VECTOR BUNDLES ON CURVES AND p-ADIC HODGE THEORY

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Contents

Introduction

This text is an introduction to our work [12] on curves and vector bundles in p-adic Hodge theory. This is a more elaborate version of the reports [13] and [14]. We give a detailed construction of the "fundamental curve of p-adic Hodge theory" together with sketches of proofs of the main properties of the objects showing up in the theory. Moreover, we explain thoroughly the classification theorem for vector bundles on the curve, giving a complete proof for rank two vector bundles. The applications to p-adic Hodge theory, the theorem "weakly admissible implies admissible" and the p-adic monodromy theorem, are not given here but can be found in [13] and [14].

We would like to thank the organizers of the EPSRC Durham Symposium "Automorphic forms and Galois representations" for giving us the opportunity to talk about this subject.

1. Holomorphic functions of the variable π

1.1. Background on holomorphic functions in a p-adic punctured disk after Lazard ([25]).

1.1.1. *The Frechet algebra* B. Let F be a complete non-archimedean field for a non trivial valuation

$$v : F \longrightarrow \mathbb{R} \cup \{+\infty\},$$

with characteristic p residue field. We note $|\cdot| = p^{-v(\cdot)}$ the associated absolute value. Consider the open punctured disk

$$\mathbb{D}^* = \{0 < |z| < 1\} \subset \mathbb{A}^1_F$$

as a rigid analytic space over F, where z is the coordinate on the affine line. If $I \subset]0,1[$ is a compact interval set

$$\mathbb{D}_I = \{|z| \in I\} \subset \mathbb{D}^*,$$

an annulus that is an affinoïd domain in $\mathbb{D}^*$ if $I = [\rho_1, \rho_2]$ with $\rho_1, \rho_2 \in \sqrt{|F^\times|}$. There is an admissible affinoïd covering

$$\mathbb{D}^* = \bigcup_{I \subset]0,1[} \mathbb{D}_I$$

The first author acknowledges support from ANR-10-BLAN-0114 "ArShiFo".

where I goes through the preceding type of compact intervals.

Set now

$$\begin{aligned} \mathrm{B} &= \mathcal{O}(\mathbb{D}^*) \\ &= \Big\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in F,\ \forall \rho \in]0,1[\lim_{|n|\to+\infty} |a_n|\rho^n = 0 \Big\} \end{aligned}$$

the ring of holomorphic functions on $\mathbb{D}^*$. In the preceding description of B, one checks the infinite set of convergence conditions associated to each $\rho \in]0,1[$ can be rephrased in the following two conditions

$$(1) \qquad \begin{cases} \liminf\limits_{n\to+\infty} \frac{v(a_n)}{n} & \geq 0 \\ \lim\limits_{n\to+\infty} \frac{v(a_{-n})}{n} & = +\infty. \end{cases}$$

For $\rho \in]0,1[$ and $f = \sum_n a_n z^n \in \mathrm{B}$ set

$$|f|_\rho = \sup_{n\in\mathbb{Z}} \{|a_n|\rho^n\}.$$

If $\rho = p^{-r}$ with $r > 0$ one has $|f|_\rho = p^{-v_r(f)}$ with

$$v_r(f) = \inf_{n\in\mathbb{Z}} \{v(a_n) + nr\}.$$

Then $|\cdot|_\rho$ is the Gauss supremum norm on the annulus $\{|z| = \rho\}$. It is in fact a *multiplicative* norm, that is to say v_r *is a valuation*. Equipped with the set of norms $(|\cdot|_\rho)_{\rho\in]0,1[}$, B is a Frechet algebra. The induced topology is the one of uniform convergence on compact subsets of the Berkovich space associated to $\mathbb{D}^*$. If $I \subset]0,1[$ is a compact interval then

$$\mathrm{B}_I = \mathcal{O}(\mathbb{D}_I)$$

equipped with the set of norms $(|\cdot|_\rho)_{\rho\in I}$ is a Banach algebra. In fact if $I = [\rho_1, \rho_2]$ then by the maximum modulus principle, for $f \in \mathrm{B}_I$

$$\sup_{\rho\in I} |f_\rho| = \sup\{|f|_{\rho_1}, |f|_{\rho_2}\}.$$

One then has

$$\mathrm{B} = \varprojlim_{I\subset]0,1[} \mathrm{B}_I$$

as a Frechet algebra written as a projective limit of Banach algebras.

For $f = \sum_n a_n z^n \in \mathrm{B}$ one has

$$|f|_1 = \lim_{\rho\to1} |f|_\rho = \sup_{n\in\mathbb{Z}} |a_n| \in [0, +\infty].$$

We will later consider the following closed sub-$\mathcal{O}_F$-algebra of B

$$\begin{aligned} \mathrm{B}^+ &= \{f \in \mathrm{B} \mid \|f\|_1 \leq 1\} \\ &= \Big\{ \sum_{n\in\mathbb{Z}} a_n z^n \in \mathrm{B} \mid a_n \in \mathcal{O}_F \Big\} \\ &= \Big\{ \sum_{n\in\mathbb{Z}} a_n z^n \mid a_n \in \mathcal{O}_F,\ \lim_{n\to+\infty} \frac{v(a_{-n})}{n} = +\infty \Big\}. \end{aligned}$$

Set now

$$\begin{aligned} \mathrm{B}^b &= \{f \in \mathrm{B} \mid \exists N \in \mathbb{N},\ z^N f \in \mathcal{O}(\mathbb{D}) \text{ and is bounded on } \mathbb{D}\} \\ &= \Big\{ \sum_{n\gg-\infty} a_n z^n \mid a_n \in F,\ \exists C\ \forall n\ |a_n| \leq C \Big\}. \end{aligned}$$

This is a dense sub-algebra of B. In particular *one can find back* B *from* B^b *via completion with respect to the norms* $(|\cdot|_\rho)_{\rho\in]0,1[}$. In the same way

$$\begin{aligned}\mathrm{B}^{b,+} &= \mathrm{B}^b \cap \mathrm{B}^+ \\ &= \Big\{ \sum_{n\gg-\infty} a_n z^n \mid a_n \in \mathcal{O}_F \Big\} \\ &= \mathcal{O}_F[\![z]\!][\tfrac{1}{z}]\end{aligned}$$

is dense in B^+ and thus *one can find back* B^+ *from* $\mathrm{B}^{b,+}$ *via completion.*

1.1.2. *Zeros and growth of holomorphic functions.* Recall the following properties of holomorphic functions overs $\mathbb{C}$. Let f be holomorphic on the open disk of radius 1 (one could consider the punctured disk but we restrict to this case to simplify the exposition). For $\rho \in [0,1[$ set

$$M(\rho) = \sup_{|z|=\rho} |f(z)|.$$

The following properties are verified:

- the function $\rho \mapsto -\log M(\rho)$ is a concave function of $\log\rho$ (Hadamard),
- if $f(0)\neq 0$, f has no zeros on the circle of radius ρ and $(a_1,\dots,a_n)$ are its zeros counted with multiplicity in the disk of radius ρ, then as a consequence of Jensen's formula

$$-\log|f(0)| \geq \sum_{i=1}^n (-\log|a_i|) - n\rho - \log M(\rho).$$

In the non-archimedean setting we have an exact formula linking the growth of an holomorphic function and its zeros. For this, recall the formalism of the Legendre transform. Let

$$\varphi : \mathbb{R} \longrightarrow]-\infty,+\infty]$$

be a convex decreasing function. Define the Legendre transform of φ as the concave function (see figure 1)

$$\begin{aligned}\mathscr{L}(\varphi) :]0,+\infty[&\longrightarrow [-\infty,+\infty[\\ \lambda &\longmapsto \inf_{x\in\mathbb{R}}\{\varphi(x)+\lambda x\}.\end{aligned}$$

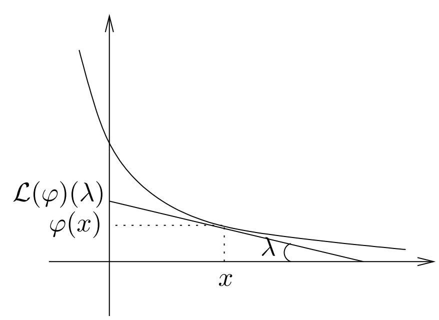

FIGURE 1. The Legendre transform of φ evaluated at the slope λ where by definition the slope is the opposite of the derivative (we want the slopes to be the valuations of the roots for Newton polygons).

If φ_1,φ_2 are convex decreasing functions as before define $\varphi_1 * \varphi_2$ as the convex decreasing function defined by

$$(\varphi_1 * \varphi_2)(x) = \inf_{a+b=x}\{\varphi_1(a)+\varphi_2(b)\}.$$

We have the formula

$$\mathscr{L}(\varphi_1 * \varphi_2) = \mathscr{L}(\varphi_1) + \mathscr{L}(\varphi_2).$$

One can think of the Legendre transform as being a "tropicalization" of the Laplace transform:

$$(\mathbb{R},+,\times) \xrightarrow{\text{tropicalization}} (\mathbb{R},\inf,+)$$
$$\text{Laplace transform} \rightsquigarrow \text{Legendre transform}$$
$$\text{usual convolution} * \rightsquigarrow \text{tropical } * \text{ just defined}$$

The function φ is a polygon, that is to say piecewise linear, if and only if $\mathscr{L}(\varphi)$ is a polygon. Moreover in this case:

- the slopes of $\mathscr{L}(\varphi)$ are the x-coordinates of the breakpoints of φ,
- the x-coordinates of the breakpoints of $\mathscr{L}(\varphi)$ are the slopes of φ.

Thus $\mathscr{L}$ and its inverse give a duality

$$\text{slopes} \longleftrightarrow x\text{-coordinates of breakpoints.}$$

From these considerations one deduces that if φ_1 and φ_2 are convex decreasing polygons such that $\forall i = 1,2,\ \forall \lambda \in]0,+\infty[,\ \mathscr{L}(\varphi_i)(\lambda) \neq -\infty$ then the slopes of $\varphi_1 * \varphi_2$ are obtained by concatenation from the slopes of φ_1 and the ones of φ_2.

For $f = \sum_{n\in\mathbb{Z}} a_n z^n \in \mathrm{B}$ set now

$$\mathrm{Newt(f)} = \text{decreasing convex hull of } \{(\mathrm{n}, \mathrm{v}(\mathrm{a_n}))\}_{\mathrm{n}\in\mathbb{Z}}.$$

This is a polygon with integral x-coordinate breakpoints. Moreover the function $r \mapsto v_r(f)$ defined on $]0,+\infty[$ is the Legendre transform of Newt(f).

Then, the statement of the "p-adic Jensen formula" is the following: the slopes of Newt(f) are the valuations of the zeros of f (with multiplicity).

Example 1.1. *Take $f \in \mathcal{O}(\mathbb{D})$, $f(0) \neq 0$. Let $\rho = p^{-r} \in]0,1[$ and $(a_1,\dots,a_n)$ be the zeros of f in the ball $\{|z| \leq \rho\}$ counted with multiplicity. Then, as a consequence of the fact that $r \mapsto v_r(f)$ is the Legendre transform of a polygon whose slopes are the valuations of the roots of f, one has the formula (see figure 2)*

$$v(f(0)) = v_r(f) - nr + \sum_{i=1}^{n} v(a_i).$$

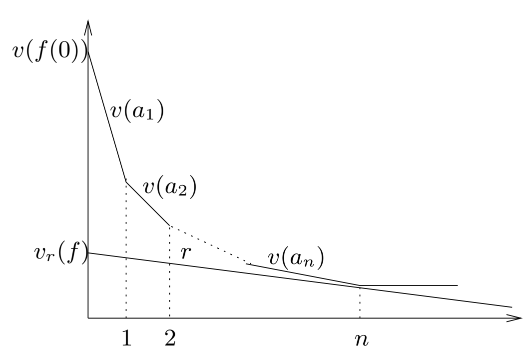

FIGURE 2. An illustration of the "p-adic Jensen formula". The numbers over each line are their slopes.

Finally, remark that B^+ is characterized in terms of Newton polygons:

$$\mathrm{B}^+ = \{f \in \mathrm{B} \mid \mathrm{Newt(f)} \subset \text{upper half plane } \mathrm{y} \geq 0\}.$$

1.1.3. *Weierstrass products.* For a compact interval $I = [\rho_1, \rho_2] \subset]0,1[$ with $\rho_1, \rho_2 \in \sqrt{|F^\times|}$ the ring B_I is a P.I.D. with $\mathrm{Spm}(\mathrm{B}_I) = |\mathbb{D}_I|$. In particular $\mathrm{Pic}(\mathbb{D}_I) = 0$. Now let's look at $\mathrm{Pic}(\mathbb{D}^*)$. In fact in the following we will only be interested in the submonoid of effective line bundles

$$\mathrm{Pic}^+(\mathbb{D}^*) = \{[\mathscr{L}] \in \mathrm{Pic}(\mathbb{D}^*) \mid H^0(\mathbb{D}^*, \mathscr{L}) \neq 0\}.$$

Set

$$\mathrm{Div}^+(\mathbb{D}^*) = \Big\{ D = \sum_{x \in |\mathbb{D}^*|} m_x[x] \mid m_x \in \mathbb{N},\ \forall I \subset]0,1[\text{ compact } \ \mathrm{supp}(D) \cap \mathbb{D}_I \text{ is finite} \Big\}$$

the monoid of effective divisors on $\mathbb{D}^*$. There is an exact sequence

$$0 \to \mathrm{B} \setminus \{0\}/\mathrm{B}^\times \xrightarrow{\mathrm{div}} \mathrm{Div}^+(\mathbb{D}^*) \longrightarrow \mathrm{Pic}^+(\mathbb{D}^*) \longrightarrow 0.$$

We are thus led to the question: for $D \in \mathrm{Div}^+(\mathbb{D}^*)$, does there exist $f \in \mathrm{B} \setminus \{0\}$ such that $\mathrm{div}(f) = D$?

This is of course the case if $\mathrm{supp}(D)$ is finite. Suppose thus it is infinite. We will suppose moreover F is algebraically closed (the discrete valuation case is easier but this is not the case we are interested in) and thus $|\mathbb{D}^*| = \mathfrak{m}_F \setminus \{0\}$ where $\mathfrak{m}_F$ is the maximal ideal of $\mathcal{O}_F$.

Suppose first there exists $\rho_0 \in]0,1[$ such that $\mathrm{supp}(D) \subset \{0 < |z| \leq \rho_0\}$. Then we can write

$$D = \sum_{i \geq 0} [a_i],\ a_i \in \mathfrak{m}_F \setminus \{0\},\ \lim_{i \to +\infty} |a_i| = 0.$$

The infinite product

$$\prod_{i=0}^{+\infty} \left(1 - \frac{a_i}{z}\right),$$

converges in the Frechet algebra B and its divisor is D.

We are thus reduced to the case $\mathrm{supp}(D) \subset \{\rho_0 < |z| < 1\}$ for some $\rho_0 \in]0,1[$. But if we write

$$D = \sum_{i \geq 0} [a_i],\ \lim_{i \to +\infty} |a_i| = 1$$

then neither of the infinite products "$\prod_{i \geq 0} \left(1 - \frac{a_i}{z}\right)$" or "$\prod_{i \geq 0} \left(1 - \frac{z}{a_i}\right)$" converges. Recall that over $\mathbb{C}$ this type of problem is solved by introducing renormalization factors. Typically, if we are looking for a holomorphic function f on $\mathbb{C}$ such that $\mathrm{div}(f) = \sum_{n \in \mathbb{N}} [-n]$ then the product "$z \prod_{n \in \mathbb{N}} \left(1 + \frac{z}{n}\right)$" does not converge but $z \prod_{n \in \mathbb{N}} \left[\left(1 + \frac{z}{n}\right) e^{-\frac{z}{n}}\right] = \frac{1}{e^{\gamma z} \Gamma(z)}$ does. In our non archimedean setting this problem has been solved by Lazard.

Theorem **1.2** (Lazard [25])**.** *If F is spherically complete then there exists a sequence $(h_i)_{i \geq 0}$ of elements of* $\mathrm{B}^\times$ *such that the product $\prod_{i \geq 0} [(z - a_i).h_i]$ converges.*

Thus, if F is spherically complete $\mathrm{Pic}^+(\mathbb{D}^*) = 0$ (and in fact $\mathrm{Pic}(\mathbb{D}^*) = 0$). In the preceding problem, it is easy to verify that for any F there always exist renormalization factors $h_i \in \mathrm{B} \setminus \{0\}$ such that $\prod_{i \geq 0} [(z - a_i).h_i]$ converges and thus an $f \in \mathrm{B} \setminus \{0\}$ such that $\mathrm{div}(f) \geq D$. The difficulty is thus to introduce renormalization factors that do not add any new zero.

Let's conclude this section with a trick that sometimes allows us not to introduce any renormalization factors. Over $\mathbb{C}$ this is the following. Suppose we are looking for a holomorphic function on $\mathbb{C}$ whose divisor is $\sum_{n \in \mathbb{Z}} [n]$. The infinite product "$z \prod_{n \in \mathbb{Z} \setminus \{0\}} \left(1 - \frac{z}{n}\right)$" does not converge. Nevertheless, regrouping the terms, the infinite product $z \prod_{n \geq 1} \left[\left(1 - \frac{z}{n}\right)\left(1 + \frac{z}{n}\right)\right] = \frac{\sin \pi z}{\pi}$ converges. In the non archimedean setting there is a case where this trick works. This is the following. Suppose $E|\mathbb{Q}_p$ is a finite extension and $\overline{E}$ is an algebraic closure of E. Let $\mathcal{LT}$ be a Lubin-Tate group law over $\mathcal{O}_E$. Its logarithm $\log_{\mathcal{LT}}$ is a rigid analytic function on the open disk $\mathbb{D}$ with zeros the torsion points of the Lubin-Tate group law,

$$\mathcal{LT}[\pi^\infty] = \{x \in \mathfrak{m}_{\overline{E}} \mid \exists n \geq 1,\ [\pi^n]_{\mathcal{LT}}(x) = 0\}.$$

The infinite product

$$z\prod_{\zeta\in\mathcal{LT}[\pi^\infty]\setminus\{0\}}\Big(1-\frac{z}{\zeta}\Big)$$

does not converge since in this formula $|\zeta|\to 1$. Nevertheless, the infinite product

$$z\prod_{n\geq 1}\Big[\prod_{\zeta\in\mathcal{LT}[\pi^n]\setminus\mathcal{LT}[\pi^{n-1}]}\Big(1-\frac{z}{\zeta}\Big)\Big]$$

converges in the Frechet algebra of holomorphic functions on $\mathbb{D}$ and equals $\log_{\mathcal{LT}}$. This is just a reformulation of the classical formula

$$\log_{\mathcal{LT}}=\lim_{n\to+\infty}\pi^{-n}[\pi^n]_{\mathcal{LT}}.$$

1.2. **Analytic functions in mixed characteristic.**

1.2.1. *The rings* B *and* B^+. Let E be a local field with uniformizing element π and finite residue field $\mathbb{F}_q$. Thus, either E is a finite extension of $\mathbb{Q}_p$ or $E=\mathbb{F}_q((\pi))$. Let $F|\mathbb{F}_q$ be a valued complete extension for a non trivial valuation $v:F\to\mathbb{R}\cup\{+\infty\}$. Suppose moreover F is perfect (in particular v is not discrete).

