \ ,$$

i.e., for any $d \in \mathbb{Z}$, a line bundle $\mathcal{L}$ is of degree d if and only $\mathcal{L} \simeq \mathcal{O}_X(d)_\pi$.

In particular, if π' is any other uniformizing parameter, $\mathcal{O}_X(1)_{\pi'}$ is isomorphic (not canonically) to $\mathcal{O}_X(1)_\pi$[3].

Let h be a positive integer. We may consider

$$X_h = \mathrm{Proj} \bigoplus_{d \in \mathbb{N}} P_{h,\pi,d} \ \text{ with } P_{h,\pi,d} = \left\{\varphi^h(u) = \pi^d u\right\} \ .$$

If E_h denotes the unramified extension of E whose residue field is the unique extension of degree h of the residue field k_E of E which is contained in F, we see that $X_h = X_{F,E,h}$. It is a complete regular curve whose field of definition is E_h.

If $x \in P_{\pi,d}$ then $x \in P_{h,\pi,dh}$. It it easy to see that the induced map

$$\oplus P_{\pi,d} \to \oplus P_{h,\pi,d}$$

induces a morphism

$$\nu_h : X_h \to X$$

which is a cyclic cover of degree h identifying $X_{F,h}$ with $X \times_{\mathrm{Spec}\ E} \mathrm{Spec}\ E_h$.

For each $\lambda \in \mathbb{Q}$, if $\lambda = d/h$, with $d, h \in \mathbb{Z}$ relatively prime and $h > 0$, we set

$$\mathcal{O}_X(\lambda)_\pi = (\nu_h)_*\big(\mathcal{O}_{X_{F,h}}(d)_\pi\big) \ .$$

This is a vector bundle over X of rank h and degree d, hence of slope λ.

THEOREM 4.2. *For any non-zero coherent $\mathcal{O}_X$-module $\mathcal{F}$, the Harder-Narasimhan filtration on $\mathcal{F}$ splits (non canonically). Moreover, if $\lambda \in \mathbb{Q}$, then $\mathcal{F}$ is stable (resp. semistable) of slope λ if and only if $\mathcal{F} \simeq \mathcal{O}_X(\lambda)_\pi$ (resp. there is an integer $n > 0$ such that $\mathcal{F} \simeq \mathcal{O}_X(\lambda)_\pi^{\oplus n}$).*

COROLLARY 4.4.1. *The functor*

$$\left\{\textit{finite dimensional } E\textit{-vector spaces}\right\} \to$$
$$\left\{\textit{semistable vector bundles of slope } 0 \textit{ over } X\right\}$$

sending V to $V \otimes_E \mathcal{O}_X$ is an equivalence of tannakian categories. The functor

$$\mathcal{F} \mapsto H^0(X, \mathcal{F})$$

is a quasi-inverse.

The proof of the theorem is easily reduced to the proof of the corollary. By dévissage, one sees that it is enough to prove the following statement:

[3] When F is not algebraically closed, this result remains true if and only if the residue field k_F of F is algebraically closed.

LEMME 4.2.1. *Let h be a positive integer and $\mathcal{F}$ be a vector bundle extension of $\mathcal{O}_X(1)$ by $\mathcal{O}_X(-1/h)$. Then*

$$H^0(X,\mathcal{F}) \neq 0 \ .$$

This lemma can be deduced by elementary manipulations on modifications of vector bundles from:

PROPOSITION 4.4.2. *Let h be a positive integer and*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{E} \to 0$$

a short exact sequence of coherent $\mathcal{O}_X$-modules, with $\mathcal{E}$ torsion of length 1. Then:
i) If $\mathcal{F} \simeq \mathcal{O}_X(1/h)$, then $\mathcal{F}' \simeq \mathcal{O}_X^h$.
ii) If $\mathcal{F}' \simeq \mathcal{O}_X^h$, then $\mathcal{F} \simeq \mathcal{O}_X(1/r) \oplus \mathcal{O}_X^{h-r}$ for some r with $1 \leq r \leq h$.

Let C be the residue field of X at the closed point which is the support of $\mathcal{E}$. This is an algebraically closed extension of E, complete with respect to an absolute value extending the given absolute value on E. This proposition can be translated:
i) in terms of *Banach-Colmez spaces* over C, i.e. the "Espaces de Banach de dimension finie" introduced by Colmez [**Col02**],
ii) or in terms of free B-modules equipped with a φ-semi-linear automorphism,
iii) or in terms of Barsotti-Tate groups over $\mathcal{O}_C$.
This leads to three different proofs of the proposition which becomes a consequence of the work of Colmez (*loc. cit.*) or of Kedlaya ([**Ke05**], [**Ke08**]) or of a result of Laffaille ([**Laf79**], also proved in [**GH94**]) for the first part and of Drinfel'd ([**Dr76**], also proved in [**Laf85**]) for the second part.

A consequence of the previous theorem is that *the geometric étale π_1 of the curve X is trivial*. More precisely:

PROPOSITION 4.4.3. *Let $X' \to X$ be a finite étale morphism and $E' = H^0(X',\mathcal{O}_{X'})$. The natural morphism*

$$X' \to X \times_{\text{Spec } E} SpecE'$$

is an isomorphism.

4.5. The topology on $\mathcal{O}_X$. The multiplicative norms $|\ |_\rho$ for $0 < \rho < 1$ extend to the fraction field of B. For each open subset U of X, we endow the ring $\Gamma(U,\mathcal{O}_X) \subset \text{Frac}(B)$ with the topology defined by the restriction of this family of norms. The transition maps

$$\Gamma(U,\mathcal{O}_X) \to \Gamma(V,\mathcal{O}_X)$$

for $V \subset U$ open is obviously continuous. This endows $\mathcal{O}_X$ with a natural structure of sheaf of locally convex E-algebras [4] which plays an important role in the study of O_X-representations of certain topological groups.

[4]A *locally convex E-vector space* is a topological E-vector space whose topology can be defined by a family of semi-norms.

4.6. $\mathcal{O}_X$-representations. We denote by $\mathcal{G}_F$ the group of continuous automorphisms of the field F (an automorphism of the field F is continuous if and only if it sends the valuation of F to a a strictly positive multiple of it). We equip $\mathcal{G}_F$ and its subgroups with the pointwise convergence topology, that is to say the weakest topology making the applications

$$\begin{array}{ccc} \mathcal{G}_F & \longrightarrow & F \\ g & \longmapsto & g(x) \end{array}$$

continuous when x goes through F. If $F = \widehat{\overline{F_0}}$ where F_0 is complete valued then $\mathrm{Gal}(\overline{F_0}|F_0) \subset \mathcal{G}_F$ is a closed subgroup and the induced topology on $\mathrm{Gal}(\overline{F_0}|F_0)$ is the usual profinite topology. By functoriality, $\mathcal{G}_F$ acts on X. We'll need slightly more. The action of $\mathcal{G}_F$ on $\mathcal{O}_X$ is *continuous*, i.e., for any open subset U of X, the subgroup

$$\mathcal{G}_{F,U} = \{ g \in \mathcal{G}_F \mid g(U) = U \}$$

is a closed subgroup of $\mathcal{G}_F$ and the natural map

$$\mathcal{G}_{F,U} \times \Gamma(U, \mathcal{O}_X) \to \Gamma(U, \mathcal{O}_X)$$

is continuous.

Let H be any closed subgroup of $\mathcal{G}_F$. We explained in §1.1 what is a $\mathcal{O}_X$-representations of H. We now use the topology on the sheaf $\mathcal{O}_X$ to put a continuity condition on these representations. More precisely if $\mathcal{E}$ is an $\mathcal{O}_X$-representation of H we require, for any open subset $U$of X, the natural map

$$H_U \times \Gamma(U, \mathcal{E}) \to \Gamma(U, \mathcal{E})$$

(where $H_U = \{h \in H \mid h(U) = U\}$) to be continuous.

From now on an $\mathcal{O}_X$-representation of H will mean a continuous one.

5. Galois descent

5.1. The curve X when F may not be algebraically closed. We don't assume anymore F algebraically closed and we consider the curve

$$X = X_{F,E} = \mathrm{Proj}\ P_\pi\ .$$

We choose an algebraic closure $\overline{F}$ of F and we set $H = \mathrm{Gal}(\overline{F}/F)$. The absolute value $|\ |$ of F extends uniquely to $\overline{F}$ and to its completion $\widetilde{F}$ (which is algebraically closed). We set

$$\widetilde{B} = B_{\widetilde{F},E}\ ,\ \widetilde{P}_\pi = P_{\widetilde{F},E,\pi,}\ \text{ and }\ \widetilde{X} = X_{\widetilde{F},E} = \mathrm{Proj}\ \widetilde{P}_\pi\ .$$

The action of H on $\overline{F}$ extends uniquely to a continuous action on $\widetilde{F}$ and by functoriality to a continuous action on $\widetilde{B}$ and $\widetilde{P}_\pi$. As we may identify H with a closed subgroup of the group $\mathcal{G}_{\widetilde{F}}$ of continuous automorphisms of the field $\widetilde{F}$, H also acts on the curve $\widetilde{X}$.