Let $\mathscr{E}|E$ be the unique complete unramified extension of E inducing the extension $F|\mathbb{F}_q$ on the residue fields, $\mathcal{O}_{\mathscr{E}}/\pi\mathcal{O}_{\mathscr{E}}=F$. There is a Teichmüller lift $[\cdot]:F\to\mathcal{O}_{\mathscr{E}}$ and

$$\mathscr{E}=\Big\{\sum_{n\gg-\infty}[x_n]\pi^n\mid x_n\in F\Big\}\ \text{(unique writing)}.$$

If $\mathrm{char}E=p$ then $[\cdot]$ is additive, $\mathscr{E}|F$, and $\mathscr{E}=F((\pi))$. If $E|\mathbb{Q}_p$ then

$$\mathscr{E}=W_{\mathcal{O}_E}(F)[\tfrac{1}{\pi}]=W(F)\otimes_{W(\mathbb{F}_q)}E$$

the ramified Witt vectors of F. There is a Frobenius φ acting on $\mathscr{E}$,

$$\varphi\Big(\sum_n[x_n]\pi^n\Big)=\sum_n[x_n^q]\pi^n.$$

If $E|\mathbb{Q}_p$ then on $W(F)\otimes_{W(\mathbb{F}_q)}E$ one has $\varphi=\varphi_{\mathbb{Q}_p}^f\otimes\mathrm{Id}$ where in this formula $q=p^f$ and $\varphi_{\mathbb{Q}_p}$ is the usual Frobenius of the Witt vectors. In this case the addition law of $W_{\mathcal{O}_E}(F)$ is given by

$$\sum_{n\geq 0}[x_n]\pi^n+\sum_{n\geq 0}[y_n]\pi^n=\sum_{n\geq 0}[P_n(x_0,\dots,x_n,y_0,\dots,y_n)]\pi^n \tag{2}$$

where $P_n\in\mathbb{F}_q\big[X_i^{q^{i-n}},Y_j^{q^{j-n}}\big]_{0\leq i,j\leq n}$ are generalized polynomials. The multiplication law is given in the same way by such kind of generalized polynomials.

Definition 1.3.

(1) Define

$$\begin{aligned}\mathrm{B}^b&=\Big\{\sum_{n\gg-\infty}[x_n]\pi^n\in\mathscr{E}\mid\exists C,\ \forall n\ |x_n|\leq C\Big\}\\ \mathrm{B}^{b,+}&=\Big\{\sum_{n\gg-\infty}[x_n]\pi^n\in\mathscr{E}\mid x_n\in\mathcal{O}_F\Big\}\\ &=W_{\mathcal{O}_E}(\mathcal{O}_F)\big[\tfrac{1}{\pi}\big]\ \textit{if}\ E|\mathbb{Q}_p\\ &=\mathcal{O}_F[\![\pi]\!][\tfrac{1}{\pi}]\ \textit{if}\ E=\mathbb{F}_q((\pi)).\end{aligned}$$

(2) For $x=\sum_n[x_n]\pi^n\in\mathrm{B}^b$ and $r\geq 0$ set

$$v_r(x)=\inf_{n\in\mathbb{Z}}\{v(x_n)+nr\}.$$

If $\rho=q^{-r}\in]0,1]$ set $|x|_\rho=q^{-v_r(x)}$.

(3) For $x = \sum_n [x_n]\pi^n \in \mathrm{B}^b$ *set*

$$\mathrm{Newt(x)} = \textit{decreasing convex hull of } \{(\mathrm{n}, \mathrm{v(x_n)})\}_{\mathrm{n}\in\mathbb{Z}}.$$

In the preceding definition one can check that the function v_r does not depend on the choice of a uniformizing element π. In the equal characteristic case, that is to say $E = \mathbb{F}_q((\pi))$, setting $z = \pi$ one finds back the rings defined in section 1.1.

For $x \in \mathrm{B}^b$ the function $r \mapsto v_r(x)$ defined on $]0, +\infty[$ is the Legendre transform of Newt(x). One has $v_0(x) = \lim_{r\to 0} v_r(x)$. The Newton polygon of x is $+\infty$ exactly on $]-\infty, v_\pi(x)[$ and moreover $\lim_{+\infty} \mathrm{Newt(x)} = \mathrm{v_0(x)}$. One has to be careful that since the valuation of F is not discrete, this limit is not always reached, that is to say Newt(x) may have an infinite number of strictly positive slopes going to zero. One key point is the following proposition whose proof is not very difficult but needs some work.

Proposition **1.4.** *For* $r \geq 0$*,* v_r *is a valuation on* B^b*.*

Thus, for all $\rho \in]0, 1]$, $|\cdot|_\rho$ is a multiplicative norm. One deduces from this that for all $x, y \in \mathrm{B}^b$,

$$\mathrm{Newt(xy)} = \mathrm{Newt(x)} * \mathrm{Newt(y)}$$

(see 1.1.2). In particular the slopes of Newt(xy) are obtained by concatenation from the slopes of Newt(x) and the ones of Newt(y). For example, as a consequence, if $a_1, \ldots, a_n \in \mathfrak{m}_F \setminus \{0\}$, then

$$\mathrm{Newt}\big((\pi - [\mathrm{a_1}]) \ldots (\pi - [\mathrm{a_n}])\big)$$

is $+\infty$ on $]-\infty, 0[$, 0 on $[n, +\infty[$ and has non-zero slopes $v(a_1), \ldots, v(a_n)$.

Definition 1.5. *Define*

- $\mathrm{B} =$ *completion of* B^b *with respect to* $(|\cdot|_\rho)_{\rho\in]0,1[}$*,*
- $\mathrm{B}^+ =$ *completion of* $\mathrm{B}^{b,+}$ *with respect to* $(|\cdot|_\rho)_{\rho\in]0,1[}$*,*
- *for* $I \subset]0, 1[$ *a compact interval* $\mathrm{B}_I =$ *completion of* B^b *with respect to* $(|\cdot|_\rho)_{\rho\in I}$*.*

The rings B and B^+ are E-Frechet algebras and B^+ is the closure of $\mathrm{B}^{b,+}$ in B. Moreover, if $I = [\rho_1, \rho_2] \subset]0, 1[$, for all $f \in \mathrm{B}$

$$\sup_{\rho\in I} |f|_\rho = \sup\{|f|_{\rho_1}, |f|_{\rho_2}\}$$

because the function $r \mapsto v_r(f)$ is concave. Thus, B_I is an E-Banach algebra. As a consequence, the formula

$$\mathrm{B} = \varprojlim_{I\subset]0,1[} \mathrm{B}_I$$

expresses the Frechet algebra B as a projective limit of Banach algebras.

Remark 1.6. *Of course, the preceding rings are not new and appeared for example under different names in the work of Berger (*[2]*) and Kedlaya (*[21]*). The new point of view here is to see them as rings of holomorphic functions of the variable* π*. In particular, the fact that* v_r *is a valuation (prop. 1.4) had never been noticed before.*

The Frobenius φ extends by continuity to automorphisms of B and B^+, and for $[\rho_1, \rho_2] \subset]0, 1[$ to an isomorphism $\varphi : \mathrm{B}_{[\rho_1,\rho_2]} \xrightarrow{\sim} \mathrm{B}_{[\rho_1^q,\rho_2^q]}$.

Remark 1.7. *In the case* $E = \mathbb{F}_q((\pi))$*, setting* $z = \pi$ *as in section 1.1, the Frobenius* φ *just defined is given by* $\varphi\big(\sum_n x_n z^n\big) = \sum_n x_n^q z^n$*. This is thus an arithmetic Frobenius, the geometric one being* $\sum_n x_n z^n \mapsto \sum_n x_n z^{qn}$*.*

The ring B^+ satisfies a particular property. In fact, if $x \in \mathrm{B}^{b,+}$ and $r \geq r' > 0$ then

$$v_{r'}(x) \geq \frac{r'}{r}\, v_r(x). \tag{3}$$

Thus, if $0 < \rho \leq \rho' < 1$ we have

$$\mathrm{B}^+_{[\rho,\rho']} = \mathrm{B}^+_\rho \subset \mathrm{B}^+_{\rho'}$$

where for a compact interval $I \subset]0,1[$ we note B^+_I for the completion of $\mathrm{B}^{b,+}$ with respect to the $(|\cdot|_\rho)_{\rho\in I}$, and $\mathrm{B}^+_\rho := \mathrm{B}^+_{\{\rho\}}$. One deduces that for any $\rho_0 \in]0,1[$, $\mathrm{B}^+_{\rho_0}$ is stable under φ and

$$\mathrm{B}^+ = \bigcap_{n\geq 0} \varphi^n(\mathrm{B}^+_{\rho_0})$$

the biggest sub-algebra of $\mathrm{B}^+_{\rho_0}$ on which φ is bijective.

Suppose $E = \mathbb{Q}_p$ and choose $\rho \in |F^\times| \cap]0,1[$. Let $a \in F$ such that $|a| = \rho$. Define

$$\mathrm{B}^+_{cris,\rho} = \big(p\text{-adic completion of the P.D. hull of the ideal } W(\mathcal{O}_F)[a] \text{ of } W(\mathcal{O}_F)\ \big) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p.$$

This depends only on ρ since $W(\mathcal{O}_F)[a] = \{x \in W(\mathcal{O}_F) \mid |x|_0 \geq \rho\}$. We thus have

$$\mathrm{B}^+_{cris,\rho} = \widehat{\Big(W(\mathcal{O}_F)\Big[\tfrac{[a^n]}{n!}\Big]_{n\geq 1}\Big)}\Big[\frac{1}{p}\Big].$$

One has

$$\mathrm{B}^+_{\rho^p} \subset \mathrm{B}^+_{cris,\rho} \subset \mathrm{B}^+_{\rho^{p-1}}.$$

From this one deduces

$$\mathrm{B}^+ = \bigcap_{n\geq 0} \varphi^n(\mathrm{B}^+_{cris,\rho}),$$

the ring usually denoted "B^+_{rig}" in p-adic Hodge theory. The ring $\mathrm{B}^+_{cris,\rho}$ appears naturally in comparison theorems where it has a natural interpretation in terms of crystalline cohomology. But the structure of the ring B^+ is simpler as we will see. Moreover, if $k \subset \mathcal{O}_F$ is a perfect sub-field, $K_0 = W(k)_\mathbb{Q}$ and (D,φ) a k-isocrystal, then

$$\big(D \otimes_{K_0} \mathrm{B}^+_{cris,\rho}\big)^{\varphi=\mathrm{Id}} = \big(D \otimes_{K_0} \mathrm{B}^+\big)^{\varphi=\mathrm{Id}}$$

because φ is bijective on D. Replacing $\mathrm{B}^+_{cris,\rho}$ by B^+ is thus harmless most of the time.

Remark 1.8. *Suppose $E|\mathbb{Q}_p$ and let $(x_n)_{n\in\mathbb{Z}}$ be a sequence of $\mathcal{O}_F$ such that $\lim\limits_{n\to+\infty} \frac{v(x_{-n})}{n} = +\infty$. Then, the series*

$$\sum_{n\in\mathbb{Z}} [x_n]\pi^n$$

converges in B^+. But:

- *we don't know if each element of B^+ is of this form,*
- *for such an element of B^+ we don't know if such a writing is unique,*
- *we don't know if the sum or product of two element of this form is again of this form.*

The same remark applies to B.

Nevertheless, there is a sub E-vector space of B^+ where the preceding remark does not apply. One can define for any $\mathcal{O}_E$-algebra R the group of (ramified) Witt bivectors $\mathrm{B}W_{\mathcal{O}_E}(R)$. Elements of $\mathrm{B}W_{\mathcal{O}_E}(R)$ have a Teichmüller expansion that is infinite on the left and on the right. One has

$$\begin{aligned} \mathrm{B}W_{\mathcal{O}_E}(\mathcal{O}_F) &:= \varprojlim_{\substack{\mathfrak{a}\subset\mathcal{O}_F \\ \text{non zero ideal}}} \mathrm{B}W_{\mathcal{O}_E}(\mathcal{O}_F/\mathfrak{a}) \\ &= \Big\{ \sum_{n\in\mathbb{Z}} V_\pi^n[x_n] \mid x_n \in \mathcal{O}_F,\ \liminf_{n\to-\infty} v(x_n) > 0 \Big\} \\ &= \Big\{ \sum_{n\in\mathbb{Z}} [y_n]\pi^n \mid y_n \in \mathcal{O}_F,\ \liminf_{n\to-\infty} q^n v(x_n) > 0 \Big\} \subset \mathrm{B}^+. \end{aligned}$$

The point is that in $\mathrm{B}W_{\mathcal{O}_E}(\mathcal{O}_F)$

$$\sum_{n\in\mathbb{Z}} [x_n]\pi^n + \sum_{n\in\mathbb{Z}} [y_n]\pi^n = \sum_{n\in\mathbb{Z}} \Big[\lim_{k\to+\infty} P_k(x_{n-k},\ldots,x_n,y_{n-k},\ldots,y_n)\Big]\pi^n$$

where the generalized polynomials

$$P_k \in \mathbb{F}_q\big[X_i^{q^{i-k}}, Y_j^{q^{j-k}}\big]_{1\leq i,j\leq k}$$

give the addition law of the Witt vectors as in formula (2) and the limits in the preceding formulas exist thanks to [15], prop.1.1 chap. II, and the convergence condition appearing in the definition of the bivectors.

In fact, periods of π-divisible $\mathcal{O}_E$-modules (that is to say p-divisible groups when $E = \mathbb{Q}_p$) lie in $\mathrm{B}W_{\mathcal{O}_E}(\mathcal{O}_F)$, and $\mathrm{B}W_{\mathcal{O}_E}(\mathcal{O}_F)$ contains all periods whose Dieudonné-Manin slopes lie in $[0,1]$. In equal characteristic, when $E = \mathbb{F}_q((\pi))$, there is no restriction on Dieudonné-Manin slopes of formal $\mathcal{O}_E$-modules (what we call here a formal $\mathcal{O}_E$-module is a Drinfeld module in dimension 1). This gives a meta-explanation to the fact that in equal characteristic all elements of B have a unique power series expansion and the fact that this may not be the case in unequal characteristic.

1.2.2. *Newton polygons.* Since the elements of B may not be written uniquely as a power series $\sum_{n\in\mathbb{Z}}[x_n]\pi^n$, we need a trick to define the Newton polygon of such elements. The following proposition is an easy consequence of the following Dini type theorem: if a sequence of concave functions on $]0,+\infty[$ converges point-wise then the convergence is uniform on all compact subsets of $]0,+\infty[$ (but not on all $]0,+\infty[$ in general).

Proposition **1.9.** *If $(x_n)_{n\geq 0}$ is a sequence of B^b that converges to $x \in \mathrm{B}\setminus\{0\}$ then for all $I \subset]0,1[$ compact,*

$$\exists N,\ n \geq N \text{ and } q^{-r} \in I \Longrightarrow v_r(x_n) = v_r(x).$$

One deduces immediately:

Corollary **1.10.** *For $x \in \mathrm{B}$ the function $r \mapsto v_r(x)$ is a concave polygon with integral slopes.*

This leads us to the following definition.

Definition 1.11. *For $x \in \mathrm{B}$, define* Newt(x) *as the inverse Legendre transform of the function $r \mapsto v_r(x)$.*

Thus, Newt(x) is a polygon with integral x-coordinate breakpoints. Moreover, if $(\lambda_i)_{i\in\mathbb{Z}}$ are its slopes, where λ_i is the slope on the segment $[i, i+1]$ (we set $\lambda_i = +\infty$ if Newt(x) is $+\infty$ on this segment), then

$$\lim_{i\to+\infty} \lambda_i = 0 \text{ and } \lim_{i\to-\infty} \lambda_i = +\infty.$$

In particular $\lim\limits_{-\infty} \mathrm{Newt(x)} = +\infty$. Those properties of Newt(x) are the only restrictions on such polygons.

Remark 1.12. *If $x_n \underset{n\to+\infty}{\longrightarrow} x$ in B with $x_n \in \mathrm{B}^b$ and $x \neq 0$ then one checks using proposition 1.9 that in fact for any compact subset K of $\mathbb{R}$, there exists an integer N such that for $n \geq N$,* $\mathrm{Newt(x_n)_{|K}} = \mathrm{Newt(x)_{|K}}$. *The advantage of definition 1.11 is that it makes it clear that* Newt(x) *does not depend on the choice of a sequence of B^b going to x.*

Example 1.13.

(1) If $(x_n)_{n\in\mathbb{Z}}$ is a sequence of F satisfying the two conditions of formula (1) then the polygon $\mathrm{Newt}\big(\sum_{n\in\mathbb{Z}}[x_n]\pi^n\big)$ *is the decreasing convex hull of $\{(n, v(x_n))\}_{n\in\mathbb{N}}$.*

(2) If $(a_n)_{n\geq 0}$ is a sequence of $F^\times$ going to zero then the infinite product $\prod_{n\geq 0}\big(1 - \frac{[a_n]}{\pi}\big)$ converges in B and its Newton polygon is zero on $[0,+\infty[$ and has slopes the $(v(a_n))_{n\geq 0}$ on $]-\infty, 0]$.

Of course the Newton polygon of x does not give more informations than the polygon $r \mapsto v_r(x)$. But it is much easier to visualize and its interest lies in the fact that we can appeal to our geometric intuition from the usual case of holomorphic functions recalled in section 1.1 to guess and prove results. Here is a typical application: the proof of the following proposition is not very difficult once you have convinced yourself it has to be true by analogy with the usual case of holomorphic functions.

Proposition **1.14.** *We have the following characterizations:*

(1) $\mathrm{B}^+ = \{x \in \mathrm{B} \mid \mathrm{Newt(x)} \geq 0\}$
(2) $\mathrm{B}^b = \{x \in \mathrm{B} \mid \mathrm{Newt(x)}$ *is bounded below and* $\exists \mathrm{A},\ \mathrm{Newt(x)}_{|]-\infty,\mathrm{A}]} = +\infty\}$.
(3) *The algebra* $\{x \in \mathrm{B} \mid \exists A,\ \mathrm{Newt(x)}_{|]-\infty,\mathrm{A}]} = +\infty\}$ *is a subalgebra of* $W_{\mathcal{O}_E}(F)[\frac{1}{\pi}]$ *equal to* $\big\{ \sum_{n\gg -\infty} [x_n]\pi^n \mid \liminf\limits_{n\to+\infty} \frac{v(x_n)}{n} \geq 0 \big\}$.

This has powerful applications that would be difficult to obtain without Newton polygons. For example one obtains the following.