THEOREM 5.1. *i) The natural maps*

$$B \to \widetilde{B}^H \text{ and } P_\pi \to \widetilde{P}_\pi^H$$

are isomorphisms.

ii) The map $P_\pi \to \widetilde{P}_\pi$ induces a morphism of schemes

$$\nu : \widetilde{X} \to X$$

independent of the choice of π.

iii) Define the degree of any closed point $x \in X$ *by*

$$\deg(x) = \textit{cardinality of } \nu^{-1}(x) \ .$$

Then X *is a complete regular curve defined over* E.

iv) The morphism ν *induces an isomorphism*

$$\mathrm{Div}(X) \to (\mathrm{Div}(\widetilde{X}))^H \ .$$

Let H^* be *the group of characters of* H, i.e. the group of continuous homomorphisms from H to the multiplicative group E^* of E. If $D \in \mathrm{Div}^+(X) = (\mathrm{Div}^+(\widetilde{X}))^H$ is an effective divisor of degree $d \in \mathbb{N}$ and if $u \in \widetilde{P}_{\pi,d}$ is a generator of the homogeneous ideal of $\widetilde{P}$ corresponding to D, there is $\xi_D \in H^*$ such that, for all $h \in H$,

$$h(u) = \xi_D(h)u$$

and ξ_D is independent of the choice of u. The map $D \mapsto \xi_D$ extends uniquely to an homomorphism of groups

$$\mathrm{Div}(X) \to H^* \ .$$

This map induces an isomorphism

$$Pic^0(X) \to H^* \ .$$

More precisely,

Proposition 5.1.1. *Let* $\mathcal{K} = \mathcal{O}_{X,\eta}$ *the function field of* X. *The sequence*

$$0 \to E^* \to \mathcal{K}^* \to \mathrm{Div}(X) \to \mathbb{Z} \times H^* \to 0 \ ,$$

where $\mathrm{Div}(X) \to \mathbb{Z} \times H^*$ *is the map sending* D *to* $(\deg(D), \xi_D)$, *is exact.*

Moreover, for all $\xi_0 \in H^*$, *there exists an infinite set of effective divisors* D *of degree* 1 *such that* $\xi_D = \xi_0$.

If $\mathcal{F}$ is a coherent $\mathcal{O}_X$-module (resp. a vector bundle over X), then $\nu^*\mathcal{F}$ may be viewed as an $\mathcal{O}_{\widetilde{X}}$-representation of H (resp. an H equivariant vector bundle over $\widetilde{X}$).

Conversely, if $\mathcal{E}$ is an $\mathcal{O}_{\widetilde{X}}$-representation of H, we define the $\mathcal{O}_X$-module $\mathcal{E}^H$ by setting, for all open subset U of X

$$\Gamma(U, \mathcal{E}^H) = \Gamma(\nu^{-1}(U), \mathcal{E})^H$$

(and obvious restriction maps).

Theorem 5.2. *The functor*

$$\nu^* : \big\{\textit{coherent } \mathcal{O}_X\textit{-modules}\ \big\} \to \big\{\mathcal{O}_{\widetilde{X}}\textit{-representations of } H\big\}$$

is an equivalence of tensor categories, respecting the rank, the degree and the Harder-Narasimhan filtration.

For any $\mathcal{O}_{\widetilde{X}}$*-representation* $\mathcal{E}$ *of* H, *the* $\mathcal{O}_X$*-module* $\mathcal{E}^H$ *is coherent. The functor*

$$\mathcal{E} \mapsto \mathcal{E}^H$$

is a quasi-inverse of the functor $\mathcal{F} \mapsto \nu^*\mathcal{F}$.

5.2. The étale fundamental group. Let F' be a finite extension of F and E' be a finite extension of E.

– When, the residue field $k_{E'}$ is embedded in k_F we have defined the curve $X_{F',E'}$ and the natural morphism

$$X_{F,E'} \longrightarrow X_{F',E} \otimes_E E'$$

is an isomorphism.

– Therefore, we may define in general the curve $X_{F',E'}$ by

$$X_{F',E'} = X_{F',E} \otimes_E E' \ .$$

We have

$$X_{F',E} = \text{Proj } P_{F',E,\pi}$$

and the obvious map $P_{F,E,\pi} \to P_{F',E,\pi}$ induces a morphism

$$X_{F',E} \to X$$

which is a finite étale cover of X of degree $[F' : F]$, independent of the choice of π. Therefore

$$X_{F',E'} \to X$$

is a finite étale cover of X of degree $[F' : F].[E' : E]$.

Choose a closed point $\widetilde{x} = \text{Spec } C$ of $\widetilde{X}$. Then C is algebraically closed and we denote by $\overline{x}$ the geometric point of X

$$\text{Spec } C \to \widetilde{X} \to X \ .$$

Let $\mathcal{I}$ the set of pairs (F', E') with F' be a finite Galois extension of F contained in the field $F(C)$ introduced in §2.4 and E' a finite Galois extension of E contained in C.

The inclusion $F' \to F(C)$ induces an extension of the morphism

$$\overline{x} : \text{Spec } C \to X$$

to a morphism of X-schemes

$$\text{Spec } C \to X_{F',E} \ ,$$

which, using the inclusion $E' \to C$, extends also to a morphism of X-schemes

$$\text{Spec } C \to X_{F',E'} \ .$$

Proposition 5.2.1. *For each $(F', E') \in \mathcal{I}$, the morphism $X_{F',E'} \to X$ is a finite étale Galois cover whose Galois group is* $\text{Gal}(F'/F) \times \text{Gal}(E'/E)$.

Moreover the projective system

$$(X_{F',E'} \to X)_{(F',E') \in \mathcal{I}}$$

(with obvious transition maps) induces an isomorphism

$$\pi_1^{et}(X, \overline{x}) \to \text{Gal}(E^s/E) \times \text{Gal}(\overline{F}/F) \ ,$$

where E^s (resp. $\overline{F}$) denote the separable closure of E in C (resp. of F in $F(C)$).

In particular, the geometric étale π_1 of X may be identified with $\text{Gal}(\overline{F}/F)$.

6. de Rham G_K-equivariant vector bundles

In this section, K is a field of characteristic 0 which is the fraction field of a complete discrete valuation ring $\mathcal{O}_K$ whose residue field k is perfect of characteristic $p > 0$. We choose an algebraic closure $\overline{K}$ of K and we set $G_K = \mathrm{Gal}(\overline{K}/K)$. We denote by C the completion of $\overline{K}$. This is an algebraically closed field, therefore it is a strictly p-perfect field and the field $F = F(C)$ is algebraically closed.

6.1. The curve $X = X_{F(C),\mathbb{Q}_p}$. We consider the curve

$$X = X_{F,\mathbb{Q}_p} .$$

We set

$$B = B_{F,\mathbb{Q}_p} \text{ and } B^+ = B^+_{F,\mathbb{Q}_p} .$$

We have

$$X = \mathrm{Proj}\ P_p \ \text{ with } P_p = \bigoplus_{d\in\mathbb{N}} P_{p,d} \ \text{ and } \ P_{p,d} = \left\{u \in B \mid \varphi(u) = p^d u\right\} .$$

The natural map $P_p \to B$ is injective, with image contained in B^+, and we identify P_p with its image.

As $F = F(C)$, we have a canonical continuous surjective homomorphism of $\mathbb{Q}_p$-algebras

$$\theta : B \to C$$

(the restriction of θ to B^b is the map $\sum_{n>\gg-\infty}[a_n]p^n \mapsto \sum_{n\gg-\infty} a_n^{(0)}p^n$).

We fix $\varpi \in F$ such that $\varpi^{(0)} = p$. Then the kernel of θ is the principal ideal generated by $p - [\varpi]$. As usual in p-adic Hodge theory [**Fon94a**], we denote B^+_{dR} the completion of $B^{b,+}$ for the $(p - [\varpi])$-adic topology. This is also the completion of B (or of B^+) for the $\ker\theta$-adic topology. As θ is G_K-equivariant, the action of G_K on B extends to B^+_{dR}.

As usual (*loc. cit.*), we fix $\varepsilon \in F$ such that $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$. We set

$$t = \log([\varepsilon]) = \sum_{n=1}^{+\infty}(-1)^{n+1}\frac{([\varepsilon]-1)^n}{n} \ \in B^+ .$$

Then t is a generator of the $\mathbb{Q}_p$-line $P_{p,1} \cap \ker\theta$. The homogeneous ideal of P_p generated by t defines a closed point ∞ of X which is the image in $|X|$ of the maximal ideal $\ker\theta$ of B.

Therefore ∞ is fixed under G_K, its residue field is C and the completion of the discrete valuation ring $\mathcal{O}_{X,\infty}$ is B^+_{dR}. We set

$$X_e = X\backslash\{\infty\} .$$

This is an affine open subset, stable under G_K. We see that

$$B_e := \Gamma(X_e, \mathcal{O}_X) = \left\{\text{homogeneous elements of degree 0 of } P_p[\tfrac{1}{t}]\ \right\}$$

is a principal ideal domain. We set

$$B_{cr} = B^+[\tfrac{1}{t}] .$$

The Frobenius φ on B^+ extends uniquely to an automorphism of B_{cr} and we have

$$B_e = \left\{b \in B_{cr} \mid \varphi(b) = b\right\} .$$

Remark. The ring B^+ is sometimes denoted $\widetilde{\mathbf{B}}^+_{rig}$ (e.g. [**Ber02**], §1 where $F = F(C)$ is denoted $\widetilde{\mathbf{E}}$, though A is denoted $\widetilde{\mathbf{A}}$ and $B^{b,+}$ is denoted $\widetilde{\mathbf{B}}^+$).Traditionally [**Fon94a**], one defines the ring A_{cris} as the p-adic completion of the divided power envelop of the ring A with respect to the ideal generated by $p - [\varpi]$ and $B^+_{cris} = A_{cris}[1/p]$. The inclusion of $A[1/p] = B^{b,+}$ into B^+ extends by continuity to a canonical injective map from B^+ into B^+_{cris}. Hence, we may identify B^+ with a subring of B^+_{cris} and $B^+[1/t]$ with a subring of $B_{cris} = B^+_{cris}[1/t]$. We then have

$$B^+ = \cap_{n\in\mathbb{N}}\varphi^n(B^+_{cris}) \text{ and } B^+[\tfrac{1}{t}] = \cap_{n\in\mathbb{N}}\varphi^n(B_{cris}) \ ,$$

so, we have also

$$B_e = \big\{b \in B_{cris} \mid \varphi(b) = b\big\}$$

and the definition of B_e given here agrees with the definition of [**FP94**], chap.I, §3.3.

6.2. B_e-representations of G_K. Recall that a B_e-representation of G_K is a B_e-module of finite type equipped with a semi-linear and continuous action of G_K. Those are the (continuous) $\mathcal{O}_{X_e}$-representations of G_K. They form an abelian category. A G_K-equivariant vector bundle over Spec B_e is a B_e-representation of G_K such that the underlying B_e-module is locally free, hence free as B_e is a principal domain. It turns out that this condition is automatic:

Proposition 6.2.1. *The B_e-module underlying any B_e-representation of G_K is torsion free. The category of B_e-representations of G_K is an abelian category.*

Granted what we already know, the proof of this proposition is easy: The second assertion results from the first. To show the first assertion, it is enough to show, that if V is a B_e-representation of G_K such that the underlying B_e-module is a torsion module, then $V = 0$. We observe that the annihilator of V is a non-zero ideal $\mathfrak{a}$ stable under G_K. Then $\mathfrak{a}$ is the product of finitely many maximal ideals. If $\mathfrak{m}$ is one of them, for all $g \in G_K$, $g(\mathfrak{m})$ must contain $\mathfrak{a}$. But the maximal ideals corresponds to the closed points of $X_e = X\backslash\{\infty\}$ and one can show that ∞, which is fixed under G_K, is the unique closed point of X whose orbit under G_K is finite. Therefore $\mathfrak{a} = B_e$ and $V = 0$.

Remarks. (1) This result implies that the tensor category of B_e-representations is a tannakian $\mathbb{Q}_p$-linear category.

(2) It is easy to see that $B_e^* = \mathbb{Q}_p^*$. This implies that any continuous 1-cocycle

$$\alpha : G_K \to (B_e)^*$$

takes its values in $\mathbb{Q}_p^*$. It means that, if V is a one dimensional B_e-representation, the $\mathbb{Q}_p$-line generated by a basis of V over B_e is stable under G_K. In other words, any one dimensional B_e-representation of G_K comes by scalar extension from a one dimensional p-adic representation of G_K.

6.3. Vector bundles and their cohomology. Let $\mathcal{F}$ be a coherent $\mathcal{O}_X$-module. Then

– the B_e-module

$$\mathcal{F}_e = \Gamma(X_e, \mathcal{F})$$

is of finite type,

– the completion $\mathcal{F}^+_{dR}$ of the fiber at ∞ is a B^+_{dR}-module of finite type,

– we have a canonical isomorphism

$$\iota_{\mathcal{F}} : B_{dR} \otimes_{B_e} \mathcal{F}_e \to B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+$$

With an obvious definition for the morphisms, the triples

$$(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$$

with $\mathcal{F}_e$ a B_e-module of finite type, $\mathcal{F}_{dR}^+$ a B_{dR}^+-module of finite type and

$$\iota_{\mathcal{F}} : B_{dR} \otimes_{B_e} \mathcal{F}_e \to B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+$$

an isomorphism of B_{dR}^+-modules form a tensor abelian category. The correspondence

$$\mathcal{F} \mapsto (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$$

just defined induces a tensor equivalence of categories. We use it to identify these two categories.

Then $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$ is a vector bundle if and only if $\mathcal{F}_e$ is free over B_e and $\mathcal{F}_{dR}^+$ is free over B_{dR}^+. In this case, to give $\iota_{\mathcal{F}}$ is the same as giving an isomorphism from $\mathcal{F}_{dR}^+$ onto a B_{dR}^+-*lattice* of $B_{dR} \otimes_{B_e} \mathcal{F}_e$, i.e. a sub-$B_{dR}^+$-module of finite type generating $B_{dR} \otimes_{B_e} \mathcal{F}_e$ as a B_{dR} vector space.

Therefore, we may as well see a vector bundle over X as a pair

$$(\mathcal{F}_e, \mathcal{F}_{dR}^+)$$

where $\mathcal{F}_e$ is a free B_e-module of finite rank and $\mathcal{F}_{dR}^+$ is a B_{dR}^+-lattice in $\mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e$.

The cohomology of $\mathcal{F}$ is easy to compute: we have an exact sequence

$$0 \to H^0(X, \mathcal{F}) \to \mathcal{F}_e \oplus \mathcal{F}_{dR}^+ \to \mathcal{F}_{dR} \to H^1(X, \mathcal{F}) \to 0$$

where the middle map is $(b, b') \mapsto b - b'$. In the special case of $\mathcal{O}_X$, we have $H^0(X, \mathcal{O}_X) = \mathbb{Q}_p$ and $H^1(X, \mathcal{O}_X) = 0$, giving rise to the "fundamental exact sequence of p-adic Hodge theory"

$$0 \to \mathbb{Q}_p \to B_e \oplus B_{dR}^+ \to B_{dR} \to 0 \ .$$

6.4. G_K-equivariant vector bundles. As ∞ is stable under G_K, we see that:

– We may identity the abelian tensor category of $\mathcal{O}_X$-representations of G_K with the category of triples

$$(\mathcal{F}_e, \mathcal{F}_{dR}^+, \iota_{\mathcal{F}})$$

where

i) $\mathcal{F}_e$ is a B_e-representation of G_K,

ii) $\mathcal{F}_{dR}^+$ is a B_{dR}-representation of G_K,

iii)
$$\iota_{\mathcal{F}} : B_{dR} \otimes_{B_e} \mathcal{F}_e \to B_{dR} \otimes_{B_{dR}^+} \mathcal{F}_{dR}^+$$

is a G_K-equivariant isomorphism of B_{dR} vector spaces.

– We may identify the category of G_K-equivariant vector bundles over X to the category of pairs

$$(\mathcal{F}_e, \mathcal{F}_{dR}^+)$$

where

i) $\mathcal{F}_e$ is a B_e-representation of G_K,

ii) $\mathcal{F}_{dR}^+$ is a G_K-stable B_{dR}^+-lattice in $\mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e$.

The category of such pairs has already been considered by Berger [**Ber08**].

Remark. Let $\mathcal{F}$ be an $\mathcal{O}_X$-representation of G_K. The fact that ∞ is the only closed point of X whose orbit under G_K is finite implies that the torsion of $\mathcal{F}$, if any, is concentrated at ∞. If $\mathcal{F}$ is a vector bundle, i.e. is torsion free and if $\mathcal{G}$ is a G_K-equivariant *modification of* $\mathcal{F}$ (i.e. $\mathcal{F}$ and $\mathcal{G}$ have the same generic fiber), we have $\mathcal{G}_e = \mathcal{F}_e$ though $\mathcal{G}_{dR}^+$ may be any G_K-stable B_{dR}^+-lattice of $\mathcal{F}_{dR}$.

6.5. The hierarchy of $\mathcal{O}_X$-representations. Let $B^?$ be any topological ring equipped with a continuous action of G_K. We say that a $B^?$-representation V of G_K is *trivial* if the natural map

$$B^? \otimes_{(B^?)^{G_K}} V^{G_K} \to V$$

is an isomorphism.