Corollary **1.15.**

(1) $\mathrm{B}^\times = \big(\mathrm{B}^b\big)^\times = \big\{x \in \mathrm{B}^b \mid \mathrm{Newt(x)}$ *has* 0 *as its only non infinite slope*$\big\}$.
(2) *One has* $\mathrm{B}^{\varphi=\pi^d} = 0$ *for* $d<0$, $\mathrm{B}^{\varphi=\mathrm{Id}} = E$ *and for* $d \geq 0$,

$$\mathrm{B}^{\varphi=\pi^d} = \big(\mathrm{B}^+\big)^{\varphi=\pi^d}.$$

Typically, the second point is obtained in the following way. If $x \in \mathrm{B}$ satisfies $\varphi(x) = \pi^d x$ then $\mathrm{Newt}(\varphi(\mathrm{x})) = \mathrm{Newt}(\pi^{\mathrm{d}}\mathrm{x})$ that is to say Newt(x) satisfies the functional equation $q\mathrm{Newt(x)(t)} = \mathrm{Newt(x)(t-d)}$. By solving this functional equation and applying proposition 1.14 one finds the results.

2. The space $|Y|$

2.1. **Primitive elements.** We would like to see the Frechet algebra B defined in the preceding section as an algebra of holomorphic functions on a "rigid analytic space" Y. This is of course the case if $E = \mathbb{F}_q((\pi))$ since we can take $Y = \mathbb{D}^*$ a punctured disk as in section 1.1. This is not the case anymore when $E|\mathbb{Q}_p$, at least as a Tate rigid space. But nevertheless we can still define a topological space $|Y|$ that embeds in the Berkovich space $\mathcal{M}(\mathrm{B})$ of rank 1 continuous valuations on B. It should be thought of as the set of classical points of this "space" Y that would remain to construct.

To simplify the exposition, in the following we always assume $E|\mathbb{Q}_p$, that is to say we concentrate on the most difficult case. When $E = \mathbb{F}_q((\pi))$, all stated results are easy to obtain by elementary manipulation and are more or less already contained in the backgrounds of section 1.1.

Definition 2.1.

(1) *An element* $x = \sum_{n\geq 0}[x_n]\pi^n \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ *is primitive if* $x_0 \neq 0$ *and there exists an integer* n *such that* $x_n \in \mathcal{O}_F^\times$. *For such a primitive element* x *we define* $\deg(x)$ *as the smallest such integer* n.
(2) *A primitive element of strictly positive degree is irreducible if it can not be written as a product of two primitive elements of strictly lower degree.*

If k_F is the residue field of $\mathcal{O}_F$ there is a projection

$$W_{\mathcal{O}_E}(\mathcal{O}_F) \twoheadrightarrow W_{\mathcal{O}_E}(k_F).$$

Then, x is primitive if and only if $x \notin \pi W_{\mathcal{O}_E}(\mathcal{O}_F)$ and its projection $\tilde{x} \in W_{\mathcal{O}_E}(k_F)$ is non-zero. For such an x, $\deg(x) = v_\pi(\tilde{x})$. We deduce from this that the product of a degree d by a degree d' primitive element is a degree $d+d'$ primitive element. Degree 0 primitive elements are the units of $W_{\mathcal{O}_E}(\mathcal{O}_F)$. Any primitive degree 1 element is irreducible.
In terms of Newton polygons, $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ is primitive if and only if $\mathrm{Newt(x)(0)} \neq +\infty$ and $\mathrm{Newt(x)(t)} = 0$ for $t \gg 0$.

Definition 2.2. *Define* $|Y|$ *to be the set of primitive irreducible elements modulo multiplication by an element of* $W_{\mathcal{O}_E}(\mathcal{O}_F)^\times$.

There is a degree function

$$\deg : |Y| \to \mathbb{N}_{\geq 1}$$

given by the degree of any representative of a class in $|Y|$. If x is primitive note $\bar{x} \in \mathcal{O}_F$ for its reduction modulo π. We have $|\bar{x}| = |\bar{y}|$ if $y \in W_{\mathcal{O}_E}(\mathcal{O}_F)^\times .x$. There is thus a function

$$\begin{aligned} \|\cdot\| : |Y| &\longrightarrow \;]0,1[\\ W_{\mathcal{O}_E}(\mathcal{O}_F)^\times .x &\longmapsto |\bar{x}|^{1/\deg(x)}. \end{aligned}$$

A primitive element $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ of strictly positive degree is irreducible if and only if the ideal generated by x is prime. In fact, if $x = yz$ with $y, z \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ and x primitive then projecting to $W_{\mathcal{O}_E}(k_F)$ and $\mathcal{O}_F$ the preceding equality one obtains that y and z are primitive. There is thus an embedding

$$|Y| \subset \operatorname{Spec}(W_{\mathcal{O}_E}(\mathcal{O}_F)).$$

The Frobenius φ induces a bijection

$$\varphi : |Y| \xrightarrow{\sim} |Y|$$

that leaves invariant the degree and satisfies

$$\|\varphi(y)\| = \|y\|^q.$$

Remark 2.3. *When $E = \mathbb{F}_q((\pi))$, replacing $W_{\mathcal{O}_E}(\mathcal{O}_F)$ by $\mathcal{O}_F[\![z]\!]$ in the preceding definitions (we set $z = \pi$) there is an identification $|Y| = |\mathbb{D}^*|$. In fact, according to Weierstrass, any irreducible primitive $f \in \mathcal{O}_F[\![z]\!]$ has a unique irreducible unitary polynomial $P \in \mathcal{O}_F[z]$ in its $\mathcal{O}_F[\![z]\!]^\times$-orbit satisfying: $P(0) \neq 0$ and the roots of P have absolute value < 1. Then for $y \in |\mathbb{D}^*|$, $deg(y) = [k(y) : F]$ and $\|y\|$ is the distance from y to the origin of the disk $\mathbb{D}$.*

2.2. **Background on the ring $\mathscr{R}$.** For an $\mathcal{O}_E$-algebra A set

$$\mathscr{R}(A) = \left\{ \left(x^{(n)}\right)_{n \geq 0} \mid x^{(n)} \in A,\ \left(x^{(n+1)}\right)^q = x^{(n)} \right\}.$$

If A is π-adic, I is a closed ideal of A such that A is $I + (\pi)$-adic, then the reduction map induces a bijection

$$\mathscr{R}(A) \xrightarrow{\sim} \mathscr{R}(A/I)$$

with inverse given by

$$\left(x^{(n)}\right)_{n \geq 0} \longmapsto \left(\lim_{k \to +\infty} \left(\widehat{x^{(n+k)}} \right)^{q^k} \right)_{n \geq 0}.$$

where $\widehat{x^{(n+k)}} \in A$ is any lift of $x^{(n+k)} \in A/I$, and the preceding limit is for the $I + (\pi)$-adic topology. In particular, applying this for $I = (\pi)$, we deduce that the set valued functor $\mathscr{R}$ factorizes canonically as a functor

$$\mathscr{R} : \pi\text{-adic } \mathcal{O}_E\text{-algebras} \longrightarrow \text{perfect } \mathbb{F}_q\text{-algebras}.$$

If $W_{\mathcal{O}_E}$ stands for the (ramified) Witt vectors there is then a couple of adjoint functors

$$\pi\text{-adic } \mathcal{O}_E\text{-algebras} \underset{W_{\mathcal{O}_E}}{\overset{\mathscr{R}}{\rightleftarrows}} \text{perfect } \mathbb{F}_q\text{-algebras}$$

where $W_{\mathcal{O}_E}$ is left adjoint to $\mathscr{R}$ and the adjunction morphisms are given by:

$$\begin{aligned} A &\xrightarrow{\sim} \mathscr{R}\big(W_{\mathcal{O}_E}(A)\big) \\ a &\longmapsto \left(\left[a^{q^{-n}}\right]\right)_{n \geq 0} \end{aligned}$$

and

$$\begin{aligned} \theta : W_{\mathcal{O}_E}(\mathscr{R}(A)) &\longrightarrow A \\ \sum_{n \geq 0} [x_n]\pi^n &\longmapsto \sum_{n \geq 0} x_n^{(0)} \pi^n. \end{aligned}$$

If $L|\mathbb{Q}_p$ is a complete valued extension for a valuation $w : L \to \mathbb{R} \cup \{+\infty\}$ extending a multiple of the p-adic valuation, then $\mathscr{R}(L)$ equipped with the valuation

$$x \longmapsto w(x^{(0)})$$

is a characteristic p perfect complete valued field with ring of integers $\mathscr{R}(\mathcal{O}_L)$ (one has to be careful that the valuation on $\mathscr{R}(L)$ may be trivial). It is not very difficult to prove that if L is algebraically closed then $\mathscr{R}(L)$ is too. A reciprocal to this statement will be stated in the following sections.

2.3. **The case when is F algebraically closed.**

Theorem **2.4.** *Suppose F is algebraically closed. Let $\mathfrak{p} \in Spec(W_{\mathcal{O}_E}(\mathcal{O}_F))$ generated by a degree one primitive element. Set*

$$A = W_{\mathcal{O}_E}(\mathcal{O}_F)/\mathfrak{p}$$

and $\theta : W_{\mathcal{O}_E}(\mathcal{O}_F) \twoheadrightarrow A$ the projection. The following properties are satisfied:

(1) *There is an isomorphism*

$$\begin{array}{ccc} \mathcal{O}_F & \xrightarrow{\sim} & \mathscr{R}(A) \\ x & \longmapsto & \Big(\theta\big([x^{q^{-n}}]\big)\Big)_{n\geq 0}. \end{array}$$

(2) *The map*

$$\begin{array}{ccc} \mathcal{O}_F & \longrightarrow & A \\ x & \longmapsto & \theta([x]) \end{array}$$

is surjective.

(3) *There is a unique valuation w on A such that for all $x \in \mathcal{O}_F$,*

$$w(\theta([x])) = v(x).$$

Moreover, $A[\frac{1}{\pi}]$ equipped with the valuation w is a complete algebraically closed extension of E with ring of integers A. There is an identification of valued fields $F = \mathscr{R}\big(A[\frac{1}{\pi}]\big)$.

(4) *If $\underline{\pi} \in \mathscr{R}(A)$ is such that $\underline{\pi}^{(0)} = \pi$ then*

$$\mathfrak{p} = ([\underline{\pi}] - \pi).$$

Remark 2.5. *One can reinterpret points (2) and (4) of the preceding theorem in the following way. Let x be primitive of degree 1. Then:*

- *we have a Weierstrass division in $W_{\mathcal{O}_E}(\mathcal{O}_F)$: given $y \in W_{\mathcal{O}_E}(\mathcal{O}_F)$, there exists $z \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ and $a \in \mathcal{O}_F$ such that*

$$y = zx + [a],$$

- *we have a Weierstrass factorization of x: there exists $u \in W_{\mathcal{O}_E}(\mathcal{O}_F)^\times$ and $b \in \mathcal{O}_F$ such that*

$$x = u.(\pi - [b]).$$

One has to be careful that, contrary to the classical situation, the remainder term a is not unique in such a Weierstrass division. Similarly, b is not uniquely determined by x in the Weierstrass factorization.

Indications on the proof of theorem 2.4. Statement (1) is an easy consequence of general facts about the ring $\mathscr{R}$ recalled in section 2.2: if $\mathfrak{p} = (x)$

$$\mathscr{R}(A) \xrightarrow{\sim} \mathscr{R}(A/\pi A) = \mathscr{R}(\mathcal{O}_F/\bar{x}) = \mathcal{O}_F$$

since $\mathcal{O}_F$ is perfect.

According to point (1), point (2) is reduced to proving that any element of A has a q-th root. If $E^0 = W(\mathbb{F}_q)_\mathbb{Q}$, the norm map N_{E/E^0} induces a norm map $W_{\mathcal{O}_E}(\mathcal{O}_F) \to W(\mathcal{O}_F)$ sending a primitive element of degree 1 to a primitive element of degree 1. Using this one can reduce the problem to the case $E = \mathbb{Q}_p$. Suppose $p \neq 2$ to simplify. Then any element of $1 + p^2 W(\mathcal{O}_F)$ has a p-th root. Using this fact plus some elementary manipulations one is reduced to solving some explicit equations in the truncated Witt vectors of length 2, $W_2(\mathcal{O}_F)$. One checks this is possible, using the fact that F is algebraically closed (here this hypothesis is essential, the hypothesis F perfect is not sufficient).

In fact, the preceding proof gives that for any integer n, any element of A has an n-th root, the case when n is prime to p being easier than the case $n = p$ we just explained (since then any element of $1 + pW_{\mathcal{O}_E}(\mathcal{O}_F)$ has an n-th root).

Point (4) is an easy consequence of the following classical characterization of $\ker\theta$: an element $y = \sum_{n\geq 0}[y_n]\pi^n \in \ker\theta$ such that $y_0^{(0)} \in \pi A^\times$ is a generator of $\ker\theta$. In fact, if y is such an element then $\ker\theta = (y) + \pi\ker\theta$ and one concludes $\ker\theta = (y)$ by applying the π-adic Nakayama lemma ($\ker\theta$ is π-adically closed).

In point (3), the difficulty is to prove that the complete valued field $L = A[\frac{1}{\pi}]$ is algebraically closed, other points following easily from point (2). Using the fields of norms theory one verifies that L contains an algebraic closure of $\mathbb{Q}_p$. More precisely, one can suppose thanks to point (4) that $\mathfrak{p} = (\pi - [\underline{\pi}])$. There is then an embedding $\mathbb{F}_q((\underline{\pi})) \subset F$ that induces $F' := \widehat{\overline{\mathbb{F}_q((\underline{\pi}))}} \subset F$. This induces a morphism $L' = W_{\mathcal{O}_E}(\mathcal{O}_{F'})\left[\frac{1}{\pi}\right]/(\pi - [\underline{\pi}]) \to L$. But thanks to the fields of norms theory, L' is the completion of an algebraic closure of E. In particular, L contains all roots of unity. Let us notice that since $\mathcal{O}_L/\pi\mathcal{O}_L = \mathcal{O}_F/\underline{\pi}\mathcal{O}_F$, the residue field of L is the same as the one of F and is thus algebraically closed. Now, we use the following proposition that is well known in the discrete valuation case thanks to the theory of ramification groups (those ramification groups do not exist in the non discrete valuation case, but one can define some ad hoc one to obtain the proposition).

Proposition **2.6.** *Let K be a complete valued field for a rank 1 valuation and $K'|K$ a finite degree Galois extension inducing a trivial extension on the residue fields. Then the group $Gal(K'|K)$ is solvable.*

Since for any integer n any element of L has an n-th root, one concludes L is algebraically closed using Kümmer theory. □

Note now $\mathcal{O}_C = W_{\mathcal{O}_E}(\mathcal{O}_F)/\mathfrak{p}$ with fraction field C. One has to be careful that the valuation w on C extends only a multiple of the π-adic valuation of E: $q^{-w(\pi)} = \|\mathfrak{p}\|$. The quotient morphism $\mathrm{B}^{b,+} \longrightarrow C$ extends in fact by continuity to surjective morphisms

$$\begin{array}{ccccccc} \mathrm{B}^{b,+} & \hookrightarrow & \mathrm{B}^{b} & \hookrightarrow & \mathrm{B} & \hookrightarrow & \mathrm{B}_I \\ & \searrow & \downarrow & \swarrow & & \swarrow & \\ & & C & & & & \end{array}$$

where $I \subset]0,1[$ is such that $\|\mathfrak{p}\| \in I$. This is a consequence of the inequality

$$q^{-w(f)} \leq |f|_\rho$$

for $f \in \mathrm{B}^b$ and $\rho = \|\mathfrak{p}\|$. If $\mathfrak{p} = (x)$ then all kernels of those surjections are the principal ideals generated by x in those rings.

Convention: we will now see $|Y|^{\deg=1}$ as a subset of $Spm(\mathrm{B})$. For $\mathfrak{m} \in |Y|^{\deg=1}$, we note

$$C_\mathfrak{m} = \mathrm{B}/\mathfrak{m},\ \theta_\mathfrak{m} : \mathrm{B} \twoheadrightarrow C_\mathfrak{m}$$

and $v_\mathfrak{m}$ the valuation such that

$$v_\mathfrak{m}(\theta_\mathfrak{m}([x])) = v(x).$$

Theorem **2.7.** *If F is algebraically closed then the primitive irreducible elements are of degree one.*

Remark 2.8. *On can reinterpret the preceding theorem as a factorization statement: if $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ is primitive of degree d then*

$$x = u.(\pi - [a_1])\ldots(\pi - [a_d]), \quad u \in W_{\mathcal{O}_E}(\mathcal{O}_F)^\times,\ a_1, \ldots, a_d \in \mathfrak{m}_F \setminus \{0\}.$$

Indications on the proof of theorem 2.7. Let $f \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ be primitive. The theorem is equivalent to saying that f "has a zero in $|Y|^{\deg=1}$" that is to say there exists $\mathfrak{m} \in |Y|^{\deg=1}$ such that $\theta_{\mathfrak{m}}(f) = 0$. The method to construct such a zero is a Newton type method by successive approximations. To make it work we need to know it converges in a sense that has to be specified. We begin by proving the following.

Proposition **2.9.** *For $\mathfrak{m}_1, \mathfrak{m}_2 \in |Y|^{\deg=1}$ set*

$$d(\mathfrak{m}_1, \mathfrak{m}_2) = q^{-v_{\mathfrak{m}_1}(a)} \quad \text{if } \theta_{\mathfrak{m}_1}(\mathfrak{m}_2) = \mathcal{O}_{C_{\mathfrak{m}_1}} a.$$

Then:

(1) *This defines an ultrametric distance on $|Y|^{\deg=1}$.*
(2) *For any $\rho \in]0,1[$, $\left(|Y|^{\deg=1, \|\cdot\| \geq \rho}, d\right)$ is a complete metric space.*

Remark 2.10. *In equal characteristic, if $E = \mathbb{F}_q((\pi))$, then $|Y| = |\mathbb{D}^*| = \mathfrak{m}_F \setminus \{0\}$ and this distance is the usual one induced by the absolute value $|\cdot|$ of F.*

The approximation algorithm then works like this. We define a sequence $(\mathfrak{m}_n)_{n\geq 1}$ of $|Y|^{\deg=1}$ such that:

- $(\|\mathfrak{m}_n\|)_{n\geq 1}$ is constant,
- it is a Cauchy sequence,
- $\lim\limits_{n\to+\infty} v_{\mathfrak{m}_n}(f) = +\infty$.

Write $f = \sum_{k\geq 0}[x_k]\pi^k$. The Newton polygon of f as defined in section 1.2.2 is the same as the Newton polygon of $g(T) = \sum_{k\geq 0} x_k T^k \in \mathcal{O}_F[\![T]\!]$. Let $z \in \mathfrak{m}_F$ be a root of $g(T)$ with valuation the smallest one among the valuations of the roots of $g(T)$ (that is to say the smallest non zero slope of Newt(f)). Start with $\mathfrak{m}_1 = (\pi - [z]) \in |Y|^{\deg=1}$. If $\mathfrak{m}_n$ is defined, $\mathfrak{m}_n = (\xi)$ with ξ primitive of degree 1, we can write

$$f = \sum_{k\geq 0} [a_k]\xi^k$$

in $W_{\mathcal{O}_E}(\mathcal{O}_F)$ (this is a consequence of point (2) of theorem 2.4 and the fact that $W_{\mathcal{O}_E}(\mathcal{O}_F)$ is ξ-adic). We check the power series $h(T) = \sum_{k\geq 0} a_k T^k \in \mathcal{O}_F[\![T]\!]$ is primitive of degree d. Let z be a root of $h(T)$ of maximal valuation. Then $\xi - [z]$ is primitive of degree 1 and we set $\mathfrak{m}_{n+1} = (\xi - [z])$.

We then prove the sequence $(\mathfrak{m}_n)_{n\geq 1}$ satisfies the required properties. □

2.4. **Parametrization of $|Y|$ when F is algebraically closed.** Suppose F is algebraically closed. We see $|Y|$ as a subset of Spm(B). As we saw, for any $\mathfrak{m} \in |Y|$ there exists $a \in \mathfrak{m}_F \setminus \{0\}$ such that $\mathfrak{m} = (\pi - [a])$. The problem is that such an a is not unique. Moreover, given $a, b \in \mathfrak{m}_F \setminus \{0\}$, there is no simple rule to decide whether $(\pi - [a]) = (\pi - [b])$ or not.

Here is a solution to this problem. Let $\mathcal{LT}$ be a Lubin-Tate group law over $\mathcal{O}_E$. We note

$$Q = [\pi]_{\mathcal{LT}} \in \mathcal{O}_E[\![T]\!]$$

and $\mathcal{G}$ the associated formal group on Spf$(\mathcal{O}_E)$. We have

$$\mathcal{G}(\mathcal{O}_F) = \Big(\mathfrak{m}_F, \underset{\mathcal{LT}}{+}\Big).$$

The topology induced by the norms $(|\cdot|_\rho)_{\rho\in]0,1[}$ on $W_{\mathcal{O}_E}(\mathcal{O}_F)$ is the "weak topology" on the coefficients of the Teichmüller expansion, that is to say the product topology via the bijection

$$\begin{array}{ccc} \mathcal{O}_F^{\mathbb{N}} & \xrightarrow{\sim} & W_{\mathcal{O}_E}(\mathcal{O}_F) \\ (x_n)_{n\geq 0} & \longmapsto & \sum\limits_{n\geq 0} [x_n]\pi^n. \end{array}$$

If $a \in \mathfrak{m}_F \setminus \{0\}$, this coincides with the $([a], \pi)$-adic topology. Moreover $W_{\mathcal{O}_E}(\mathcal{O}_F)$ is complete, that is to say closed in B. If

$$W_{\mathcal{O}_E}(\mathcal{O}_F)^{00} = \Big\{ \sum_{n\geq 0} [x_n]\pi^n \mid x_0 \in \mathfrak{m}_F \Big\} = \{x \in W_{\mathcal{O}_E}(\mathcal{O}_F) \mid x \bmod \pi \in \mathfrak{m}_F\},$$

then

$$\mathcal{G}\big(W_{\mathcal{O}_E}(\mathcal{O}_F)\big) = \Big(W_{\mathcal{O}_E}(\mathcal{O}_F)^{00}, \underset{\mathcal{LT}}{+}\Big).$$

One verifies the following proposition.

Proposition **2.11.** *For $x \in \mathfrak{m}_F$, the limit*

$$[x]_Q := \lim_{n\to+\infty} [\pi^n]_{\mathcal{LT}}\Big(\Big[x^{q^{-n}}\Big]\Big)$$

exists in $W_{\mathcal{O}_E}(\mathcal{O}_F)$, reduces to x modulo π, and defines a "lift"

$$[\cdot]_Q : \mathcal{G}(\mathcal{O}_F) \hookrightarrow \mathcal{G}(W_{\mathcal{O}_E}(\mathcal{O}_F)).$$

The usual Teichmüller lift $[\cdot]$ is well adapted to the multiplicative group law: $[xy] = [x].[y]$. The advantage of the twisted Teichmüller lift $[\cdot]_Q$ is that it is more adapted to the Lubin-Tate one:

$$[x]_Q \underset{\mathcal{LT}}{+} [y]_Q = \big[x \underset{\mathcal{LT}}{+} y\big]_Q.$$

When $E = \mathbb{Q}_p$ and $\mathcal{LT} = \widehat{\mathbb{G}}_m$ one has $[x]_Q = [1+x] - 1$.

Definition 2.12. *For $\epsilon \in \mathfrak{m}_F \setminus \{0\}$ define $u_\epsilon = \dfrac{[\epsilon]_Q}{[\epsilon^{1/q}]_Q} \in W_{\mathcal{O}_E}(\mathcal{O}_F)$.*

This is a primitive degree one element since it is equal to the power series $\frac{Q(T)}{T}$ evaluated at $[\epsilon^{1/q}]_Q$. For example, if $E = \mathbb{Q}_p$ and $\mathcal{LT} = \widehat{\mathbb{G}}_m$, setting $\epsilon' = 1 + \epsilon$ one has

$$u_\epsilon = 1 + \Big[\epsilon'^{\frac{1}{p}}\Big] + \cdots + \Big[\epsilon'^{\frac{p-1}{p}}\Big].$$

Proposition **2.13.** *There is a bijection*

$$\begin{array}{rcl} \big(\mathcal{G}(\mathcal{O}_F) \setminus \{0\}\big)/\mathcal{O}_E^\times & \xrightarrow{\sim} & |Y| \\ \mathcal{O}_E^\times.\epsilon & \longmapsto & (u_\epsilon). \end{array}$$

The inverse of this bijection is given by the following rule. For $\mathfrak{m} \in |Y|$, define

$$X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}) = \big\{(x_n)_{n\geq 0} \mid x_n \in \mathcal{G}(\mathcal{O}_{C_\mathfrak{m}}),\ \pi.x_{n+1} = x_n\big\}.$$

More generally, $X(\mathcal{G})$ will stand for the projective limit "$\varprojlim_{n\geq 0} \mathcal{G}$" where the transition mappings are multiplication by π (one can give a precise geometric meaning to this but this is not our task here, see [12] for more details). The reduction modulo π map induces a bijection

$$X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}) \xrightarrow{\sim} X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}/\pi\mathcal{O}_{C_\mathfrak{m}})$$

with inverse given by

$$(x_n)_n \mapsto \Big(\lim_{k\to+\infty} \pi^k \hat{x}_{n+k}\Big)_n$$

where $\hat{x}_{n+k}$ is any lift of x_{n+k}. But $[\pi]_{\mathcal{LT}}$ modulo π is the Frobenius Frob_q. We thus have

$$X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}/\pi\mathcal{O}_{C_\mathfrak{m}}) = \mathcal{G}\big(\mathscr{R}(\mathcal{O}_{C_\mathfrak{m}}/\pi\mathcal{O}_{C_\mathfrak{m}})\big) = \mathcal{G}(\mathcal{O}_F)$$

since

$$O_F \xrightarrow[\theta_\mathfrak{m}\circ[\cdot]]{\sim} \mathscr{R}(\mathcal{O}_{C_\mathfrak{m}}) \xrightarrow{\sim} \mathscr{R}(\mathcal{O}_{C_\mathfrak{m}}/\pi\mathcal{O}_{C_\mathfrak{m}}).$$

The Tate module

$$T_\pi(\mathcal{G}) = \{(x_n)_{n\geq 0} \in X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}) \mid x_0 = 1\} \subset X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}})$$

embeds thus in $\mathcal{G}(\mathcal{O}_F)$. This is a rank 1 sub-$\mathcal{O}_E$-module. The inverse of the bijection of proposition 2.13 sends $\mathfrak{m}$ to this $\mathcal{O}_E$-line in $\mathcal{G}(\mathcal{O}_F)$.

2.5. **The general case: Galois descent.** Let now F be general (but still perfect), that is to say not necessarily algebraically closed. Let $\overline{F}$ be an algebraic closure of F with Galois group $G_F = \mathrm{Gal}(\overline{F}|F)$. Since the field F will vary we now put a subscript in the preceding notation to indicate this variation. The set $\big|Y_{\widehat{\overline{F}}}\big|$ is equipped with an action of G_F.

Theorem **2.14.** *For $\mathfrak{p} \in |Y_F| \subset Spec(W_{\mathcal{O}_E}(\mathcal{O}_F))$ set*

$$L = \big(W_{\mathcal{O}_E}(\mathcal{O}_F)/\mathfrak{p}\big)\big[\tfrac{1}{\pi}\big]$$

and

$$\theta : \mathrm{B}_F^{b,+} \twoheadrightarrow L.$$

Then:

(1) There is a unique valuation w on L such that for $x \in \mathcal{O}_F$,

$$w(\theta([x])) = v(x).$$

(2) (L, w) is a complete valued extension of E.
(3) (L, w) is perfectoïd in the sense that the Frobenius of $\mathcal{O}_L/\pi\mathcal{O}_L$ is surjective.
(4) Via the embedding

$$\begin{array}{ccc} \mathcal{O}_F & \hookrightarrow & \mathscr{R}(L) \\ a & \longmapsto & \Big(\theta\big(\big[a^{q^{-n}}\big]\big)\Big)_{n\geq 0}f \end{array}$$

one has $\mathscr{R}(L)|F$ and this extension is of finite degree

$$[\mathscr{R}(L) : F] = \deg \mathfrak{p}.$$

Remark 2.15. *What we call here perfectoid is what we called "strictly p-perfect" in* [13] *and* [14]*. The authors decided to change their terminology because meanwhile the work* [29] *appeared.*

Indications on the proof. The proof is based on theorems 2.4 and 2.7 via a Galois descent argument from $\big|Y_{\widehat{\overline{F}}}\big|$ to $|Y_F|$. For this we need the following.

Theorem **2.16.** *One has $\mathfrak{m}_F.H^1\big(G_F, \mathcal{O}_{\widehat{\overline{F}}}\big) = 0$.*

Using Tate's method ([31]) this theorem is a consequence of the following "almost etalness" statement whose proof is much easier than in characteristic 0.

Proposition **2.17.** *If $L|F$ is a finite degree extension then $\mathfrak{m}_F \subset tr_{L/F}(\mathcal{O}_L)$.*

Sketch of proof. The trace $\mathrm{tr}_{L/F}$ commutes with the Frobenius $\varphi = \mathrm{Frob}_q$. Choosing $x \in \mathcal{O}_L$ such that $\mathrm{tr}_{L/F}(x) \neq 0$ one deduces that

$$\lim_{n\to+\infty} |\mathrm{tr}_{L/F}(\varphi^{-n}(x))| = 1.$$

□

From theorem 2.16 one deduces that $H^1\big(G_F, \mathfrak{m}_{\widehat{\overline{F}}}\big) = 0$ which implies

$$H^1\big(G_F, 1 + W_{\mathcal{O}_E}\big(\mathfrak{m}_{\widehat{\overline{F}}}\big)\big) = 0 \tag{4}$$

where

$$1 + W_{\mathcal{O}_E}\big(\mathfrak{m}_{\widehat{\overline{F}}}\big) = \ker\Big(W_{\mathcal{O}_E}\big(\mathcal{O}_{\widehat{\overline{F}}}\big)^\times \longrightarrow W_{\mathcal{O}_E}\big(k_{\overline{F}}\big)^\times\Big).$$

Finally, let us notice that thanks to Ax's theorem (that can be easily deduced from 2.17) one has

$$W_{\mathcal{O}_E}\big(\mathcal{O}_{\widehat{\overline{F}}}\big)^{G_F} = W_{\mathcal{O}_E}(\mathcal{O}_F).$$

Proposition **2.18.** *Let $x \in W_{\mathcal{O}_E}\big(\mathcal{O}_{\widehat{\overline{F}}}\big)$ be a primitive element such that $\forall \sigma \in G_F$, $(\sigma(x)) = (x)$ as an ideal of $W_{\mathcal{O}_E}\big(\mathcal{O}_{\widehat{\overline{F}}}\big)$. Then, there exists $y \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ such that $(x) = (y)$.*

Proof. If $\tilde{x} \in W_{\mathcal{O}_E}(k_{\overline{F}})$ is the projection of x via $W_{\mathcal{O}_E}(\mathcal{O}_{\widehat{\overline{F}}}) \to W_{\mathcal{O}_E}(k_{\overline{F}})$, up to multiplying x by a unit, one can suppose $\tilde{x} = \pi^{\deg x}$. Looking at the cocycle $\sigma \mapsto \frac{\sigma(x)}{x}$, the proposition is then a consequence of the vanishing (4). □

Let now $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ be primitive irreducible of degree d. There exist $y_1, \dots, y_r \in W_{\mathcal{O}_E}\big(\mathcal{O}_{\widehat{\overline{F}}}\big)$ primitive of degree 1 satisfying $(y_i) \neq (y_j)$ for $i \neq j$, $a_1, \dots, a_r \in \mathbb{N}_{\geq 1}$ and $u \in W_{\mathcal{O}_E}(\mathcal{O}_{\widehat{\overline{F}}})^\times$ such that

$$x = u.\prod_{i=1}^{r} y_i^{a_i}.$$

The finite subset

$$\big\{(y_i)\big\}_{1\leq i\leq r} \subset |Y_{\widehat{\overline{F}}}|$$

is stable under G_F. Using proposition 2.18 and the irreducibility of x one verifies that this action is transitive and $a_1 = \cdots = a_r = 1$. In particular one has $r = d$. Note $\mathfrak{m}_i = (y_i)$ and $C_{\mathfrak{m}_i}$ the associated algebraically closed residue field. Let $K|F$ be the degree d extension of F in $\overline{F}$ such that

$$G_K = \mathrm{Stab}_{G_F}(\mathfrak{m}_1).$$

One has

$$\mathrm{B}^{b,+}_{\widehat{\overline{F}}}/(x) = \prod_{i=1}^{d} C_{\mathfrak{m}_i}$$

and thus

$$\Big(\mathrm{B}^{b,+}_{\widehat{\overline{F}}}/(x)\Big)^{G_F} = C_{\mathfrak{m}_1}^{G_K}.$$

Now, one verifies using that x is primitive that if $a \in \mathfrak{m}_F \setminus \{0\}$ then

$$\big(W_{\mathcal{O}_E}(\mathcal{O}_F)/(x)\big)\big[\tfrac{1}{\pi}\big] = \big(W_{\mathcal{O}_E}(\mathcal{O}_F)/(x)\big)\Big[\tfrac{1}{[a]}\Big].$$

Theorem 2.16 implies that for such an a,

$$[a].H^1\big(G_F, W_{\mathcal{O}_E}(\mathcal{O}_{\widehat{\overline{F}}})\big) = 0.$$

One deduces from this that

$$L = \mathrm{B}^{b,+}_F/(x) = \Big(\mathrm{B}^{b,+}_{\widehat{\overline{F}}}/(x)\Big)^{G_F}$$

that is thus a complete valued field. Moreover

$$\mathscr{R}(L) = \mathscr{R}\Big(C_{\mathfrak{m}_1}^{G_K}\Big) = \mathscr{R}(C_{\mathfrak{m}_1})^{G_K} = \widehat{\overline{F}}^{G_K} = K.$$

Other statements of theorem 2.14 are easily deduced in the same way. □

In the preceding theorem the quotient morphism $\theta : \mathrm{B}^{b,+}_F \twoheadrightarrow L$ extends by continuity to a surjection $\mathrm{B}_F \twoheadrightarrow L$ with kernel the principal ideal $\mathrm{B}_F\mathfrak{p}$. *From now on, we will see* $|Y_F|$ *as a subset of Spm*(B). *If* $\mathfrak{m} \in |Y_F|$ *we note*

$$L_{\mathfrak{m}} = \mathrm{B}_F/\mathfrak{m},\ \theta_{\mathfrak{m}} : \mathrm{B} \twoheadrightarrow L_{\mathfrak{m}}.$$

The preceding arguments give the following.

Theorem **2.19.** *Let* $\big|Y_{\widehat{\overline{F}}}\big|^{G_F-\mathrm{fin}}$ *be the elements of* $\big|Y_{\widehat{\overline{F}}}\big|$ *whose* G_F*-orbit is finite. There is a surjection*

$$\beta : \big|Y_{\widehat{\overline{F}}}\big|^{G_F-\mathrm{fin}} \longrightarrow |Y_F|$$

whose fibers are the G_F*-orbits:*

$$\big|Y_{\widehat{\overline{F}}}\big|^{G_F-\mathrm{fin}}/G_F \xrightarrow{\sim} |Y_F|.$$

Moreover:

- *for* $\mathfrak{m} \in |Y_F|$, *one has*
$$\#\beta^{-1}(\mathfrak{m}) = [\mathscr{R}(L_\mathfrak{m}) : F] = \deg \mathfrak{m},$$
- *for* $\mathfrak{n} \in |Y_{\widehat{\overline{F}}}|^{G_F-\mathrm{fin}}$, *if* $\mathfrak{m} = \beta(\mathfrak{n})$, *one has an extension* $C_\mathfrak{n}|L_\mathfrak{m}$ *that identifies* $C_\mathfrak{n}$ *with the completion of the algebraic closure* $\overline{L}_\mathfrak{m}$ *of* $L_\mathfrak{m}$ *and*
$$Gal(\overline{F}|\mathscr{R}(L_\mathfrak{m})) \xrightarrow{\sim} Gal(\overline{L}_\mathfrak{m}|L_\mathfrak{m}).$$

Remark 2.20.

(1) *One has to be careful that* $\theta_\mathfrak{m}(W_{\mathcal{O}_E}(\mathcal{O}_F)) \subset \mathcal{O}_{L_\mathfrak{m}}$ *is only an order. It is equal to* $\mathcal{O}_{L_\mathfrak{m}}$ *if and only if* $\deg \mathfrak{m} = 1$.
(2) *Contrary to the case when* F *is algebraically closed, in general an* $\mathfrak{m} \in |Y_F|$ *of degree* 1 *is not generated by an element of the form* $\pi - [a]$, $a \in \mathfrak{m}_F \setminus \{0\}$.

2.6. **Application to perfectoid fields.** Reciprocally, given a complete valued field $L|E$ for a rank 1 valuation, it is perfectoid if and only if the morphism
$$\theta : W_{\mathcal{O}_E}(\mathscr{R}(\mathcal{O}_L)) \longrightarrow \mathcal{O}_L$$
is surjective. In this case one can check that the kernel of θ is generated by a primitive degree 1 element. The preceding considerations thus give the following.

Theorem **2.21.**

(1) *There is an equivalence of categories between perfectoid fields* $L|E$ *and the category of couples* $(F, \mathfrak{m})$ *where* $F|\mathbb{F}_q$ *is perfectoid and* $\mathfrak{m} \in |Y_F|$ *is of degree* 1.
(2) *In the preceding equivalence, if* L *corresponds to* $(F, \mathfrak{m})$, *the functor* $\mathscr{R}$ *induces an equivalence between finite étale* L*-algebras and finite étale* F*-algebras. The inverse equivalence sends the finite extension* $F'|F$ *to* $\mathrm{B}_{F'}/\mathrm{B}_{F'}\mathfrak{m}$.

Example 2.22.

(1) *The perfectoid field* $L = \widehat{\mathbb{Q}_p(\zeta_{p^\infty})}$ *corresponds to* $F = \widehat{\mathbb{F}_p((T))^{\mathrm{perf}}}$ *and* $\mathfrak{m} = \big(1 + [T^{1/p} + 1] + \cdots + [T^{\frac{p-1}{p}} + 1]\big)$.
(2) *Choose* $\underline{p} \in \mathscr{R}(\overline{\mathbb{Q}}_p)$ *such that* $\underline{p}^{(0)} = p$. *The perfectoid field* $L = \widehat{M}$ *with* $M = \cup_{n \geq 0} \mathbb{Q}_p(\underline{p}^{(n)})$ *corresponds to* $F = \widehat{\mathbb{F}_p((T))^{\mathrm{perf}}}$ *and* $\mathfrak{m} = ([T] - p)$.