We introduce the ring

$$B_{lcr} = B_{cr}[\log([\varpi])]$$

of polynomials in the indeterminate $\log([\varpi])$ with coefficients in B_{cr}.

Consider the continuous maps

$$\chi : G_K \to \mathbb{Z}_p^* \quad \text{and} \quad \eta : G_K \to \mathbb{Z}_p$$

such that, for all $g \in \mathcal{G}_K$,

$$g(t) = \chi(g)t \quad \text{and} \quad g(\varpi) = \varpi\varepsilon^{\eta(g)} \ .$$

The action of G_K on B^+ extends to B_{lcr} by setting, for all $g \in G_K$,

$$g(\tfrac{1}{t}) = \tfrac{1}{\chi(g)t} \quad \text{and} \quad g(\log([\varpi]) = \log([\varpi]) + \eta(g)t \ .$$

We say that a B_e-representation V is *de Rham* (resp. *log-crystalline*, resp. *crystalline*) if the representation $B_{dR} \otimes_{B_e} V$ (resp. $B_{lcr} \otimes_{B_e} V$, resp. $B_{cr} \otimes_{B_e} V$) is trivial. We say that V is *potentially log-crystalline* if there is a finite extension L of K contained in $\overline{K}$ such that V, viewed as a B_e-representation of $G_L = \mathrm{Gal}(\overline{K}/L)$ is log-crystalline.

For any property which makes sense for a B_e-representation, we say that a G_K-equivariant vector bundle $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}_{dR}^+)$ over X_E satisfies this property if $\mathcal{F}_e$ does.

The following result is easy to prove:

Proposition 6.5.1. *Let*

$$0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$$

a short exact sequence of B_e-representations or of G_K-equivariant vector bundles. If $\mathcal{F}$ is de Rham (resp. potentially log-crystalline, resp. log-crystalline, resp. crystalline), so are $\mathcal{F}'$ and $\mathcal{F}''$.

Therefore we may say that an $\mathcal{O}_X$-representation of G_K is *de Rham* (resp. *potentially log-crystalline*, resp. *log-crystalline*, resp. *crystalline*) if it is isomorphic to a quotient of a G_K-equivariant vector bundle which has this property.

It is easy to show (see more details in §6.7 below) that:

– if $\mathcal{F}_1$ and $\mathcal{F}_2$ are two $\mathcal{O}_X$-representations of G_K having one of those four properties, then any sub-$\mathcal{O}_X$-representation of $\mathcal{F}_1$, any quotient of $\mathcal{F}_1$, the representation $\mathcal{F}_1 \otimes \mathcal{F}_2$ and $\mathcal{L}_{\mathcal{O}_X}(\mathcal{F}_1, \mathcal{F}_2)$ have the same properties,

– we have the implications

$$\text{crystalline} \implies \text{log-crystalline} \implies \text{potentially log-crystalline} \implies \text{de Rham}.$$

It is a deep result (see §7 below) that, conversely, any de Rham $\mathcal{O}_X$-representation is potentially log-crystalline.

6.6. Log-crystalline B_e-representations and (φ, N)-modules. Let $K_0 = \text{Frac}\, W(k)$. One can show that

$$(B_{lcr})^{G_K} = K_0 \ .$$

If V is any B_e-representation of G_K, we set

$$\mathcal{D}_{lcr}(V) = (B_{lcr} \otimes_{B_e} V)^{G_K} \ .$$

This is a K_0-vector space and we denote

$$\alpha_V : B_{lcr} \otimes_{K_0} \mathcal{D}_{lcr}(V) \to B_{lcr} \otimes_{B_e} V$$

the B_{lcr}-linear map deduced by scalar extension from the inclusion $\mathcal{D}_{lcr}(V) \subset B_{lcr} \otimes_{B_e} V$.

By definition V is log-crystalline if and only if α_V is bijective. It is not hard to see that α_V is injective, that the dimension over K_0 of $\mathcal{D}_{lcr}(V)$ is $\leq$ the rank of V over B_e and that equality holds if and only if α_V is bijective (this last statement comes from the fact that any B_e-representation of G_K of rank one comes, by scalar extension, from a one dimensional p-adic representation of G_K and that any non-zero element $b \in B_{lcr}$ such that the $\mathbb{Q}_p$-vector space generated by b is stable under G_K is invertible).

The Frobenius φ on B_+ extends to B_{lcr} by setting

$$\varphi(\tfrac{1}{t}) = \tfrac{1}{pt} \quad \text{and} \quad \varphi(\log([\varpi])) = p \log([\varpi]) \ .$$

One denotes $N : B_{lcr} \to B_{lcr}$ the unique B^+-derivation such that $N(\log([\varpi])) = -1$. We get

$$N\varphi = p\varphi N \ .$$

The action of φ and of N commute with the action of G_K. On K_0 we have $N = 0$ and the Frobenius φ is the absolute Frobenius, i.e. the unique continuous automorphism inducing $x \mapsto x^p$ on the residue field.

A *(φ, N)-module over k* is a finite dimensional K_0-vector space D equipped with two operators

$$\varphi, N : D \rightrightarrows D$$

with φ semi-linear with respect to the action of φ on K_0 and bijective, N K_0-linear and $N\varphi = p\varphi N$.

With an obvious definition of the morphisms, the (φ, N)-modules over k form an abelian category $Mod(\varphi, N)_k$. It has an obvious structure of a tannakian $\mathbb{Q}_p$-linear category.

Let V be a B_e-representation of G_K. The free B_{lcr}-module $B_{lcr} \otimes_{B_e} V$ is equipped with operators φ and N by setting

$$\varphi(b \otimes v) = \varphi(b) \otimes v \text{ and } N(b \otimes v) = Nb \otimes v \quad \text{if } b \in B_{lcr} \text{ and } v \in V \ ,$$

commuting with the action of G_K. Therefore

$$\mathcal{D}_{lcr}(V) = (B_{lcr} \otimes_{B_e} V)^{G_K}$$

is stable under φ and N and becomes a (φ, N)-module over k.

If D is a (φ, N)-module over k, then G_K, φ and N act on $B_{lcr} \otimes_{K_0} D$ via

$$g(b \otimes x) = g(b) \otimes x \ , \varphi(b \otimes x) = \varphi(b) \otimes \varphi(x) \ \ N(b \otimes x) \\ = Nb \otimes x + b \otimes Nx \text{ for } g \in G_K, b \in B_{lcr}, x \in D.$$

It is easy to see that the B_e-module

$$\mathcal{V}_{lcr}(D) = \big\{ v \in B_{lcr} \otimes_{K_0} D \mid \varphi_E(v) = v \text{ and } Nv = 0 \big\}$$

is free of rank equal to the dimension of D over K_0, hence is a B_e-representation of G_K.

Let $\mathrm{Rep}_{B_e,lcr}(G_K)$ be the full sub-category of the category $\mathrm{Rep}_{B_e}(G_K)$ of B_e-representations of G_K whose objects are the representations which are log-crystalline. The proof of the following statement is straightforward and formal:

THEOREM 6.1. *For any (φ, N)-module D over k, the B_e-representation $\mathcal{V}_{lcr}(D)$ of G_K is log-crystalline. The functor*

$$\mathcal{V}_{lcr} : Mod(\varphi, N)_k \to \mathrm{Rep}_{B_e,lcr}(G_K)$$

is an equivalence of categories and the functor

$$V \mapsto \mathcal{D}_{lcr}(V)$$

is a quasi-inverse.

REMARKS. (1) It is easy to see that a B_e-representation V of G_K is crystalline if and only if it is log-crystalline and $N = 0$ on $\mathcal{D}_{lcr}(V)$.

(2) The relation $N\varphi = p\varphi N$ implies that N is nilpotent on any object of $Mod(\varphi, N)_k$ and that the kernel of N is a sub-object.

In particular, the semi-simplification of a log-crystalline B_e-representation of G_K is a crystalline B_e-representation of G_K. If k is algebraically closed, the full sub-category $Mod(\varphi)_k$ of $Mod(\varphi, N)_k$ whose objects are those on which $N = 0$ is semi-simple ([**Man63**], §2). Therefore a B_e-representation of G_K is crystalline if and only if it is log-crystalline and semi-simple.

(3) The category $\mathrm{Rep}_{B_e,lcr}(G_K)$ is a *tannakian subcategory of* $\mathrm{Rep}_{B_e}(G_K)$, i.e. it is stable under taking sub-objects, quotients, direct sums, tensor products, internal hom and contains the unit representation B_e. *The functor $\mathcal{V}_{lcr}$ is an equivalence of tannakian categories.*

Let $I_K \subset G_K$ the inertia subgroup. We have $C^{I_K} = \widehat{K}_{nr}$, the p-adic completion of the maximal unramified extension of K contained in $\overline{K}$. The algebraic closure of $\widehat{K}_{nr}$ in C is a dense subfield of C and I_K can be identified with the Galois group of this algebraic closure over $\widehat{K}_{nr}$.