Remark 2.23. *In the preceding correspondence,* F *is maximally complete if and only if* L *is. In particular, one finds back the formula given in* [26] *for* p*-adic maximally complete fields: they are of the form* $W(\mathcal{O}_F)\left[\frac{1}{p}\right]/([x] - p)$ *where* F *is maximally complete of characteristic* p *and* $x \in F^\times$ *satisfies* $v(x) > 0$.

Remark 2.24. *Suppose* $F = \widehat{\overline{\mathbb{F}_p((T))}}$. *One can ask what are the algebraically closed residue fields* $C_\mathfrak{m}$ *up to isomorphism when* $\mathfrak{m}$ *goes through* $|Y_F|$. *Let us note* $\mathbb{C}_p$ *the completion of an algebraic closure of* $\mathbb{Q}_p$. *Thanks to the fields of norms theory it appears as a residue field* $C_\mathfrak{m}$ *for some* $\mathfrak{m} \in |Y_F|$. *The question is: is it true that for all* $\mathfrak{m} \in |Y_F|$, $C_\mathfrak{m} \simeq \mathbb{C}_p$? *The authors do not know the answer to this question. They know that for each integer* $n \geq 1$, $\mathcal{O}_{C_\mathfrak{m}}/p^n\mathcal{O}_{C_\mathfrak{m}} \simeq \mathcal{O}_{\mathbb{C}_p}/p^n\mathcal{O}_{\mathbb{C}_p}$ *but in a non canonical way.*

As a consequence of the preceding theorem we deduce almost étalness for characteristic 0 perfectoid fields.

Corollary **2.25.** *For* $L|E$ *a perfectoid field and* $L'|L$ *a finite extension we have*
$$\mathfrak{m}_L \subset tr_{L'|L}(\mathcal{O}_{L'}).$$

Proof. Set $F' = \mathscr{R}(L')$ and $F = \mathscr{R}(L)$. If L corresponds to $\mathfrak{m} \in |Y_F|$, $\mathfrak{a} = \{x \in \mathcal{O}_F \mid |x| \leq \|\mathfrak{m}\|\}$ and $\mathfrak{a}' = \{x \in \mathcal{O}_{F'} \mid |x| \leq \|\mathfrak{m}\|\}$ we have identifications
$$\begin{aligned} \mathcal{O}_L/\pi\mathcal{O}_L &= \mathcal{O}_F/\mathfrak{a} \\ \mathcal{O}_{L'}/\pi\mathcal{O}_{L'} &= \mathcal{O}_{F'}/\mathfrak{a}'. \end{aligned}$$

According to point (2) of the preceding theorem, with respect to those identifications the map $\mathrm{tr}_{L'|L}$ modulo π is induced by the map $\mathrm{tr}_{F'|F}$. The result is thus a consequence of proposition 2.17. □

Remark 2.26.

(1) *Let K be a complete valued extension of $\mathbb{Q}_p$ with discrete valuation and perfect residue field and $M|K$ be an algebraic infinite degree arithmetically profinite extension. Then, by the fields of norms theory ([32]), $L = \widehat{M}$ is perfectoid and point (2) of the preceding theorem is already contained in [32].*

(2) *In [29] Scholze has obtained a different proof of point (2) of theorem 2.21 and of corollary 2.25.*

Using Sen's method ([30]), corollary 2.25 implies the following.

Theorem **2.27.** *Let $L|E$ be a perfectoid field with algebraic closure $\overline{L}$. Then the functor $V \mapsto V \otimes_L \widehat{\overline{L}}$ induces an equivalence of categories between finite dimensional L-vector spaces and finite dimensional $\widehat{\overline{L}}$-vector spaces equipped with a continuous semi-linear action of $Gal(\overline{L}|L)$. An inverse is given by the functor $W \mapsto W^{Gal(\overline{L}|L)}$.*

Using this theorem one deduces by dévissage the following that we will use later.

Theorem **2.28.** *Let $\mathfrak{m} \in |Y_F|$ and note*

$$\mathrm{B}^+_{\widehat{\overline{F}},dR,\mathfrak{m}} = \prod_{\substack{\mathfrak{m}' \in |Y_{\widehat{\overline{F}}}| \\ \beta(\mathfrak{m}')=\mathfrak{m}}} \mathrm{B}^+_{\widehat{\overline{F}},dR,\mathfrak{m}'}$$

the $\mathrm{B}_{\widehat{\overline{F}}}\mathfrak{m}$-adic completion of $\mathrm{B}_{\widehat{\overline{F}}}$ (see def. 3.1). Then the functor

$$M \longmapsto M \otimes_{\mathrm{B}^+_{F,dR,\mathfrak{m}}} \mathrm{B}^+_{\widehat{\overline{F}},dR,\mathfrak{m}}$$

induces an equivalence of categories between finite type $\mathrm{B}^+_{F,dR,\mathfrak{m}}$-modules and finite type $\mathrm{B}^+_{\widehat{\overline{F}},dR,\mathfrak{m}}$-modules equipped with a continuous semi-linear action of $Gal(\overline{F}|F)$. An inverse is given by the functor $W \mapsto W^{Gal(\overline{F}|F)}$.

3. Divisors on Y

3.1. **Zeros of elements of** B**.** We see $|Y|$ as a subset of Spm(B). For $\mathfrak{m} \in |Y|$, we set

$$L_\mathfrak{m} = \mathrm{B}/\mathfrak{m} \text{ and } \theta_\mathfrak{m} : \mathrm{B} \twoheadrightarrow L_\mathfrak{m}.$$

We note $v_\mathfrak{m}$ the valuation on $L_\mathfrak{m}$ such that

$$v_\mathfrak{m}(\theta_\mathfrak{m}([a])) = v(a).$$

One has

$$\|\mathfrak{m}\| = q^{-v_\mathfrak{m}(\pi)/\deg \mathfrak{m}}$$

where $\|.\|$ was defined after definition 2.2.

Definition 3.1. *For $\mathfrak{m} \in |Y|$ define $\mathrm{B}^+_{dR,\mathfrak{m}}$ as the $\mathfrak{m}$-adic completion of* B*.*

The ring $\mathrm{B}^+_{dR,\mathfrak{m}}$ is a discrete valuation ring with residue field $L_\mathfrak{m}$ and the natural map $\mathrm{B} \to \mathrm{B}^+_{dR,\mathfrak{m}}$ is injective. We note again $\theta_\mathfrak{m} : \mathrm{B}^+_{dR,\mathfrak{m}} \to L_\mathfrak{m}$. We note

$$\mathrm{ord}_\mathfrak{m} : \mathrm{B}^+_{dR,\mathfrak{m}} \longrightarrow \mathbb{N} \cup \{+\infty\}$$

its normalized valuation.

Example 3.2. *If $E = \mathbb{F}_q((\pi))$ then $|Y| = |\mathbb{D}^*|$ and if $\mathfrak{m} \in |Y|$ corresponds to $x \in |\mathbb{D}^*|$ then $\mathrm{B}^+_{dR,\mathfrak{m}} = \widehat{\mathcal{O}}_{\mathbb{D}^*,x}$.*

Theorem **3.3.** *For $f \in \mathrm{B}$, the non-zero finite slopes of* $\mathrm{Newt(f)}$ *are the* $-\log_q \|\mathfrak{m}\|$ *with multiplicity* $ord_{\mathfrak{m}}(f)\deg(\mathfrak{m})$ *where* $\mathfrak{m}$ *goes through the elements of* $|Y|$ *such that* $\theta_{\mathfrak{m}}(f)=0$.

Indications on the proof. It suffices to prove that for any finite non-zero slope λ of Newt(f) there exists $\mathfrak{m} \in |Y|$ such that

$$q^{-\lambda} = \|\mathfrak{m}\| \ \text{ and } \ \theta_{\mathfrak{m}}(f)=0.$$

As in proposition 2.9 there is a metric d on $|Y|$ such that for all $\rho \in]0,1[$, $\{\|\mathfrak{m}\| \geq \rho\}$ is complete. For $\mathfrak{m}_1, \mathfrak{m}_2 \in |Y|$, if $\theta_{\mathfrak{m}_1}(\mathfrak{m}_2) = \mathcal{O}_{L_{\mathfrak{m}_1}} x$ then

$$d(\mathfrak{m}_1,\mathfrak{m}_2) = q^{-v_{\mathfrak{m}_1}(x)/\deg \mathfrak{m}_2}.$$

We begin with the case when

$$f = \sum_{n\geq 0} [x_n]\pi^n \in W_{\mathcal{O}_E}(\mathcal{O}_F).$$

If $d \geq 0$ set

$$f_d = \sum_{n=0}^{d} [x_n]\pi^n.$$

For $d \gg 0$, λ appears as a slope of $\mathrm{Newt(f_d)}$ with the same multiplicity as in Newt(f). For each d, $f_d = [a_d].g_d$ for some $a_d \in \mathcal{O}_F$ and $g_d \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ primitive. Thanks to the preceding results, we already know the result for each g_d. Thus, setting

$$X_d = \{\mathfrak{m} \in |Y| \mid \|m\| = q^{-\lambda} \text{ and } \theta_{\mathfrak{m}}(f_d)=0\},$$

we know that for $d \gg 0$, $X_d \neq \emptyset$. Moreover $\#X_d$ is bounded when d varies. Now, if $\mathfrak{m} \in X_d$, looking at $v_{\mathfrak{m}}(\theta_{\mathfrak{m}}(f_{d+1}))$, one verifies that there exists $\mathfrak{m}' \in X_{d+1}$ such that

$$d(\mathfrak{m},\mathfrak{m}') \leq q^{-\frac{(d+1)\lambda - v(x_0)}{\#X_{d+1}}}.$$

From this one deduces there exists a Cauchy sequence $(\mathfrak{m}_d)_{d\gg 0}$ where $\mathfrak{m}_d \in X_d$. If $\mathfrak{m} = \lim_{d\to+\infty} \mathfrak{m}_d$ then $\theta_{\mathfrak{m}}(f)=0$. This proves the theorem when $f \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ and thus when $f \in \mathrm{B}^b$.

The general case is now obtained in the same way by approximating $f \in \mathrm{B}$ by a converging sequence of elements of B^b. □

Example 3.4. *As a corollary of the preceding theorem and proposition 1.14, for $f \in \mathrm{B}\setminus\{0\}$ one has $f \in \mathrm{B}^\times$ if and only if for all $\mathfrak{m} \in |Y|$, $\theta_{\mathfrak{m}}(f) \neq 0$.*

3.2. **A factorization of elements of** B **when** F **is algebraically closed.** Set

$$\begin{aligned}\mathrm{B}_{[0,1[} &= \{f \in \mathrm{B} \mid \exists A,\ \mathrm{Newt(f)}_{|]-\infty,A]} = +\infty\} \\ &= \Big\{ \sum_{n\gg -\infty} [x_n]\pi^n \in W_{\mathcal{O}_E}(F)\big[\tfrac{1}{\pi}\big] \ \Big|\ \liminf_{n\to+\infty} \frac{v(x_n)}{n} \geq 0 \Big\}\end{aligned}$$

(see prop.1.14 for this equality). Suppose F is algebraically closed. Given $f \in \mathrm{B}$, applying theorem 3.3, if $\lambda_1,\dots,\lambda_r > 0$ are some slopes of Newt(f) one can write

$$f = g.\prod_{i=1}^{r}\big(1 - \frac{[a_i]}{\pi}\big)$$

where $v(a_i) = \lambda_i$. Using this one proves the following.

Theorem **3.5.** *Suppose F is algebraically closed. For $f \in \mathrm{B}$ there exists a sequence $(a_i)_{i\geq 0}$ of elements of $\mathfrak{m}_F$ going to zero and $g \in \mathrm{B}_{[0,1[}$ such that*

$$f = g.\prod_{i=0}^{+\infty}\Big(1 - \frac{[a_i]}{\pi}\Big).$$

If moreover $f \in \mathrm{B}^+$ there exists such a factorization with $g \in W_{\mathcal{O}_E}(\mathcal{O}_F)$.

3.3. Divisors and closed ideals of B.

Definition 3.6. *Define*

$$\mathrm{Div}^+(Y) = \Big\{ \sum_{\mathfrak{m}\in|Y|} a_{\mathfrak{m}}[\mathfrak{m}] \ \big|\ \forall I \subset]0,1[\text{ compact } \{\mathfrak{m} \mid a_{\mathfrak{m}} \neq 0 \text{ and } \|\mathfrak{m}\| \in I\} \text{ is finite} \Big\}.$$

For $f \in \mathrm{B}\setminus\{0\}$ *set*

$$\mathrm{div}(f) = \sum_{\mathfrak{m}\in|Y|} ord_{\mathfrak{m}}(f)[\mathfrak{m}] \in \mathrm{Div}^+(Y).$$

Definition 3.7. *For* $D \in \mathrm{Div}^+(Y)$ *set* $\mathfrak{a}_{-D} = \{f \in \mathrm{B} \mid \mathrm{div}(f) \geq D\}$*, an ideal of* B.

For each $\mathfrak{m} \in |Y|$, the function $\mathrm{ord}_{\mathfrak{m}} : \mathrm{B} \to \mathbb{N}\cup\{+\infty\}$ is upper semi-continuous. From this one deduces that the ideal $\mathfrak{a}_{-D}$ is closed in B.

Theorem **3.8.** *The map* $D \mapsto \mathfrak{a}_{-D}$ *induces an isomorphism of monoids between* $\mathrm{Div}^+(Y)$ *and the monoid of closed non-zero ideals of* B*. Moreover,* $D \leq D'$ *if and only if* $\mathfrak{a}_{-D'} \subset \mathfrak{a}_{-D}$*.*

If $\mathfrak{a}$ is a closed ideal of B then

$$\mathfrak{a} \xrightarrow{\sim} \varprojlim_{I\subset]0,1[} \mathrm{B}_I\mathfrak{a}.$$

The result is thus a consequence of the following.

Theorem **3.9.** *For a compact non empty interval* $I \subset]0,1[$*:*

- *if* $I = \{\rho\}$ *with* $\rho \notin |F^\times|$ *then* B_I *is a field*
- *if not then the ring* B_I *is a principal ideal domain with maximal ideals* $\{\mathrm{B}_I\mathfrak{m} \mid \mathfrak{m} \in |Y|,\ \|\mathfrak{m}\| \in I\}$*.*

Sketch of proof. The proof of this theorem goes as follows. First, given $f \in \mathrm{B}_I$, one can define a bounded Newton polygon

$$\mathrm{Newt_I(f)}.$$

If $f \in \mathrm{B}^b$ then $\mathrm{Newt_I(f)}$ is obtained from $\mathrm{Newt(f)}$ by removing the slopes λ such that $q^{-\lambda} \notin I$ (if there is no such slope we define $\mathrm{Newt_I(f)}$ as the empty polygon). Now, if $f \in \mathrm{B}$ the method used in definition 1.11 to define the Newton polygon does not apply immediately. For example, if $I = \{\rho\}$ and $f \in \mathrm{B}^b$ then $|f|_\rho$ does not determine $\mathrm{Newt_I(f)}$ (more generally, if $I = [q^{-\lambda_1}, q^{-\lambda_2}]$ one has the same problem with the definition of the pieces of $\mathrm{Newt_I(f)}$ where the slopes are λ_1 and λ_2). But one verifies that if $f_n \underset{n\to+\infty}{\longrightarrow} f$ in B_I with $f_n \in \mathrm{B}^b$ then for $n \gg 0$, $\mathrm{Newt_I(f_n)}$ is constant and does not depend on the sequence of B^b going to f.

Then, if $f \in \mathrm{B}_I$ we prove a theorem that is analogous to theorem 3.3: the slopes of $\mathrm{Newt_I(f)}$ are the $-\log_q \|\mathfrak{m}\|$ with multiplicity $\mathrm{ord}_{\mathfrak{m}}(f)\deg\mathfrak{m}$ where $\|\mathfrak{m}\| \in I$ and $\theta_{\mathfrak{m}}(f) = 0$. This gives a factorization of any $f \in \mathrm{B}_I$ as a product

$$f = g.\prod_{i=1}^r \xi_i$$

where the ξ_i are irreducible primitive, $\|\xi_i\| \in I$, and $g \in \mathrm{B}_I$ satisfies $\mathrm{Newt_I(g)} = \emptyset$.

Finally, we prove that if $f \in \mathrm{B}_I$ satisfies $\mathrm{Newt_I(f)} = \emptyset$ then $f \in \mathrm{B}_I^\times$. For $f \in \mathrm{B}^b$ this is verified by elementary manipulations. Then if $f_n \underset{n\to+\infty}{\longrightarrow} f$ in B_I with $f_n \in \mathrm{B}^b$, since $\mathrm{Newt_I(f)} = \emptyset$ for $n \gg 0$ one has $\mathrm{Newt_I(f_n)} = \emptyset$. But then for $n \gg 0$ and $\rho \in I$,

$$|f_{n+1}^{-1} - f_n^{-1}|_\rho = |f_{n+1}|_\rho^{-1}.|f_n|_\rho^{-1}.|f_{n+1} - f_n|_\rho \underset{n\to+\infty}{\longrightarrow} 0.$$

Thus the sequence $(f_n^{-1})_{n\gg 0}$ of B_I converges towards an inverse of f. □

Example 3.10. *For* $f, g \in \mathrm{B}\setminus\{0\}$*,* f *is a multiple of* g *if and only if* $\mathrm{div}(f) \geq \mathrm{div}(g)$*. In particular there is an injection of monoids*

$$\mathrm{div} : \mathrm{B}\setminus\{0\}/\mathrm{B}^\times \hookrightarrow \mathrm{Div}^+(Y).$$

Corollary **3.11.** *The set $|Y|$ is the set of closed maximal ideals of* B*.*

Remark 3.12. *Even when F is spherically complete we do not know whether* $\mathrm{div} : \mathrm{B}^\times \to \mathrm{Div}^+(Y)$ *is surjective or not (see 1.2).*

4. Divisors on $Y/\varphi^{\mathbb{Z}}$

4.1. **Motivation.** Suppose we want to classify φ-modules over B, that is to say free B-modules equipped with a φ-semi-linear automorphism. This should be the same as vector bundles on

$$Y/\varphi^{\mathbb{Z}}$$

where Y is this "rigid" space we did not really define but that should satisfy

- $\Gamma(Y, \mathcal{O}_Y) = \mathrm{B}$
- $|Y|$ is the set of "classical points" of Y.

Whatever this space Y is, since $\|\varphi(\mathfrak{m})\| = \|\mathfrak{m}\|^q$, φ acts in a proper discontinuous way without fixed point on it. Thus, $Y/\varphi^{\mathbb{Z}}$ should have a sense as a "rigid" space. Let's look in more details at what this space $Y/\varphi^{\mathbb{Z}}$ should be.