If V is any B_e-representation of G_K, denote by $Res_{I_K}(V)$ the B_e-representation of I_K which is V with the action of I_K deduced from the inclusion of I_K into G_K.

If $\overline{k}$ is the residue field of $\widehat{K}_{nr}$, and $G_k = \mathrm{Gal}(\overline{k}/k) = G_K/I_K$, we have

$$\mathcal{D}_{lcr}(V) = (\mathcal{D}_{lcr}(Res_{I_K}(V)))^{G_k} \ .$$

From the fact that, if $\widehat{K}_{0,nr}$ is the fraction field of $W(\overline{k})$ and D is a finite dimensional $\widehat{K}_{0,nr}$ vector space equipped with a semi-linear and continuous action of G_k, the natural map

$$\widehat{K}_{0,nr} \otimes_{K_0} D^{G_k} \to D$$

is an isomorphism, we deduce:

PROPOSITION 6.6.1. *Let V be a B_e-representation of G_K. Then V is log-crystalline if and only if $Res_{I_K}(V)$ is log-crystalline.*

6.7. Log-crystalline vector bundles and filtered (φ, N)-modules. As B^+ is separated for the $\ker\theta$-adic topology, we may view B^+ as a subring of B^+_{dR} and $B_{cr} = B^+[1/t]$ as a sub B_e-algebra of $B_{dR} = B^+_{dR}[1/t]$.

Extending the p-adic logarithm by deciding that $\log(p) = 0$, one can identify B_{lcr} with a sub-B_{cr}-algebra of B_{dR} by setting

$$\log([\varpi]) = \log([\varpi]/p) = -\sum_{n=1}^{+\infty} \frac{(p - [\varpi])^n}{np^n} .$$

If $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}^+_{dR})$ is a G_K-equivariant vector bundle over X, and if $\mathcal{F}_{dR} = B_{dR} \otimes_{B_e} \mathcal{F}_e = B_{dR} \otimes_{B^+_{dR}} \mathcal{F}^+_{dR}$, we set

$$\mathcal{D}_{lcr}(\mathcal{F}) = \mathcal{D}_{lcr}(\mathcal{F}^e) = (B_{lcr} \otimes_{B_e} \mathcal{F}^e)^{G_K} \text{ and } \mathcal{D}_{dR}(\mathcal{F}) = (\mathcal{F}_{dR})^{G_K}$$

If $\mathcal{F}$ is of rank r, then:

i) $\mathcal{D}_{lcr}(\mathcal{F})$ is a (φ, N)-module over K_0 whose dimension over K_0 is $\leq r$ with equality if and only if $\mathcal{F}$ is log-crystalline.

ii) The natural map

$$B_{dR} \otimes_K D_{dR}(\mathcal{F}) \to \mathcal{F}_{dR}$$

is always injective, therefore the K-vector space $D_{dR}(\mathcal{F})$ is of dimension $\leq r$ with equality if and only if $\mathcal{F}$ is de Rham.

We see also that $\mathcal{D}_{dR}(\mathcal{F})$ is a *filtered K-vector space*, i.e. a finite dimensional K-vector space Δ equipped with a decreasing filtration, indexed by $\mathbb{Z}$, by sub K vector spaces

$$\ldots \supset F^{i-1}\Delta \supset F^i\Delta \supset F^{i+1}\Delta \supset \ldots$$

such that $F^i\Delta = 0$ for $i \gg 0$ and $= \Delta$ for $i \ll 0$: The filtration is defined by

$$F^i\mathcal{D}_{dR}(\mathcal{F}) = (F^iB_{dR} \otimes_{B^+_{dR}} \mathcal{F}^+_{dR})^{G_K}$$

where $F^iB_{dR} = B^+_{dR}t^i$ is the fractional ideal of the discrete valuation ring B^+_{dR} which is the i^{th} power of its maximal ideal.

The inclusion $K \otimes_{K_0} B_{lcr} \to B_{dR}$ induces an injective map

$$K \otimes_{K_0} \mathcal{D}_{lcr}(\mathcal{F}) \to \mathcal{D}_{dR}(\mathcal{V}) .$$

For dimension reasons, *if $\mathcal{F}$ is log-crystalline, this map is an isomorphism, $\mathcal{F}$ is de Rham* and the pair $\mathcal{D}_{lcr,K}(\mathcal{F})$ consisting of $\mathcal{D}_{lcr}(\mathcal{F})$ and the filtration on $K \otimes_{K_0} \mathcal{D}_{lcr}(\mathcal{F})$ induced by this isomorphism is a *filtered (φ, N)-module over K* (cf. [**Fon94b**]), i.e. it is a finite dimensional K_0-vector space D, equipped with operators φ, N giving to D the structure of a (φ, N)-module over k, plus a filtration F (i.e. a structure of filtered K vector space) on the K vector space $D_K = K \otimes_{K_0} D$.

A *morphism of filtered (φ, N)-modules over K*

$$f : (D, F) \to (D', F)$$

is a K_0-linear map commuting with φ and N and such that, if $f_K : D_K \to D'_K$ is the K-linear map deduced from f by scalar extension, then $f_K(F^i D_K) \subset F^i D'_K$ for all $i \in \mathbb{Z}$.

The category $\mathrm{MF}_K(\varphi, N)$ of filtered (φ, N)-modules over K is an additive $\mathbb{Q}_p$-linear category.

If there is no risk of confusion on the filtration, we write $D = (D, F)$ for any object (D, F) of $\mathrm{MF}_K(\varphi, N)$. The following result is now obvious:

Theorem 6.2. *The functor*

$$\mathcal{D}_{lcr,K} : \{\text{log-cryst. } G_K\text{-equiv. vector bundles over } X\} \to \mathrm{MF}_K(\varphi, N)$$

is an equivalence of categories. A quasi-inverse is given by the functor $\mathcal{F}_{lcr}$ defined by

$$\mathcal{F}_{lcr,K}(D) = (\mathcal{V}_{lcr}(D), F^0(B_{dR} \otimes_K D_K))$$

where $\mathcal{V}_{lcr}(D)$ is the B_e-representation of G_K associated to the (φ, N)-module over k underlying D and

$$F^0(B_{dR} \otimes_K D_K)) = \sum_{i \in \mathbb{Z}} F^i B_{dR} \otimes_K F^{-i} D_K \subset B_{dR} \otimes_K D_K = B_{dR} \otimes_{B_e} \mathcal{V}_{lcr}(D) \ .$$

Remarks. (1) We say that a sequence of morphisms of log-crystalline G_K-equivariant vector bundles over X is *exact* if the underlying sequence of $\mathcal{O}_X$-modules is exact. Similarly we say that a sequence of morphisms

$$\ldots \to (D', F) \to (D, F) \to (D'', F) \to \ldots$$

of $\mathrm{MF}_K(\varphi, N)$ is *exact* if, for any $i \in \mathbb{Z}$, the induced sequence of K-vector spaces

$$\ldots F^i D'_K \to F^i D_K \to F^i D''_K \ldots$$

is exact.

With these definitions (or rather with the restriction of this definition to short exact sequences) these two categories are exact categories ([**Qui73**], §2). The functors $\mathcal{D}_{lcr,K}$ and $\mathcal{F}_{lcr,K}$ turn exact sequences into exact sequences.

(2) The category of G_K-equivariant vector bundles over X and the category $\mathrm{MF}_K(\varphi, N)$ both have a natural structure of a $\mathbb{Q}_p$-linear tensor category ([**Fon94b**], §4.3.4, for the later). The functors $\mathcal{F}_{lcr,K}$ and $\mathcal{V}_{lcr,K}$ are tensor functors.

(3) Let $\mathcal{F}$ be a log-crystalline G_K-equivariant vector bundle over X and let $D = \mathcal{D}_{lcr}(V)$. If $\mathcal{G}$ is a G_K-equivariant modification of $\mathcal{F}$, then $\mathcal{G}$ is still log-crystalline and $\mathcal{D}_{lcr}(\mathcal{G}) = D$. Therefore, to give such a modification is the same as changing the filtration on D_K.

(4) We have a functor $D \to (D, F_{triv})$ from the category of (φ, N)-modules over k to $\mathrm{MF}_K(\varphi, N)$ consisting of adding to a (φ, N)-module D the trivial filtration on D_K (i.e. $F^i_{triv} D_K = D_K$ if $i \le 0$ and 0 if $i > 0$).