It is easy to classify rank 1 φ-modules over B. They are parametrized by $\mathbb{Z}$: to $n \in \mathbb{Z}$ one associates the φ-module with basis e such that $\varphi(e) = \pi^n e$. We thus should have

$$\begin{array}{ccc} \mathbb{Z} & \xrightarrow{\sim} & \mathrm{Pic}(Y/\varphi^{\mathbb{Z}}) \\ n & \longmapsto & \mathscr{L}^{\otimes n} \end{array}$$

where $\mathscr{L}$ is a line bundle such that for all $d \in \mathbb{Z}$,

$$H^0(Y/\varphi^{\mathbb{Z}}, \mathscr{L}^{\otimes d}) = \mathrm{B}^{\varphi = \pi^d}.$$

If $E = \mathbb{F}_q((\pi))$ and F is algebraically closed Hartl and Pink classified in [20] the φ-modules over B, that is to say φ-equivariant vector bundles on $\mathbb{D}^*$. The first step in the proof of their classification ([20] theo.4.3) is that if (M, φ) is such a φ-module then for $d \gg 0$

$$M^{\varphi = \pi^d} \neq 0.$$

The same type of result appears in the context of φ-modules over the Robba ring in the work of Kedlaya (see for example [22] prop.2.1.5). From this one deduces that the line bundle $\mathscr{L}$ should be ample. We are thus led to study the scheme

$$\mathrm{Proj}\Big(\bigoplus_{d \geq 0} \mathrm{B}^{\varphi = \pi^d} \Big)$$

for which one hopes it is "uniformized by Y" and allows us to study φ-modules over B. In fact if (M, φ) is such a φ-module, we hope the quasi-coherent sheaf

$$\widetilde{\Big(\bigoplus_{d \geq 0} M^{\varphi = \pi^d} \Big)}$$

is the vector bundle associated to the φ-equivariant vector bundle on Y attached to (M, φ).

4.2. **Multiplicative structure of the graded algebra P.**

Definition 4.1. *Define*

$$P = \bigoplus_{d \geq 0} \mathrm{B}^{\varphi = \pi^d}$$

as a graded E-algebra. We note $P_d = \mathrm{B}^{\varphi = \pi^d}$ the degree d homogeneous elements.

In fact we could replace B by B^+ in the preceding definition since

$$\mathrm{B}^{\varphi = \pi^d} = (\mathrm{B}^+)^{\varphi = \pi^d}$$

(coro. 1.15). One has $P_0 = E$.

Definition 4.2. *Define*

$$\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}) = \{D \in \mathrm{Div}^+(Y) \mid \varphi^* D = D\}.$$

There is an injection

$$\begin{array}{ccc} |Y|/\varphi^{\mathbb{Z}} & \hookrightarrow & \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}}) \\ \mathfrak{m} & \longrightarrow & \sum_{n\in\mathbb{Z}} [\varphi^n(\mathfrak{m})] \end{array}$$

that makes $\mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$ *a free abelian monoid on* $|Y|/\varphi^{\mathbb{Z}}$. If $x \in \mathrm{B}^{\varphi=\pi^d} \setminus \{0\}$ then $\mathrm{div}(x) \in \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$.

Theorem **4.3.** *If F is algebraically closed the morphism of monoids*

$$\mathrm{div} : \Big(\bigcup_{d\geq 0} P_d \setminus \{0\}\Big)/E^\times \longrightarrow \mathrm{Div}^+(Y/\varphi^{\mathbb{Z}})$$

is an isomorphism.

Let us note the following important corollary.

Corollary **4.4.** *If F is algebraically closed the graded algebra P is graded factorial with irreducible elements of degree* 1.

In the preceding theorem, the injectivity is an easy application of theorem 3.8. In fact, if $x \in P_d$ and $y \in P_{d'}$ are non zero elements such that $\mathrm{div}(x) = \mathrm{div}(y)$ then $x = uy$ with $u \in \mathrm{B}^\times$. But $\mathrm{B}^\times = (\mathrm{B}^b)^\times$ (see the comment after proposition 1.14). Thus,

$$u \in (\mathrm{B}^b)^{\varphi=\pi^{d-d'}} = \begin{cases} 0 \text{ if } d \neq d' \\ E \text{ if } d = d'. \end{cases}$$

The surjectivity uses Weierstrass products. For this, let $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ be a primitive degree d element and $D = \mathrm{div}(x)$ its divisor. We are looking for $f \in P_d \setminus \{0\}$ satisfying

$$\mathrm{div}(f) = \sum_{n\in\mathbb{Z}} \varphi^n(D).$$

Up to multiplying x by a unit we can suppose

$$x \in \pi^d + W_{\mathcal{O}_E}(\mathfrak{m}_F).$$

Then the infinite product

$$\Pi^+(x) = \prod_{n\geq 0} \Big(\frac{\varphi^n(x)}{\pi^d}\Big)$$

converges. For example, if $x = \pi - [a]$,

$$\Pi^+(\pi - [a]) = \prod_{n\geq 0} \Big(1 - \frac{[a^{q^n}]}{\pi}\Big).$$

One has

$$\mathrm{div}(\Pi^+(x)) = \sum_{n\geq 0} \varphi^n(D).$$

We then would like to define

$$\text{“}\Pi^-(x) = \prod_{n<0} \varphi^n(x)\text{“}$$

and then set

$$\Pi(x) = \Pi^+(x).\Pi^-(x)$$

that would satisfy $\Pi(x) \in P_d$ and $\mathrm{div}(\Pi(x)) = \sum_{n\in\mathbb{Z}} \varphi^n(D)$. But the infinite product defining $\Pi^-(x)$ does not converge. Nevertheless let us remark it satisfies formally the functional equation

$$\varphi(\Pi^-(x)) = x\Pi^-(x).$$

Moreover, if $a = x \bmod \pi$, $a \in \mathfrak{m}_F$, if we are trying to define $\Pi^-(x)$ modulo π, one should have formally

$$\prod_{n<0} \varphi^n(x) \bmod \pi = \prod_{n<0} a^{q^n} = a^{\sum_{n<0} q^n} = a^{\frac{1}{q-1}}.$$

This means that up to an $\mathbb{F}_q^\times$-multiple one would like to define $\Pi^-(x)$ modulo π as a solution of the Kümmer equation $X^{q-1} - a = 0$. Similarly, for an element $y \in 1 + \pi^k W_{\mathcal{O}_E}(\mathcal{O}_F)$ where $k \geq 1$, via the identification

$$1 + \pi^k W_{\mathcal{O}_E}(\mathcal{O}_F)/1 + \pi^{k+1} W_{\mathcal{O}_E}(\mathcal{O}_F) \xrightarrow{\sim} \mathcal{O}_F$$

if

$$y \bmod 1 + \pi^k W_{\mathcal{O}_E}(\mathcal{O}_F) \longmapsto b$$

one would have formally

$$\prod_{n<0} \varphi^n(y) \bmod 1 + \pi^{k+1} W_{\mathcal{O}_E}(\mathcal{O}_F) \longmapsto \sum_{n<0} b^{q^{-n}}$$

that is formally a solution of the Artin-Schreier equation $X^q - X - b = 0$ (the remark that one can write solutions of Artin-Schreier equations in F as such non-converging series is due to Abhyankar, see [26]).

In fact, we have the following easy proposition whose proof is by successive approximations, solving first a Kümmer and then Artin-Schreier equations.

Proposition **4.5.** *Suppose F is algebraically closed and let $z \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ be a primitive element. Up to an $E^\times$-multiple there is a unique $\Pi^-(z) \in \mathrm{B}^{b,+} \setminus \{0\}$ such that $\varphi(\Pi^-(z)) = z\Pi^-(z)$.*

Define $\Pi^-(x)$ using the preceding proposition. It is well defined up to an $E^\times$-multiple. Moreover

$$\begin{aligned} \varphi(\Pi^-(x)) = x\Pi^-(x) \quad &\Longrightarrow \quad \varphi\big(\mathrm{div}(\Pi^-(x))\big) = \mathrm{div}(\Pi^-(x)) + \underbrace{\mathrm{div}(x)}_{D} \\ &\Longrightarrow \quad \mathrm{div}(\Pi^-(x)) = \sum_{n<0} \varphi^n(D). \end{aligned}$$

Setting $\Pi(x) = \Pi^+(x)\Pi^-(x)$, this is a solution to our problem:

- $\Pi(x) \in P_d \setminus \{0\}$
- $\mathrm{div}(\Pi(x)) = \sum_{n\in\mathbb{Z}} \varphi^n(D)$.

4.3. **Weierstrass products and the logarithm of a Lubin-Tate group.** We use the notations from section 2.4. For $\epsilon \in \mathfrak{m}_F \setminus \{0\}$ and $u_\epsilon = \dfrac{[\epsilon]_Q}{[\epsilon^{1/q}]_Q}$ one has

$$\varphi^n(u_\epsilon) = \frac{[\pi^n]_{\mathcal{LT}}([\epsilon]_Q)}{[\pi^{n-1}]_{\mathcal{LT}}([\epsilon]_Q)}$$

and thus

$$\begin{aligned} \Pi^+(u_\epsilon) &= \prod_{n\geq 0} \Big(\frac{\varphi^n(u_\epsilon)}{\pi}\Big) \\ &= \frac{1}{\pi[\epsilon^{1/q}]_Q} \cdot \lim_{n\to+\infty} \pi^{-n}[\pi^n]_{\mathcal{LT}}\big([\epsilon]_Q\big) \\ &= \frac{1}{\pi[\epsilon^{1/q}]_Q} \log_{\mathcal{LT}}\big([\epsilon]_Q\big) \end{aligned}$$

where $\log_{\mathcal{LT}}$ is the logarithm of the Lubin-Tate group law $\mathcal{LT}$. Moreover, one can take

$$\Pi^-(u_\epsilon) = \pi[\epsilon^{1/q}]_Q$$

and thus

$$\Pi(u_\epsilon) = \log_{\mathcal{LT}}([\epsilon]_Q).$$

Thus, the Weierstrass product $\Pi(u_\epsilon)$ is given by the Weierstrass product expansion of $\log_{\mathcal{LT}}$ (see the end of section 1.1.3). In fact we have the following period isomorphism.

Theorem **4.6.** *The logarithm induces an isomorphism of E-Banach spaces*

$$\begin{array}{ccc} \left(\mathfrak{m}_F, \underset{\mathcal{LT}}{+}\right) & \xrightarrow{\sim} & \mathrm{B}^{\varphi=\pi} \\ \epsilon & \longmapsto & \log_{\mathcal{LT}}\left([\epsilon]_Q\right). \end{array}$$

Remark 4.7. *For $r, r' > 0$ the restrictions of v_r and $v_{r'}$ to $\mathrm{B}^{\varphi=\pi}$ induce equivalent norms. This equivalence class of norms defines the Banach space topology of the preceding theorem. This is the same topology as the one induced by the embedding $\mathrm{B}^{\varphi=\pi^d} \subset \mathrm{B}$.*
The Banach space topology on $(\mathfrak{m}_F, \underset{\mathcal{LT}}{+})$ is the one defined by the lattice $1 + \{x \in \mathfrak{m}_F \mid v(x) \geq r\}$ for any $r > 0$.

One has the formula

$$\log_{\mathcal{LT}}\left([\epsilon]_Q\right) = \lim_{n \to +\infty} \pi^n \log_{\mathcal{LT}}\left([\epsilon^{q^{-n}}]\right).$$

If $\mathcal{LT}$ is the Lubin-Tate group law whose logarithm is

$$\log_{\mathcal{LT}} = \sum_{n \geq 0} \frac{T^{q^n}}{\pi^n}$$

we then have the formula

$$\log_{\mathcal{LT}}\left([\epsilon]_Q\right) = \sum_{n \in \mathbb{Z}} \left[\epsilon^{q^{-n}}\right] \pi^n.$$

Remark 4.8. *The preceding series $\sum_{n \in \mathbb{Z}} \left[\epsilon^{q^{-n}}\right] \pi^n$ is a Witt bivector, an element of $\mathrm{BW}_{\mathcal{O}_E}(\mathcal{O}_F)$ (see the end of section 1.2.1). The fact that such a series makes sense in the Witt bivectors is an essential ingredient in the proof of theorem 4.6. For $d > 1$ we don't have such a description of the Banach space $\mathrm{B}^{\varphi=\pi^d}$.*

Suppose now $E = \mathbb{Q}_p$ and let $\mathcal{LT}$ be the formal group law with logarithm $\sum_{n \geq 0} \frac{T^{p^n}}{p^n}$. Let

$$E(T) = \exp\Big(\sum_{n \geq 0} \frac{T^{p^n}}{p^n} \Big) \in \mathbb{Z}_p[\![T]\!]$$

be the Artin-Hasse exponential:

$$E : \mathcal{LT} \xrightarrow{\sim} \widehat{\mathbb{G}}_m.$$

There is then a commutative diagram of isomorphisms

$$\begin{array}{ccc} \left(\mathfrak{m}_F, \underset{\mathcal{LT}}{+}\right) & \xrightarrow{\sim} & \mathrm{B}^{\varphi=p} \\ E \Big\downarrow \simeq & \nearrow_{\log \circ [\cdot]}^{\simeq} & \\ \left(1 + \mathfrak{m}_F, \times\right) & & \end{array}$$

where the horizontal map is $\epsilon \mapsto \sum_{n \in \mathbb{Z}} [\epsilon^{p^{-n}}] p^n$. We thus find back the usual formula: $t = \log[\epsilon]$ for $\epsilon \in 1 + \mathfrak{m}_F$. More precisely, for $\epsilon \in 1 + \mathfrak{m}_F$ and the group law $\widehat{\mathbb{G}}_m$ one has

$$u_{\epsilon - 1} = 1 + \left[\epsilon^{\frac{1}{p}}\right] + \cdots + \left[\epsilon^{\frac{p-1}{p}}\right].$$

If $\mathfrak{m} = (u_{\epsilon-1})$ then $\epsilon \in \mathscr{R}(C_\mathfrak{m})$ is a generator of $\mathbb{Z}_p(1)$. Moreover if $\rho = |\epsilon - 1|^{1-1/p}$ we have

$$\mathrm{B}^+_{cris}(C_\mathfrak{m}) = \mathrm{B}^+_{cris,\rho}$$

where $\mathrm{B}^+_{cris}(C_\mathfrak{m})$ is the crystalline ring of periods attached to $C_\mathfrak{m}$ ([16]) and $\mathrm{B}^+_{cris,\rho}$ is the ring defined at the end of section 1.2.1. Then $t = \log[\epsilon]$ is the usual period of μ_{p^∞} over $C_\mathfrak{m}$.

5. The curve

5.1. **The fundamental exact sequence.**

Using the results from the preceding section we give a new proof of the fundamental exact sequence. In fact this fundamental exact sequence is a little bit more general than the usual one. If $t \in P_1 \setminus \{0\}$ we will say t is associated to $\mathfrak{m} \in |Y|$ if $\mathrm{div}(t) = \sum_{n\in\mathbb{Z}} [\varphi^n(\mathfrak{m})]$.

Theorem **5.1.** *Suppose F is algebraically closed. Let $t_1, \dots, t_n \in P_1$ be associated to $\mathfrak{m}_1, \dots, \mathfrak{m}_n \in |Y|$ and such that for $i \neq j$, $t_i \notin E t_j$. Let $a_1, \dots, a_n \in \mathbb{N}_{\geq 1}$ and set $d = \sum_i a_i$. Then for $r \geq 0$ there is an exact sequence*

$$0 \longrightarrow P_r.\prod_{i=1}^n t_i^{a_i} \longrightarrow P_{d+r} \xrightarrow{u} \prod_{i=1}^n \mathrm{B}^+_{dR,\mathfrak{m}_i}/\mathrm{B}^+_{dR,\mathfrak{m}_i}\mathfrak{m}_i^{a_i} \longrightarrow 0.$$

Proof. For $x \in P_{d+r}$, $u(x) = 0$ if and only if

$$\mathrm{div}(x) \geq \sum_{i=1}^n a_i[\mathfrak{m}_i].$$

But since $\mathrm{div}(x)$ is invariant under φ this is equivalent to

$$\mathrm{div}(x) \geq \sum_{i=1}^n a_i \sum_{n\in\mathbb{Z}} [\varphi^n(\mathfrak{m}_i)] = \mathrm{div}\Big(\prod_{i=1}^n t_i^{a_i}\Big).$$

According to theorem 3.8 this is equivalent to

$$x = y.\prod_{i=1}^n t_i^{a_i}$$

for some $y \in \mathrm{B}$ (see example 3.10). But such an y satisfies automatically $\varphi(y) = \pi^r y$.

By induction, the surjectivity of u reduces to the case $n = 1$ and $a_1 = 1$. Let us note $\mathfrak{m} = \mathfrak{m}_1$. We have to prove that the morphism $\mathrm{B}^{\varphi=\pi} \xrightarrow{\theta_\mathfrak{m}} C_\mathfrak{m}$ is surjective. Note $\mathcal{G}$ the formal group associated to the Lubin-Tate group law $\mathcal{LT}$. We use the isomorphism

$$\mathcal{G}(\mathcal{O}_F) \xrightarrow{\sim} \mathrm{B}^{\varphi=\pi}$$

of theorem 4.6 together with the isomorphism

$$X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}) \xrightarrow{\sim} \mathcal{G}(\mathcal{O}_F)$$

of section 2.4. One verifies the composite

$$X(\mathcal{G})(\mathcal{O}_{C_\mathfrak{m}}) \longrightarrow \mathcal{G}(\mathcal{O}_F) \longrightarrow \mathrm{B}^{\varphi=\pi} \xrightarrow{\theta_\mathfrak{m}} C_\mathfrak{m}$$

is given by

$$\big(x^{(n)}\big)_{n\geq 0} \longmapsto \log_{\mathcal{LT}}\big(x^{(0)}\big).$$

We conclude since $C_\mathfrak{m}$ is algebraically closed. □

We will use the following corollary.

Corollary **5.2.** *Suppose F is algebraically closed. For $t \in P_1 \setminus \{0\}$ associated to $\mathfrak{m} \in |Y|$ there is an isomorphism of graded E-algebras*

$$\begin{array}{rcl} P/tP & \xrightarrow{\sim} & \{f \in C_\mathfrak{m}[T] \mid f(0) \in E\} \\ \sum_{d\geq 0} x_d \ mod\ tP & \longmapsto & \sum_{d\geq 0} \theta_\mathfrak{m}(x_d) T^d. \end{array}$$

5.2. **The curve when F is algebraically closed.**

Theorem **5.3.** *Suppose F is algebraically closed. The scheme $X = Proj(P)$ is an integral noetherian regular scheme of dimension 1. Moreover:*

(1) *For $t \in P_1 \setminus \{0\}$, $D^+(t) = Spec\big(P\big[\frac{1}{t}\big]_0\big)$ where $P\big[\frac{1}{t}\big]_0 = \mathrm{B}\big[\frac{1}{t}\big]^{\varphi=\mathrm{Id}}$ is a principal ideal domain.*

(2) *For $t \in P_1 \setminus \{0\}$, $V^+(t) = \{\infty_t\}$ with ∞_t a closed point of X and if t is associated to $\mathfrak{m} \in |Y|$ there is a canonical identification of D.V.R.'s*

$$\widehat{\mathcal{O}}_{X,\infty_t} = \mathrm{B}^+_{dR,\mathfrak{m}}.$$

(3) *If $|X|$ stands for the set of closed points of X, the application*

$$\begin{array}{rcl} \big(P_1 \setminus \{0\}\big)/E^\times & \longrightarrow & |X| \\ E^\times t & \longmapsto & \infty_t \end{array}$$

is a bijection.

(4) *Let us note $E(X)$ the field of rational functions on X, that is to say the stalk of $\mathcal{O}_X$ at the generic point. Then, for all $f \in E(X)^\times$ one has*

$$\deg(\mathrm{div}(f)) = 0$$

where for $x \in |X|$ we set $\deg(x) = 1$.

Sketch of proof. As a consequence of corollary 4.4 the ring $\mathrm{B}_e := P\big[\frac{1}{t}\big]_0$ is factorial with irreducible elements the $\frac{t'}{t}$ where $t' \notin Et$. To prove it is a P.I.D. it thus suffices to verify those irreducible elements generate a maximal ideal. But for such a $t' \notin Et$, if t' is associated to $\mathfrak{m}' \in |Y|$ since $\theta_{\mathfrak{m}'}(t) \neq 0$, $\theta_{\mathfrak{m}'}$ induces a morphism $\mathrm{B}_e \to C_{\mathfrak{m}'}$. Using the fundamental exact sequence one verifies it is surjective with kernel the principal ideal generated by $\frac{t'}{t}$.

Now, if $A = \{f \in C_{\mathfrak{m}}[T] \mid f(0) \in E\}$, one verifies Proj$(A)$ has only one element, the homogeneous prime ideal (0). Using corollary 5.2 one deduces that $V^+(t) \simeq \mathrm{Proj}(A)$ is one closed point of X.

We have the following description

$$\mathcal{O}_{X,\infty_t} = \Big\{\frac{x}{y} \in \mathrm{Frac}(P) \mid x \in P_d,\ y \in P_d \setminus tP_{d-1} \text{ for some } d \geq 0\Big\}$$

with uniformizing element $\frac{t}{t'}$ for some $t' \in P_1 \setminus Et$. Now, if $y \in P_d \setminus tP_{d-1}$, $y \in (\mathrm{B}^+_{dR,\mathfrak{m}})^\times$ since according to the fundamental exact sequence $\theta_{\mathfrak{m}}(y) \neq 0$. We thus have

$$\mathcal{O}_{X,\infty_t} \subset \mathrm{B}^+_{dR,\mathfrak{m}}.$$

Using the fundamental exact sequence one verifies this embedding of D.V.R.'s induces an isomorphism at the level of the residue fields. Moreover a uniformizing element of $\mathcal{O}_{X,\infty_t}$ is a uniformizing element of $\mathrm{B}^+_{dR,\mathfrak{m}}$. It thus induces $\widehat{\mathcal{O}}_{X,\infty} \xrightarrow{\sim} \mathrm{B}^+_{dR,\mathfrak{m}}$.

Other assertions of the theorem are easily verified. □

The following proposition makes clear the difference between X and $\mathbb{P}^1$ and will have important consequences on the classification of vector bundles on X. It is deduced from corollary 5.2.

Proposition **5.4.** *For a closed point $\infty \in |X|$ let $\mathrm{B}_e = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X) \subset E(X)$, v_∞ the valuation on $E(X)$ associated to ∞ and*

$$\deg = -v_{\infty|\mathrm{B}_e} : \mathrm{B}_e \longrightarrow \mathbb{N} \cup \{-\infty\}.$$

Then the couple $(\mathrm{B}_e, \deg)$ is almost euclidean in the sense that

$$\forall x, y \in \mathrm{B}_e,\ y \neq 0,\ \exists a, b \in \mathrm{B}_e \ \ x = ay + b \text{ and } \deg(b) \leq \deg(y).$$

Moreover $(\mathrm{B}_e, \deg)$ is not euclidean.

5.3. **The curve in general.** Note $\overline{F}$ an algebraic closure of F and $G_F = \mathrm{Gal}(\overline{F}|F)$. We put subscripts to indicate the dependence on the field F of the preceding constructions. The curve $X_{\widehat{\overline{F}}}$ of the preceding section is equipped with an action of G_F.

Theorem **5.5.** *The scheme $X_F = Proj(P_F)$ is an integral noetherian regular scheme of dimension 1. It satisfies the following properties.*

(1) The morphism of graded algebras $P_F \to P_{\widehat{\overline{F}}}$ induces a morphism

$$\alpha : X_{\widehat{\overline{F}}} \longrightarrow X_F$$

satisfying:

- *for $x \in |X_F|$, $\alpha^{-1}(x)$ is a finite set of closed points of $|X_{\widehat{\overline{F}}}|$*
- *for $x \in |X_{\widehat{\overline{F}}}|$:*
  - *if $G_F.x$ is infinite then $\alpha(x)$ is the generic point of X_F,*
  - *if $G_F.x$ is finite then $\alpha(x)$ is a closed point of X_F.*
- *it induces a bijection*

$$|X_{\widehat{\overline{F}}}|^{G_F-\mathrm{fin}}/G_F \xrightarrow{\sim} |X_F|$$

where $|X_{\widehat{\overline{F}}}|^{G_F-\mathrm{fin}}$ is the set of closed points with finite G_F-orbit.

(2) For $x \in |X_F|$ set $\deg(x) = \#\alpha^{-1}(x)$. Then for $f \in E(X_F)^\times$

$$\deg(\mathrm{div}(f)) = 0.$$

(3) For $\mathfrak{m} \in |Y_F|$ define

$$\mathfrak{p}_{\mathfrak{m}} = \Big\{ \sum_{d \geq \deg \mathfrak{m}} x_d \in P_F \mid x_d \in P_{F,d},\ \mathrm{div}(x_d) \geq \sum_{n\in\mathbb{Z}} [\varphi^n(\mathfrak{m})] \Big\},$$

a prime homogeneous ideal of P. Then

$$\begin{array}{ccc} |Y_F|/\varphi^{\mathbb{Z}} & \xrightarrow{\sim} & |X_F| \\ \varphi^{\mathbb{Z}}(\mathfrak{m}) & \longmapsto & \mathfrak{p}_{\mathfrak{m}} \end{array}$$

and there is an identification $\widehat{\mathcal{O}}_{X_F,\mathfrak{p}_{\mathfrak{m}}} = \mathrm{B}^+_{F,dR,\mathfrak{m}}$.

Sketch of proof. Let us give a few indications on the tools used in the proof.

Proposition **5.6.** *One has $P_{\widehat{\overline{F}}}^{G_F} = P_F$.*

Proof. The divisor of $f \in P_{\widehat{\overline{F}},d}^{G_F}$ being G_F-invariant, there exists a primitive degree d element $x \in W_{\mathcal{O}_E}(\mathcal{O}_{\widehat{\overline{F}}})$ such that $\mathrm{div}(f) = \sum_{n\in\mathbb{Z}} \varphi^n(\mathrm{div}(x))$ and for all $\sigma \in G_F$, $(\sigma(x)) = (x)$. According to proposition 2.18, one can choose $x \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ and even $x \in \pi^d + W_{\mathcal{O}_E}(\mathfrak{m}_F)$. The Weierstrass product

$$\Pi^+(x) = \prod_{n\geq 0} \Big(\frac{\varphi^n(x)}{\pi^d} \Big)$$

is convergent in B_F. Applying theorem 3.8 (see example 3.10) one finds there exists $g \in \mathrm{B}_{\widehat{\overline{F}},[0,1[}$ (see 3.2) such that

$$f = \Pi^+(x).g$$

Of course, g is G_F-invariant and one concludes since

$$\mathrm{B}_{\widehat{\overline{F}},[0,1[}^{G_F} = \mathrm{B}_{F,[0,1[}.$$

□

Let now $t \in P_{F,1} \setminus \{0\}$ and look at

$$\begin{aligned} \mathrm{B}_{F,e} &= \mathrm{B}_F\left[\tfrac{1}{t}\right]^{\varphi=\mathrm{Id}} = \Gamma(X_F \setminus V^+(t), \mathcal{O}_{X_F}) \\ \mathrm{B}_{\widehat{\overline{F}},e} &= \mathrm{B}_{\widehat{\overline{F}}}\left[\tfrac{1}{t}\right]^{\varphi=\mathrm{Id}} = \Gamma(X_{\widehat{\overline{F}}} \setminus V^+(t), \mathcal{O}_{X_{\widehat{\overline{F}}}}). \end{aligned}$$

According to the preceding proposition

$$\mathrm{B}_{F,e} = (\mathrm{B}_{\widehat{\overline{F}},e})^{G_F}.$$

We want to prove $\mathrm{B}_{F,e}$ is a Dedekind ring such that the maps $I \mapsto \mathrm{B}_{\widehat{\overline{F}},e} I$ and $J \mapsto J \cap \mathrm{B}_{F,e}$ are inverse bijections between non zero ideals of $\mathrm{B}_{F,e}$ and non-zero G_F-invariant ideals of $\mathrm{B}_{\widehat{\overline{F}},e}$. The key tool is the following cohomological computation.

Theorem **5.7.** *For $\chi : G_F \to E^\times$ a continuous character one has*

$$H^1(G_F, \mathrm{B}_{\widehat{\overline{F}},e}(\chi)) = 0$$
$$H^0\big(G_F, \mathrm{B}_{\widehat{\overline{F}},e}(\chi)\big) \neq 0$$

where

$$H^1(G_F, \mathrm{B}_{\widehat{\overline{F}},e}(\chi)) := \varinjlim_{d\geq 0} H^1\big(G_F, t^{-d} P_{\widehat{\overline{F}},d}(\chi)\big)$$

and $t^{-k} P_{\widehat{\overline{F}},d}$ is naturally an E-Banach space.

Proof. We prove that for all $d \geq 1$, $H^1(G_F, P_{\widehat{\overline{F}},d}(\chi)) = 0$. Let $\mathfrak{m} \in |Y_F|$ be associated to t. Note $\mathfrak{m}' = \mathrm{B}_{\widehat{\overline{F}}}\mathfrak{m} \in |Y_{\widehat{\overline{F}}}|$ the unique element such that $\beta(\mathfrak{m}') = \mathfrak{m}$. For $d > 1$, using the fundamental exact sequence

$$0 \longrightarrow P_{\widehat{\overline{F}},d-1}(\chi) \xrightarrow{\times t} P_{\widehat{\overline{F}},d}(\chi) \xrightarrow{\theta_{\mathfrak{m}'}} C_{\mathfrak{m}'}(\chi) \longrightarrow 0$$

of G_F-modules together with the vanishing

$$H^1(G_F, C_{\mathfrak{m}'}(\chi)) = 0$$

(theorem 2.27) one is reduced by induction to prove the case $d = 1$. Let $\mathcal{LT}$ be a Lubin-Tate group law. We have an isomorphism

$$P_{\widehat{\overline{F}},1} \simeq \Big(\mathfrak{m}_{\widehat{\overline{F}}}, \underset{\mathcal{LT}}{+}\Big).$$

For $r \in v(F^\times)_{>0}$ set

$$\mathfrak{m}_{\widehat{\overline{F}},r} = \{x \in \mathfrak{m}_{\widehat{\overline{F}}} \mid v(x) > r\}.$$

It defines a decreasing filtration of the Banach space $\Big(\mathfrak{m}_{\widehat{\overline{F}}}, \underset{\mathcal{LT}}{+}\Big)$ by sub $\mathcal{O}_E$-modules. Moreover

$$\Big(\mathfrak{m}_{\widehat{\overline{F}},r}/\mathfrak{m}_{\widehat{\overline{F}},2r}, \underset{\mathcal{LT}}{+}\Big) = \big(\mathfrak{m}_{\widehat{\overline{F}},r}/\mathfrak{m}_{\widehat{\overline{F}},2r}, +\big).$$

It thus suffices to prove that for all $r \in \mathbb{Q}_{>0}$,

$$H^1\big(G_F, \mathfrak{m}_{\widehat{\overline{F}},r}/\mathfrak{m}_{\widehat{\overline{F}},2r}(\chi)\big) = 0$$

(discrete Galois cohomology). This is deduced from theorem 2.16 which implies that all $r \in v(F^\times)_{>0}$ and $i > 0$,

$$H^i\big(G_F, \mathfrak{m}_{\widehat{\overline{F}},r}(\chi)\big) = 0.$$

To prove that $H^0(G_F, \mathrm{B}_{\widehat{\overline{F}},e}(\chi)) \neq 0$ it suffices to prove $H^0(G_F, P_{\widehat{\overline{F}},1}(\chi)) \neq 0$. One checks easily that $H^0(G_F, \widehat{\overline{F}}(\chi)) \neq 0$ and thus for all $r \in v(F^\times)_{>0}$,

$$H^0\big(G_F, \mathfrak{m}_{\widehat{\overline{F}},r}/\mathfrak{m}_{\widehat{\overline{F}},2r}(\chi)\big) \neq 0.$$

Using the vanishing

$$H^1\big(G_F, \big(\mathfrak{m}_{\widehat{\overline{F}},2r}, \underset{\mathcal{LT}}{+}\big)(\chi)\big) = 0$$

one deduces the morphism

$$H^0(G_F, \big(\mathfrak{m}_{\widehat{\overline{F}},r}, \underset{\mathcal{LT}}{+}\big)(\chi)) \longrightarrow H^0\big(G_F, \mathfrak{m}_{\widehat{\overline{F}},r}/\mathfrak{m}_{\widehat{\overline{F}},2r}(\chi)\big)$$

is surjective and concludes. □

Let now $f \in \mathrm{B}_{F,e}$. Since $H^1\big(G_F, \mathrm{B}_{\widehat{\overline{F}},e}\big) = 0$ one has

$$\mathrm{B}_{F,e}/\mathrm{B}_{F,e}f = \Big(\mathrm{B}_{\widehat{\overline{F}},e}/\mathrm{B}_{\widehat{\overline{F}},e}f\Big)^{G_F}.$$

Let

$$f = u.\prod_{i=1}^r f_i^{a_i}$$

be the decomposition of f in prime factors where $u \in \mathrm{B}^\times_{\widehat{\overline{F}},e} = E^\times$. If f_i is associated to $\mathfrak{m}_i \in |Y_{\widehat{\overline{F}}}|$ then

$$\mathrm{B}_{\widehat{\overline{F}},e}/\mathrm{B}_{\widehat{\overline{F}},e}f \simeq \prod_{i=1}^r \mathrm{B}^+_{\widehat{\overline{F}},dR,\mathfrak{m}_i}/\mathrm{B}^+_{\widehat{\overline{F}},dR,\mathfrak{m}_i}\mathfrak{m}_i^{a_i}.$$

Now, the finite subset $A = \{\mathfrak{m}_i\}_{1\leq i\leq r} \subset |Y_{\widehat{\overline{F}}}|$ is stable under G_F and defines a subset $\mathrm{B} = A/G_F \subset |Y_{\widehat{\overline{F}}}|^{G_F-\text{fin}} = |Y_F|$ (see theorem 2.19). The multiplicity function $\mathfrak{m}_i \mapsto a_i$ on A is invariant under G_F and defines a function $\mathfrak{m} \mapsto a_\mathfrak{m}$ on B. Then, according to theorem 2.28

$$\Big(\mathrm{B}_{\widehat{\overline{F}},e}/\mathrm{B}_{\widehat{\overline{F}},e}f\Big)^{G_F} \simeq \prod_{\mathfrak{m}\in A} \mathrm{B}^+_{F,dR,\mathfrak{m}}/\mathrm{B}^+_{F,dR,\mathfrak{m}}\mathfrak{m}^{a_\mathfrak{m}}$$

and the functors $I \mapsto I^{G_F}$ and $J \mapsto \big(\mathrm{B}_{\widehat{\overline{F}},e}/\mathrm{B}_{\widehat{\overline{F}},e}f\big)J$ induce inverse bijections between G_F-invariant ideals of $\mathrm{B}_{\widehat{\overline{F}},e}/\mathrm{B}_{\widehat{\overline{F}},e}f$ and ideals of $\mathrm{B}_{F,e}/\mathrm{B}_{F,e}f$.

A ring A is a Dedekind ring if and only if for all $f \in A \setminus \{0\}$ the f-adic completion of A is isomorphic to a finite product of complete D.V.R.'s. From the preceding one deduces that $\mathrm{B}_{F,e}$ is a Dedekind ring such that the applications $I \mapsto I^{G_F}$ and $J \mapsto \mathrm{B}_{\widehat{\overline{F}},e}J$ induce inverse bijections between

- non zero ideals I of $\mathrm{B}_{\widehat{\overline{F}},e}$ that are G_F-invariant and satisfy $I^{G_F} \neq 0$
- non zero ideals J of $\mathrm{B}_{F,e}$.

But if I is a non zero ideal of $\mathrm{B}_{\widehat{\overline{F}},e}$ that is G_F-invariant, $I = (f)$, since $\mathrm{B}^\times_{\widehat{\overline{F}},e} = E^\times$ there exists a continuous character

$$\chi : G_F \longrightarrow E^\times$$

such that for $\sigma \in G_F$, $\sigma(f) = \chi(\sigma)f$. According to theorem 5.7, $H^0(G_F, \mathrm{B}_{\widehat{\overline{F}},e}(\chi)) \neq 0$ and thus $I^{G_F} \neq 0$. Theorem 5.5 is easily deduced from those considerations. □

With the notations from the preceding proof, if J is a fractional ideal of $\mathrm{B}_{F,e}$ there exists $f \in \mathrm{Frac}\big(\mathrm{B}_{\widehat{\overline{F}},e}\big)$ well defined up to multiplication by $\mathrm{B}^\times_{\widehat{\overline{F}},e} = E^\times$ such that $\mathrm{B}_{\widehat{\overline{F}},e}J = \mathrm{B}_{\widehat{\overline{F}},e}f$. This ideal being stable under G_F, there exists a continuous character

$$\chi_J : G_F \longrightarrow E^\times$$

such that for all $\sigma \in G_F$, $\sigma(f) = \chi_J(\sigma)f$. The arguments used in the proof of theorem 5.5 give the following.

Theorem **5.8.** *The morphism $J \mapsto \chi_J$ induces an isomorphism*

$$\mathcal{C}\ell(\mathrm{B}_{F,e}) \xrightarrow{\sim} \mathit{Hom}(G_F, E^\times).$$

Let us remark the preceding theorem implies the following.

Theorem **5.9.** *If $F'|F$ is a finite degree extension the morphism $P_F \to P_{F'}$ induces a finite étale cover $X_{F'} \to X_F$ of degree $[F' : F]$. If moreover $F'|F$ is Galois then $X_{F'} \to X_F$ is Galois with Galois group $Gal(F'|F)$.*

5.4. **Change of the base field E.** By definition, the graded algebra P_F depends on the choice of the uniformizing element π of E. If the residue field of F is algebraically closed, the choice of another uniformizing element gives a graded algebra that is isomorphic to the preceding, but such an isomorphism is not canonical. In any case, taking the Proj, the curve X_F does not depend anymore on the choice of π. We now put a second subscript in our notations to indicate the dependence on E.