(5) Let D be a (φ, N)-module over k, and choose a basis $e_1, e_2, \ldots, e_r$ of D over K_0. If we set $\varphi(e_j) = \sum_{i=1}^r a_{ij} e_i$, the p-adic valuation of the determinant of the matrix of the a_{ij} is independent of the choice of the basis and is denoted $t_N(D)$. It is easy to see that

$$\mathrm{rank} \mathcal{V}_{lcr,K}(D, F_{triv}) = \dim_{K_0} D \text{ and } \deg \mathcal{V}_{lcr,K}(D, F_{triv}) = -t_N(D).$$

If now F is a filtration on D, so that $\mathcal{V}_{lcr,K}(D,F)$ is a modification of $\mathcal{V}_{lcr,K}(D,F_{triv})$, it's easily to see that, if $t_H(D,F)=\sum_{i\in\mathbb{Z}} i.\dim_K(F^iD_K/\ F^{i+1}D_K)$, then

$$\begin{aligned}&\text{rank } \mathcal{V}_{lcr,K}(D,F)=\text{rank } \mathcal{V}_{lcr,K}(D,F_{triv})\\ &\text{and } \deg \mathcal{V}_{lcr}(D,F)=\deg \mathcal{V}_{lcr,K}(D,F_{triv})+t_H(D,F).\end{aligned}$$

This remark suggests to define the *rank*, the *degree* and the *slope* of a non-zero filtered (φ,N)-module (D,F) over K by

$$\text{rank}(D,F)=\dim_{K_0} D\ ,\ \deg(D,F)=t_H(D,F)-t_N(D) \text{ and } \mu(D,F) = \frac{\deg(D,F)}{\text{rank}(D,F)}.$$

Let $f:(D',F)\to(D,F)$ a morphism of $\mathrm{MF}_K(\varphi,N)$, with $f_K:D'_K\to D_K$ the underlying K-linear map. We say that f is *strict* if it is strictly compatible to the filtrations, i.e. if $f_K(F^iD'_K)=F^iD_K\cap f_K(D'_K)$ for all $i\in\mathbb{Z}$. If f_K is injiective, it is equivalent to saying that f fits into a short exact sequence of $\mathrm{MF}_K(\varphi,N)$

$$0\to(D',F)\to(D,F)\to(D'',F)\to 0\ .$$

A *sub-object* (D',F) of a filtered (φ,N)-module (D,F) is a morphism $(D',F)\to(D,F)$ such that the (φ,N)-module D' is a sub-object of D.

The strict sub-objects of an object (D,F) correspond bijectively to the sub-objects of the underlying (φ,N)-module via the map

$$D'\mapsto(D',F) \text{ with } F^iD'_K=F^iD_K\cap D'_K \text{ for all } i\in\mathbb{Z}\ .$$

If (D',F) is such a sub-object, the quotient $(D,F)/(D',F)$ is the cokernel of $(D',F)\to(D,F)$.

We say that a filtered (φ,N)-module (D,F) is *semistable* if, for any non-zero sub-object (D',F) of (D,F), we have $\mu(D',F)\le\mu(D,F)$. It is enough to check it for strict sub-objects.

The following assertion is entirely formal:

PROPOSITION 6.7.1. *i) For any non-zero filtered (φ,N)-module D over K, there is a unique filtration* (called the Harder-Narasimhan filtration) *by strict sub-objects*

$$0=D_0\subset D_1\subset\ldots\subset D_{i-1}\subset D_i\subset\ldots\subset D_{m-1}\subset D_m=D$$

with each D_i/D_{i-1} non-zero and semistable such that

$$\mu(D_1/D_0)>\mu(D_2/D_1)>\ldots>\mu(D_m/D_{m-1})\ .$$

ii) The functors $\mathcal{D}_{lcr,K}$ and $\mathcal{V}_{lcr,K}$ respect the rank, the degree, the slope and the Harder-Narasimhan filtration.

6.8. p-adic Hodge theory. The corollary 4.4.1 implies that we have an equivalence of tannakian categories between p-adic representations (i.e. $\mathbb{Q}_p$-representations) of G_K and G_K-equivariant vector bundles over X which are semistable of slope 0:

$$\begin{aligned}V\to\mathcal{F}(V)=\mathcal{O}_X\otimes_{\mathbb{Q}_p}V=(B_e\otimes_{\mathbb{Q}_p}V,B^+_{dR}\otimes_{\mathbb{Q}_p}V)\\ \text{(with } \mathcal{F}\mapsto V(\mathcal{F})=H^0(X,\mathcal{F}) \text{ as a quasi-inverse).}\end{aligned}$$

We say that V is *de Rham* (resp. *potentially log-crystalline*, resp. *log-crystalline*, resp. *crystalline* if $\mathcal{F}(V)$ has this property.

Classically one introduces [**Fon94a**] the ring

$$B_{st} = B_{cris}[\log[\varpi]] \ .$$

If V is a p-adic representation of G_K, one says that V is *de Rham* (resp. *crystalline*, resp. *semistable*, resp. *potentially semistable*) if $B_{dR} \otimes_{\mathbb{Q}_p} V$ is trivial (resp. $B_{cris} \otimes_{\mathbb{Q}_p} V$ is trivial, resp. $B_{st} \otimes_{\mathbb{Q}_p} V$ is trivial, resp. there is a finite extension L of K contained in $\overline{K}$ such that V is semistable as a p-adic representation of G_L).

The origin of this terminology lies in the facts that, if Z is any proper and smooth variety over K, $i \in \mathbb{N}$ and $V = H^i_{ét}(Z_{\overline{K}}, \mathbb{Q}_p)$, then ([**Fa89**], [**Ts99**], [**Ni08**])

– the p-adic representation V is de Rham and the filtered K-vector space $D_{dR}(V) = \mathcal{D}_{dR}(\mathcal{F}(V))$ can be identified with

$$H^i_{dR}(Z) = \mathbb{H}^i(Z, \Omega^\bullet_{Z/K})$$

equipped with the Hodge filtration,

– if there exists $\mathcal{Z}$ over $\mathcal{O}_K$ proper and smooth such that

$$\mathrm{Spec}\ K \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathcal{Z} = Z \ ,$$

then V is crystalline and $D_{cris}(V) = \mathcal{D}_{lcr}(\mathcal{F}(V))$ is the i^{th}-crystalline cohomology group of the special fiber of $\mathcal{Z}$ (equality respecting the Frobenius and compatible with the filtration via the de Rham comparison isomorphism),

– if there exists $\mathcal{Z}$ over $\mathcal{O}_K$ proper and semistable such that

$$\mathrm{Spec}\ K \times_{\mathrm{Spec}\ \mathcal{O}_K} \mathcal{Z} = Z \ ,$$

then V is semistable and $D_{st}(V) = \mathcal{D}_{lcr}(\mathcal{F}(V))$ is the i^{th}-log-crystalline cohomology group of the log special fiber of $\mathbb{Z}$ (equality respecting φ and N and compatible with the filtration via the de Rham comparison isomorphism).

It is easy to check that

– the definition given in §6.5 of a de Rham and of a crystalline p-adic representation agrees with the classical definition,

– a p-adic representation V is log-crystalline (resp. potentially log-crystalline) if and only if it is semistable (resp. potentially semistable).

We made this change of terminology to avoid confusion between the two notion of semistability (semistable model of a variety and semistable vector bundle).

As a corollary of the proposition 6.7.1, denoting $\mathrm{Rep}_{\mathbb{Q}_p,lcr}(G_K)$ the full subcategory of the category $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ of p-adic representations of G_K whose objects are the log-crystalline ones and $\mathrm{MF}^0_K(\varphi, N)$ the full sub-category of $\mathrm{MF}_K(\varphi, N)$ whose objects are those which are semistable of slope 0, we get:

Theorem 6.3. *For any p-adic log-crystalline representation of G_K,*

$$D_{lcr,K}(V) = \mathcal{D}_{lcr,K}(\mathcal{O}_X \otimes V)$$

is a filtered (φ, N)-module over K which is semistable of slope 0*.*

The category $\mathrm{Rep}_{\mathbb{Q}_p,lcr}(G_K)$ is a tannakian subcategory of $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ and

$$D_{lcr,K} : \mathrm{Rep}_{\mathbb{Q}_p,lcr}(G_K) \to \mathrm{MF}^0_K(\varphi, N)$$

is an equivalence of tensor categories. The functor

$$V_{lcr,K} : \mathrm{MF}^0_K(\varphi, N) \to \mathrm{Rep}_{\mathbb{Q}_p,lcr}(G_K) \ ,$$

defined by

$$V_{lcr,K}(D) = \Gamma(X, \mathcal{V}_{lcr,K}(D)) \ ,$$

is a quasi-inverse.

This important result of p-adic Hodge theory was first proved in [**CF00**] where a filtered (φ, N)-module over K is said to be *weakly admissible* whenever it is semistable of slope 0.

7. de Rham = potentially log-crystalline

To finish, we explain the main lines of the proof of:

THEOREM 7.1. *Any p-adic representation of G_K, any B_e-representation of G_K or any G_K-equivariant vector bundle over X is de Rham if and only if it is potentially log-crystalline.*

The case of p-adic representations is another important result of p-adic Hodge theory. The first proof was given by Berger [**Ber02**] relying on Crew's conjecture first proved by André [**An02**] and Mebkhout [**Meb02**].

We know that the condition of the theorem is sufficient and it is obviously enough to show that, if $\mathcal{V}$ is a B_e-representation of G_K which is de Rham, then $\mathcal{V}$ is potentially log-crystalline.

We first reduce the proof to the case where k is algebraically closed: Let $\widehat{K}_{nr} \subset C$ the p-adic closure of the maximal unramified extension K_{nr} of K contained in $\overline{K}$. Let $\overline{\widehat{K}}_{nr}$ the algebraic closure of $\widehat{K}_{nr}$. Then $\overline{\widehat{K}}_{nr}$ is stable under the action of the inertia subgroup I_K of G_K. This gives an identification of I_K to the Galois group $\mathrm{Gal}(\overline{\widehat{K}}_{nr}/\widehat{K}_{nr})$.

PROPOSITION 7.2. *Let $\mathcal{V}$ be a B_e-representation of G_K. Then $\mathcal{V}$ is log-crystalline if and only if $\mathcal{V}$ is log-crystalline as a representation of $I_K = \mathrm{Gal}(\overline{\widehat{K}}_{nr}/\widehat{K}_{nr})$.*

Let $\overline{k}$ be the residue field of $\widehat{K}_{nr}$ and $\widehat{K}_{0,nr}$ the fraction field of $W(\overline{k})$. The group $\mathrm{Gal}(\overline{k}/k) = G_K/I_K$ acts semi-linearly on the finite dimensional $\widehat{K}_{0,nr}$ vector space

$$\mathcal{D}_{lcr,nr}(\mathcal{V}) = (B_{lcr} \otimes_{B_e} \mathcal{V})^{I_K}$$

and we have

$$\mathcal{D}_{lcr}(\mathcal{V}) = (D_{lcr,nr}(\mathcal{V}))^{G_k} \ .$$

It is well known that, if n is any positive integer, the pointed set $H^1_{cont}\,(G_k, GL_n(\widehat{K}_{0,nr}))$ is trivial. This implies that the natural map

$$\widehat{K}_{0,nr} \otimes_{K_0} \mathcal{D}_{lcr}(\mathcal{V}) \to \mathcal{D}_{lcr,nr}(\mathcal{V})$$

is an isomorphism. Therefore $\dim_{K_0} \mathcal{D}_{lcr}(\mathcal{V}) = \dim_{\widehat{K}_{0,nr}} \mathcal{D}_{lcr,nr}(\mathcal{V})$.

If r is the rank of $\mathcal{V}$ over B_e, then $\mathcal{V}$ is log-crystalline as a B_e-representation of G_K (resp. I_K) if and only if $\dim_{K_0} \mathcal{D}_{lcr}(\mathcal{V}) = r$ (resp. $\dim_{\widehat{K}_{0,nr}} \mathcal{D}_{lcr,nr}(\mathcal{V}) = r$). The proposition follows.

From now on, we assume k algebraically closed.

Let E be a finite extension of $\mathbb{Q}_p$ and $\tau : E \to K$ a $\mathbb{Q}_p$-embedding. We choose a uniformizing parameter π of E. For $d \in \mathbb{N}$, we consider the 1-dimensional E-representations of G_K

$$E\{d\}_\tau = \mathrm{Symm}^d_E V_C(\Phi_\pi) \text{ and } E\{-d\}_\tau = \text{the } E\text{-dual of } E\{d\}_\tau$$

where $V_C(\Phi_\pi)$ is the 1-dimensional representation associated to the Lubin-Tate formal group Φ_π (§4.2). If we use τ to see E as a closed subfield of C, then $V_C(\Phi_\pi) = E \otimes T_\pi(\Phi_\pi)$ where

$$T_\pi(\Phi_\pi) = \varprojlim_{n\in\mathbb{N}} \Phi_\pi(\mathcal{O}_C)_{\pi^n}$$

is the Tate module of Φ_π.

We say that a E-representation V of G_K is *τ-ordinary* if there is a decreasing filtration $(F^d_\tau V)_{d\in\mathbb{Z}}$ of V by sub-E-vector spaces stable under G_K such that $F^d V_\tau = V$ for $d \ll 0$, $F^d_\tau V = 0$ for $d \gg 0$, each $F^d_\tau V$ is stable under G_K and G_K acts trivially on $(F^d_\tau V/F^{d+1}_\tau V) \otimes_E E\{-d\}_\tau$.

If π' is an other uniformizing parameter of E, then $V_C(\Phi'_\pi)$ and $V_C(\Phi_\pi)$ are isomorphic. Therefore, the condition of being τ-ordinary is independent of the choice of π.

The theorem follows from these three propositions:

Proposition 7.3. *Any B_e-representation $\mathcal{V}$ of G_K which is potentially de Rham (i.e. de Rham as a representation of G_L for a suitably chosen finite extension L of K contained in $\overline{K}$) is de Rham.*

Proposition 7.4. *Let $\tau : E \to \mathcal{K}$ be a $\mathbb{Q}_p$-embedding of a finite extension E of $\mathbb{Q}_p$ into K. Any E-representation of G_K which is τ-ordinary is log-crystalline.*

Proposition 7.5. *Let $\mathcal{V}$ be a B_e-representation of G_K which is de Rham. There exists an integer $h_\mathcal{V} \geq 1$ such that, for any finite extension E of $\mathbb{Q}_p$ of degree divisible by $h_\mathcal{V}$ and any embedding $\tau : E \to \overline{K}$, one can find*

1) a finite extension L of K contained in $\overline{K}$ and containing $\tau(E)$,

2) a τ-ordinary E-representation V of $G_L = \mathrm{Gal}(\overline{K}/L)$,

3) a G_L-equivariant $B_e \otimes_{\mathbb{Q}_p} E$-linear bijection

$$B_e \otimes_{\mathbb{Q}_p} V \simeq E \otimes_{\mathbb{Q}_p} \mathcal{V} \ .$$

The field $\overline{K}$ is naturally embedded into B_{dR} and the proposition 7.3 becomes a formal consequence of the fact that, for any positive integer n, the pointed set $H^1(G_K, GL_n(\overline{K}))$ is trivial.

The proof of the proposition 7.4 relies on some hard computation in Galois cohomology which can be done using the techniques of Herr [**He98**] to compute Galois cohomology by the way of the theory of (ϕ, Γ)-modules [**Fon90**]. This computation has been done by Berger showing a much more general result : any extension of two semi-stable E-representations which is de Rham is semistable (unpublished, see also [**Ber02**], §6).

The proof of the proposition 7.5 runs as follows:

Say that a G_K-equivariant vector bundle $\mathcal{F} = (\mathcal{F}_e, \mathcal{F}^+_{dR})$ is *trivial at* ∞ if it is de Rham and $\mathcal{F}^+_{dR} = B^+_{dR} \otimes_K \mathcal{D}_{dR}(\mathcal{F})$.

To any B_e-representation $\mathcal{W}$ of G_K which is de Rham, setting $\mathcal{D}_{dR}(\mathcal{W}) = (B_{dR} \otimes_{B_e} \mathcal{W})^{G_K}$, one can associate to $\mathcal{W}$ the G_K-equivariant vector bundle

$$\widetilde{\mathcal{W}} = (\mathcal{W}, B_{dR}^+ \otimes_K \mathcal{D}_{dR}(\mathcal{W}))$$

which is trivial at ∞. The correspondence $\mathcal{W} \mapsto \widetilde{\mathcal{W}}$ is a functor inducing a tensor equivalence between the category of de Rham B_e-representations of G_K and G_K-equivariant vector bundles over X which are trivial at ∞.

If $\mathcal{F}$ is any de Rham G_K-equivariant vector bundle over X, then $\widetilde{\mathcal{F}}_e$ is a modification of $\mathcal{F}$ and $\mathcal{F}$ is trivial at ∞ if and only if $\widetilde{\mathcal{F}}_e = \mathcal{F}$.

Let

$$0 = \mathcal{F}_0 \subset \mathcal{F}_1 \subset \ldots \subset \mathcal{F}_{i-1} \subset \mathcal{F}_i \subset \ldots \subset \mathcal{F}_{m-1} \subset \mathcal{F}_m = \widetilde{\mathcal{V}}$$

be the Harder-Narasimhan filtration of $\widetilde{\mathcal{V}}$. By unicity of this filtration, each $\mathcal{F}_i$ is stable under G_K. Setting $\mathcal{V}_i = (\mathcal{F}_i)_e$, we get a decreasing filtration

$$0 = \mathcal{V}_0 \subset \mathcal{V}_1 \subset \ldots \subset \mathcal{V}_{i-1} \subset \mathcal{V}_i \subset \ldots \subset \mathcal{V}_{m-1} \subset \mathcal{V}_m = \mathcal{V}$$

by sub-B_e-representations of G_K. For $1 \leq i \leq m$, $\mathcal{F}_i$ and $\overline{\mathcal{F}}_i = \mathcal{F}_i/\mathcal{F}_{i-1}$ are trivial at ∞ (we have $\mathcal{F}_i = \widetilde{\mathcal{V}}_i$ and $\overline{\mathcal{F}}_i = \widetilde{\overline{\mathcal{V}}}_i$, where $\overline{\mathcal{V}}_i = \mathcal{V}_i/\mathcal{V}_{i-1}$).

Let μ_i be the slope of the semistable vector bundle $\overline{\mathcal{F}}_i$ and let $h_{\mathcal{V}}$ be the smallest positive integer such that

$$h_{\mathcal{V}}.\mu_i \in \mathbb{Z} \ \text{ for } \ 1 \leq i \leq m \ .$$

Let E be a finite extension of $\mathbb{Q}_p$ of degree h divisible by $h_{\mathcal{V}}$, τ a $\mathbb{Q}_p$-embedding of E into $\overline{K}$ and K' a finite extension of K contained in $\overline{K}$ and containing $\tau(E)$. The curve $X_E = X_{F,E}$ is a cyclic étale cover of X of degree h equipped with an action of $G_{K'}$ and the natural morphism $\nu : X_E \to X$ is $G_{K'}$-equivariant.

Choose a uniformizing parameter π of E. For each $d \in \mathbb{Z}$, the line bundle $\mathcal{O}_{X_E}(d)_\pi$ is equipped with an action of $G_{K'}$ and

$$\mathcal{O}_X(d/h)_\pi = \nu_* \mathcal{O}_{X_E}(d)_\pi$$

is a $G_{K'}$-equivariant vector bundle over X which is semistable of slope d/h. For $1 \leq i \leq m$, the $G_{K'}$-equivariant vector bundle

$$\mathcal{G}_i = Hom(\mathcal{O}_X(\mu_i)_\pi, \overline{\mathcal{F}}_i)$$

is semistable of slope 0, hence $W_i = H^0(X, \mathcal{G}_i)$ is a p-adic representation of $G_{K'}$ and $\mathcal{G}_i = \mathcal{O}_X \otimes_{\mathbb{Q}_p} W_i$.

On the other hand, $\mathcal{G}_i = \widetilde{\mathcal{W}_i}$ where $\mathcal{W}_i$ is the de Rham B_e-representation of $G_{K'}$

$$\mathcal{W}_i = \mathcal{L}_{B_e}(\Gamma(X_e, \mathcal{O}_X(\mu_i)_\pi), \mathcal{V}_i) \ ,$$

hence $\mathcal{G}_i$ is trivial at ∞. Therefore, the natural map

$$B_{dR}^+ \otimes_K (B_{dR} \otimes_{\mathbb{Q}_p} W_i)^{G_{K'}} \to B_{dR}^+ \otimes_{\mathbb{Q}_p} W_i$$

is an isomorphism. A fortiori, the natural map

$$C \otimes_K (C \otimes_{\mathbb{Q}_p} W_i)^{G_{K'}}) \to C \otimes_{\mathbb{Q}_p} W_i$$

is an isomorphism (i.e. the p-adic representation W_i of $G_{K'}$ is Hodge-Tate, with all its Hodge-Tate weights equal to 0). A deep result of Sen [**Sen73**] implies that $G_{K'}$ acts on W_i through a finite quotient. Therefore, one can find a finite extension L of K' contained in $\overline{K}$ such that G_L acts trivially on each W_i. One easily checks that it implies the existence of a positive integer r_i and of an isomorphism of G_L-equivariant vector bundles

$$f_i : (\mathcal{O}_X(\mu_i)_\pi)^{r_i} \to E \otimes_{\mathbb{Q}_p} \overline{\mathcal{F}}_i \ .$$

For all $d \in \mathbb{Z}$, there is a canonical isomorphism

$$(\mathcal{O}_X(d/h)_\pi)_e \simeq B_e \otimes_{\mathbb{Q}_p} E\{d\}_\pi$$

and therefore, for $1 \leq i \leq m$, if $\mu_i = d_i/h$, we get a G_L-equivariant $B_e \otimes_{\mathbb{Q}_p}$-linear bijection

$$B_e \otimes_{\mathbb{Q}_p} (E\{d_i\})^{r_i} \simeq E \otimes_{\mathbb{Q}_p} \overline{\mathcal{V}}_i \ .$$

In particular, this concludes the proof when $m = 1$. Assume $m \geq 2$. By induction, we may assume there is a τ-ordinary representation V' of G_L and a G_L-equivariant $B_e \otimes_{\mathbb{Q}_p}$-linear bijection

$$B_e \otimes_{\mathbb{Q}_p} V' \simeq E \otimes_{\mathbb{Q}_p} \mathcal{V}_{m-1} \ .$$

Set $B_{e,E} = B_e \otimes_{\mathbb{Q}_p} E$. We get an exact sequence of $B_{e,E}$-representations of G_L

$$0 \to B_{e,E} \otimes_E V' \to E \otimes_{\mathbb{Q}_p} \mathcal{V} \to B_{e,E} \otimes_E (E\{d_m\})^{r_m} \to 0 \ .$$

Twisting by $E\{-d_m\}$, we are reduced to show, that, if we have a short exact sequence of $B_{e,E}$-representations of G_L

$$(*) \qquad 0 \to B_{e,E} \otimes_E W' \to \mathcal{W} \to B_{e,E} \to 0$$

with W' a τ-ordinary E-representation of G_L, then $\mathcal{W}$ comes by scalar extension from an E-representation of G_L which is an extension of E by W'. Setting

$$B_{dR,E} = E \otimes_{\mathbb{Q}_p} B_{dR} \ , \ B^+_{dR,E} = E \otimes_{\mathbb{Q}_p} B^+_{dR} \text{ and } \widetilde{B}_{dR,E} = B_{dR,E}/B^+_{dR,E} \ ,$$

we get from the fundamental exact sequence (§6.3), a short exact sequence

$$0 \to E \to B_{e,E} \to \widetilde{B}_{dR,E} \to 0 \ .$$

Tensoring with W', we get an exact sequence

$$0 \to W' \to B_{e,E} \otimes_E W' \to \widetilde{B}_{dR,E} \otimes_E W' \to 0 \ ,$$

inducing an exact sequence of continuous G_L-cohomology

$$\begin{aligned} \ldots \to H^1_{cont}(G_L, W') &\to H^1_{cont}(G_L, B_{e,E} \otimes_E W') \\ &\to H^1_{cont}(G_L, \widetilde{B}_{dR,E} \otimes_E W') \to \ldots \end{aligned}$$

The short exact sequence $(*)$ defines an element $c \in H^1_{cont}(G_L, B_{e,E} \otimes_E W')$. What we need to show is that c comes from an element of $H^1_{cont}(G_L, W')$ or equivalently goes to 0 in $H^1_{cont}(G_L, \tilde{B}_{dR,E} \otimes W')$. The map

$$H^1_{cont}(G_L, B_{e,E} \otimes_E W') \to H^1_{cont}(G_L, \widetilde{B}_{dR,E} \otimes_E W')$$

factors through $H^1_{cont}(G_L, B_{dR,E} \otimes_E W')$ and this comes from the fact that the extension is de Rham which means that the image of c is already 0 in $H^1_{cont}(G_L, B_{dR,E} \otimes_E W')$.

Remark. Let $\mathcal{F}$ a de Rham G_K equivariant vector bundle over X. Choose a finite Galois extension L of K contained in $\overline{K}$ such that $\mathcal{F}$ is log-crystalline as a G_L-vector bundle. Then the (φ, N) module over L

$$\mathcal{D}_{lcr,L}(\mathcal{F})$$

is equipped with an action of $G_{L/K}$ defined in an obvious way. This give to $\mathcal{D}_{lcr,L}(\mathcal{F})$ the structure of what can be called a *filtered* $(\varphi, N, G_{L/K})$*-module over* K. The inductive limit (in a straightforward way) of the categories of filtered $(\varphi, N, G_{L/K})$-modules over K, when L runs through all the finite Galois extensions of K contained in $\overline{K}$, is the category

$$MF_K(\varphi, N, G_K)$$

of *filtered* (φ, N, G_K)*-modules over* K. This is, in an obvious way, a $\mathbb{Q}_p$-linear tensor category, with an obvious definition of the rank, the degree and the slope of any non-zero object. The Harder-Narasimhan filtration of any object can be defined.

We see that the $\mathcal{D}_{lcr,L}$'s induce a tensor equivalence of categories

$$\text{de Rham } G_K\text{-equivariant vector bundles over } X \iff Mod_K(\varphi, N, G)$$

respecting rank, degree, slopes and the Harder-Narasimhan filtration.

The restriction of this equivalence to semistable vector bundles of slope 0 leads to the “classical” equivalence ([**Fon94b**], [**Ber02**]) of categories between de Rham p-adic representations of G_K and “weakly admissible” (or semistable of slope 0) filtered (φ, N, G_K)-modules over K.

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