Proposition **5.10.** *If $E'|E$ is a finite extension with residue field contained in F there is a canonical isomorphism*

$$X_{F,E'} \xrightarrow{\sim} X_{F,E} \otimes_E E'.$$

When $E' = E_h$ the degree h unramified extension of E with residue field $\mathbb{F}_{q^h} = F^{\varphi^h=\mathrm{Id}}$ the preceding isomorphism is described in the following way. One has $\mathrm{B}_{F,E_h} = \mathrm{B}_{F,E}$ with $\varphi_{E_h} = \varphi_E^h$. Thus, taking as a uniformizing element of E_h the uniformizing element π of E, one has

$$P_{F,E_h} = \bigoplus_{d\geq 0} \mathrm{B}_{F,E}^{\varphi_E^h=\pi^d}.$$

There is thus a morphism of graded algebras

$$P_{F,E,\bullet} \longrightarrow P_{F,E_h,h\bullet}$$

where the bullet "$\bullet$" indicates the grading. It induces an isomorphism

$$P_{F,E,\bullet} \otimes_E E_h \xrightarrow{\sim} P_{F,E_h,h\bullet}$$

and thus

$$X_{F,E} \otimes_E E_h = \mathrm{Proj}(P_{F,E,\bullet} \otimes_E E_h) \xrightarrow{\sim} \mathrm{Proj}(P_{F,E_h,h\bullet}) = \mathrm{Proj}(P_{F,E_h,\bullet}) = X_{F,E_h}.$$

Suppose we have fixed algebraic closures $\overline{F}$ and $\overline{E}$. We thus have a tower of finite étale coverings of $X_{F,E}$ with Galois group $\mathrm{Gal}(\overline{F}|F) \times \mathrm{Gal}(\overline{E}|E)$

$$\left(X_{F',E'}\right)_{F',E'} \longrightarrow X_{F,E}$$

where F' goes through the set of finite extensions of F in $\overline{F}$ and E' the set of finite extensions of E in $\overline{E}$. We can prove the following.

Theorem **5.11.** *The tower of coverings $(X_{F',E'})_{F',E'} \to X_{F,E}$ is a universal covering and thus if $\bar{x}$ is a geometric point of $X_{F,E}$ then*

$$\pi_1(X_{F,E}, \bar{x}) \simeq Gal(\overline{F}|F) \times Gal(\overline{E}|E).$$

6. Vector bundles

6.1. **Generalities.**

Definition 6.1. *We note Bun_{X_F} the category of vector bundles on X_F.*

Let ∞ be a closed point of $|X_F|$, $\mathrm{B}_{dR}^+ = \widehat{\mathcal{O}}_{X,\infty}$ and

$$\mathrm{B}_e = \Gamma(X \setminus \{\infty\}, \mathcal{O}_X).$$

We note t a uniformizing element of B_{dR}^+ and $\mathrm{B}_{dR} = \mathrm{B}_{dR}^+[\frac{1}{t}]$. Let $\mathscr{C}$ be the category of couples (M, W) where W is a free B_{dR}^+-module of finite type and $M \subset W[\frac{1}{t}]$ is a sub B_e-module of finite type (that is automatically projective since torsion free) such that

$$M \otimes_{\mathrm{B}_e} \mathrm{B}_{dR} \xrightarrow{\sim} W[\tfrac{1}{t}].$$

If F is algebraically closed, the ring B_e is a P.I.D. and such an M is a free module. There is an equivalence of categories

$$\begin{array}{ccc} \mathrm{Bun}_{X_F} & \xrightarrow{\sim} & \mathscr{C} \\ \mathscr{E} & \longmapsto & \big(\Gamma(X\setminus\{\infty\},\mathscr{E}),\widehat{\mathscr{E}}_\infty\big). \end{array}$$

In particular if F is algebraically closed, isomorphism classes of rank n vector bundles are in bijection with the set

$$\mathrm{GL}_n(\mathrm{B}_e)\backslash \mathrm{GL}_n(\mathrm{B}_{dR})/\mathrm{GL}_n(\mathrm{B}_{dR}^+).$$

If $\mathscr{E}$ corresponds to the pair (M,W) then Cech cohomology gives an isomorphism

$$R\Gamma(X,\mathscr{E})\simeq \big[\ M\overset{0}{\oplus} W \xrightarrow{\partial} \overset{+1}{W[\tfrac{1}{t}]}\ \big]$$

where $\partial(x,y)=x-y$. In particular

$$\begin{array}{rcl} H^0(X,\mathscr{E}) & \simeq & M\cap W \\ H^1(X,\mathscr{E}) & \simeq & W[\tfrac{1}{t}]/W+M. \end{array}$$

6.2. **Line bundles.**

6.2.1. *Computation of the Picard group.* One has the usual exact sequence

$$\begin{array}{ccc} 0\longrightarrow E^\times \longrightarrow E(X)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(X) & \longrightarrow & \mathrm{Pic}(X)\longrightarrow 0 \\ D & \longmapsto & [\mathcal{O}_X(D)] \end{array}$$

where $\mathcal{O}_X(D)$ is the line bundle whose sections on the open subset U are

$$\Gamma(U,\mathcal{O}_X(D))=\{f\in E(X)\mid \mathrm{div}(f)_{|U}+D_{|U}\geq 0\}.$$

Since the degree of a principal divisor is zero there is thus a degree function

$$\deg:\mathrm{Pic}(X)\longrightarrow\mathbb{Z}.$$

Definition 6.2. *For $d\in\mathbb{Z}$ define*

$$\mathcal{O}_X(d)=\widetilde{P[d]},$$

a line bundle on X.

One has

$$H^0(X,\mathcal{O}_X(d))=\begin{cases} P_d \text{ if } d\geq 0 \\ 0 \text{ if } d<0. \end{cases}$$

If $d>0$ and $t\in P_d\setminus\{0\}$, $V^+(t)=D$, a degree d Weil divisor on X, then

$$\times t:\mathcal{O}_X(D)\xrightarrow{\sim}\mathcal{O}_X(d).$$

In particular for all $d\in\mathbb{Z}$,

$$\deg(\mathcal{O}_X(d))=d.$$

If F is algebraically closed, with the notations of section 6.1, $\mathrm{B}_{F,e}$ is a P.I.D. and since $\mathrm{B}_{F,e}^\times=E^\times$

$$\mathrm{Pic}(X_F)\simeq E^\times\backslash \mathrm{B}_{dR}^\times/(\mathrm{B}_{dR}^+)^\times=\mathrm{B}_{dR}^\times/(\mathrm{B}_{dR}^+)^\times \xrightarrow[\sim]{\mathrm{ord}_\infty}\mathbb{Z}.$$

We thus obtain the following.

Proposition **6.3.** *If F is algebraically closed then*

$$\deg: Pic(X_F)\xrightarrow{\sim}\mathbb{Z}$$

with inverse $d\mapsto[\mathcal{O}_X(d)]$.

Suppose now F is general. There is thus an exact sequence of G_F-modules

$$0 \longrightarrow E^\times \longrightarrow E\big(X_{\widehat{\overline{F}}}\big)^\times \xrightarrow{\text{div}} \text{Div}^0\big(X_{\widehat{\overline{F}}}\big) \longrightarrow 0.$$

But according to theorem 5.5

$$\text{Div}^0(X_F) = \text{Div}^0\big(X_{\widehat{\overline{F}}}\big)^{G_F}.$$

Applying $H^\bullet(G_F, -)$ to the preceding exact sequence one obtains a morphism

$$\begin{array}{rcl} \text{Div}^0(X_F) & \longrightarrow & H^1(G_F, E^\times) = \text{Hom}(G_F, E^\times) \\ D & \longmapsto & \chi_D. \end{array}$$

Theorem 5.8 translates in the following way.

Theorem **6.4.** *The morphism $D \mapsto \chi_D$ induces an isomorphism*

$$Pic^0(X_F) \xrightarrow{\sim} Hom(G_F, E^\times).$$

6.2.2. *Cohomology of line bundles.* Suppose F is algebraically closed. With the notations of section 6.1 the line bundle $\mathcal{O}_X(d[\infty])$ corresponds to the pair $(\text{B}_e, t^{-d}\text{B}_{dR}^+)$. The fact that $(\text{B}_e, \deg)$ is almost euclidean (5.4) is equivalent to saying that $\text{B}_{dR} = \text{B}_{dR}^+ + \text{B}_e$, that is to say $H^1(X, \mathcal{O}_X) = 0$. From this one obtains the following proposition.

Proposition **6.5.** *If F is algebraically closed,*

$$H^1(X_F, \mathcal{O}_{X_F}(d)) = \begin{cases} 0 \text{ if } d \geq 0 \\ \text{B}_{dR}^+/\text{Fil}^{-d}\text{B}_{dR}^+ + E \text{ if } d < 0. \end{cases}$$

Thus, like $\mathbb{P}^1$

$$H^1(X, \mathcal{O}_X) = 0.$$

But contrary to $\mathbb{P}^1$, $H^1(X, \mathcal{O}_X(-1))$ is non zero and even infinite dimensional isomorphic to C/E where C is the residue field at a closed point of X.

Example 6.6. *Let $t \in P_d = H^0(X, \mathcal{O}_X(d))$ be non zero. It defines an exact sequence*

$$0 \longrightarrow \mathcal{O}_X \xrightarrow{\times t} \mathcal{O}_X(d) \longrightarrow \mathscr{F} \longrightarrow 0$$

where $\mathscr{F}$ is a torsion coherent sheaf. If F is algebraically closed $H^1(X, \mathcal{O}_X) = 0$ and taking the global sections of the preceding exact sequence gives back the fundamental exact sequence (5.1).

Remark 6.7. *If F is not algebraically closed then $H^1(X_F, \mathcal{O}_{X_F}) \neq 0$ in general (see 6.31).*

6.3. **The classification theorem when F is algebraically closed.**

6.3.1. *Definition of some vector bundles.* Suppose $\overline{\mathbb{F}}_q^F$ is algebraically closed and note for all $h \geq 1$, E_h the unramified extension of E with residue field $\mathbb{F}_{q^h} = F^{\varphi_E^h = Id}$. We thus have a pro-Galois cover

$$\big(X_{F,E_h}\big)_{h \geq 1} \longrightarrow X_{F,E}$$

with Galois group $\widehat{\mathbb{Z}}$. We note $X := X_{F,E}$, $X_h := X_{F,E_h}$ and $\pi_h : X_h \to X$. If F is algebraically closed the morphism π_h is totally decomposed at each point of X:

$$\forall x \in X, \ \#\pi_h^{-1}(x) = h.$$

For $\mathscr{E} \in \text{Bun}_X$ one has

$$\begin{cases} \deg(\pi_h^*\mathscr{E}) = h \deg(\mathscr{E}) \\ \text{rk}(\pi_h^*\mathscr{E}) = \text{rk}(\mathscr{E}). \end{cases}$$

For example, $\pi_h^*\mathcal{O}_{X_h}(d) = \mathcal{O}_{X_h}(hd)$. If $\mathscr{E} \in \text{Bun}_{X_h}$ one has

$$\begin{cases} \deg(\pi_{h*}\mathscr{E}) = \deg(\mathscr{E}) \\ \text{rk}(\pi_{h*}\mathscr{E}) = h \, \text{rk}(\mathscr{E}). \end{cases}$$

Definition 6.8. *For $\lambda \in \mathbb{Q}$, $\lambda = \frac{d}{h}$ with $d \in \mathbb{Z}$, $h \in \mathbb{N}_{\geq 1}$ and $(d,h) = 1$ define*

$$\mathcal{O}_X(\lambda) = \pi_{h*}\mathcal{O}_{X_h}(d).$$

We have

$$\mu(\mathcal{O}_X(\lambda)) = \lambda$$

where $\mu = \dfrac{\deg}{\mathrm{rk}}$ is the Harder-Narasimhan slope. The following properties are satisfied

$$\begin{aligned} \mathcal{O}_X(\lambda) \otimes \mathcal{O}_X(\mu) &\simeq \bigoplus_{\text{finite}} \mathcal{O}_X(\lambda+\mu) \\ \mathcal{O}_X(\lambda)^\vee &= \mathcal{O}_X(-\lambda) \\ H^0(X, \mathcal{O}_X(\lambda)) &= 0 \text{ if } \lambda < 0 \\ \mathrm{Hom}(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) &= \bigoplus_{\text{finite}} H^0(X, \mathcal{O}_X(\mu-\lambda)) = 0 \text{ if } \lambda > \mu. \end{aligned}$$

If F is algebraically closed then if $\lambda = \frac{d}{h}$ with $(d,h) = 1$

$$\begin{aligned} H^1(X, \mathcal{O}_X(\lambda)) &= H^1(X_h, \mathcal{O}_{X_h}(d)) \\ &= 0 \text{ if } \lambda \geq 0 \end{aligned}$$

and thus

$$\begin{aligned} \mathrm{Ext}^1(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) &= \bigoplus_{\text{finite}} H^1(X, \mathcal{O}_X(\mu-\lambda)) \\ &= 0 \text{ if } \lambda \leq \mu. \end{aligned}$$

6.3.2. *Statement of the theorem.* Here is the main theorem about vector bundles. It is an analogue of Kedlaya ([21],[22]) or Hartl-Pink ([20]) classification theorems.

Theorem **6.9.** *Suppose F is algebraically closed.*

(1) The semi-stable vector bundles of slope λ on X are the direct sums of $\mathcal{O}_X(\lambda)$.
(2) The Harder-Narasimhan filtration of a vector bundle on X is split.
(3) There is a bijection

$$\begin{aligned} \{\lambda_1 \geq \cdots \geq \lambda_n \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{Q}\} &\xrightarrow{\sim} Bun_X/\sim \\ (\lambda_1, \ldots, \lambda_n) &\longmapsto \Big[\bigoplus_{i=1}^n \mathcal{O}_X(\lambda_i)\Big]. \end{aligned}$$

In this theorem, point (3) is equivalent to points (1) and (2) together. Moreover, since for $\lambda \geq \mu$ one has $\mathrm{Ext}^1(\mathcal{O}_X(\lambda), \mathcal{O}_X(\mu)) = 0$, point (2) is a consequence of point (1).

Remark 6.10. *In any category with Harder-Narasimhan filtrations ([1]), the category of semi-stable objects of slope λ is abelian with simple objects the stable objects of slope λ. The preceding theorem tells more in our particular case: this category is semi-simple with one simple object $\mathcal{O}_X(\lambda)$. One computes easily that $End(\mathcal{O}_X(\lambda)) = D_\lambda$ the division algebra with invariant λ over E. From this one deduces that the functor $\mathscr{E} \mapsto Hom(\mathcal{O}_X(\lambda), \mathscr{E})$ induces an equivalence between the abelian category of semi-simple vector bundles of slope λ and the category of finite dimensional D_λ^{opp}-vector spaces. An inverse is given by the functor $V \mapsto V \otimes_{D_\lambda} \mathcal{O}_X(\lambda)$.*

Example 6.11. *The functors $V \mapsto V \otimes_E \mathcal{O}_X$ and $\mathscr{E} \mapsto H^0(X, \mathscr{E})$ are inverse equivalences between the category of finite dimensional E-vector spaces and the category of semi-stable vector bundles of slope 0 on X.*

6.3.3. *Proof of the classification theorem: a dévissage.* We will now sketch a proof of theorem 6.9. We mainly stick to the case of rank 2 vector bundles which is less technical but contains already all the ideas of the classification theorem. Before beginning let us remark that F algebraically closed being fixed we won't prove the classification theorem for the curve X_E with fixed E but simultaneously for all curves X_{E_h}, $h \geq 1$. As before we note $X = X_E$, $X_h = X_{E_h}$ and $\pi_h : X_h \to X$. We will use the following dévissage.

Proposition **6.12.** *Let $\mathscr{E}$ be a vector bundle on X and $h \geq 1$ an integer.*

(1) *$\mathscr{E}$ is semi-stable of slope λ if and only if $\pi_h^*\mathscr{E}$ is semi-stable of slope $h\lambda$.*
(2) *$\mathscr{E} \simeq \mathcal{O}_X(\lambda)^r$ for some integer r if and only if $\pi_h^*\mathscr{E} \simeq \mathcal{O}_{X_h}(h\lambda)^{r'}$ for some integer r'.*

Proof. Since the morphism $\pi_h : X_h \to X$ is Galois with Galois group $\text{Gal}(E_h|E)$, π_h^* induces an equivalence between Bun_X and $\text{Gal}(E_h|E)$-equivariant vector bundles on X_h. Moreover, if $\mathscr{F}$ is a $\text{Gal}(E_h|E)$-equivariant vector bundle on X_h then its Harder-Narasimhan filtration is $\text{Gal}(E_h|E)$-invariant. This is a consequence of the uniqueness property of the Harder-Narasimhan filtration and the fact that for $\mathcal{G}$ a non zero vector bundle on X_h and $\tau \in \text{Gal}(E_h|E)$ one has $\mu(\tau^*\mathcal{G}) = \mu(\mathcal{G})$. From those considerations one deduces point (1). We skip point (2) that is, at the end, an easy application of Hilbert 90. □

The following dévissage is an analogue of a dévissage contained in [20] (see prop. 9.1) and [22] (see prop. 2.1.7) which is itself a generalization of Grothendieck's method for classifying vector bundles on $\mathbb{P}^1$ ([18]).

Proposition **6.13.** *Theorem 6.9 is equivalent to the following statement: for any $n \geq 1$ and any vector bundle $\mathscr{E}$ that is an extension*

$$0 \longrightarrow \mathcal{O}_X(-\tfrac{1}{n}) \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0$$

one has $H^0(X, \mathscr{E}) \neq 0$.

Proof. Let $\mathscr{E}$ be a vector bundle that is an extension as in the statement. If theorem 6.9 is true then $\mathscr{E} \simeq \bigoplus_{i \in I} \mathcal{O}_X(\lambda_i)$ but since $\deg(\mathscr{E}) = 0$, for an index $i \in I$, $\lambda_i \geq 0$. We thus have $H^0(X, \mathscr{E}) \neq 0$ since for $\lambda \geq 0$, $H^0(X, \mathcal{O}_X(\lambda)) \neq 0$.

In the other direction, let $\mathscr{E}$ be a semi-stable vector bundle on X. Up to replacing X by X_h and $\mathscr{E}$ by $\pi_h^*\mathscr{E}$ for $h \gg 1$, one can suppose $\mu(\mathscr{E}) \in \mathbb{Z}$ (here we use proposition 6.12). Up to replacing $\mathscr{E}$ by a twist $\mathscr{E} \otimes \mathcal{O}_X(d)$ for some $d \in \mathbb{Z}$ one can moreover suppose that

$$\mu(\mathscr{E}) = 0.$$

Suppose now that $\text{rk}\, \mathscr{E} = 2$ (the general case works the same but is more technical). Let $\mathscr{L} \subset \mathscr{E}$ be a sub line bundle of maximal degree (here sub line bundle means locally direct factor). Since $\mathscr{E}$ is semi-stable of slope 0, $\deg \mathscr{L} \leq 0$. Writing $\mathscr{L} \simeq \mathcal{O}_X(-d)$ with $d \geq 0$, we see that $\mathscr{E}$ is an extension

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_X(d) \longrightarrow 0. \tag{5}$$

If $d = 0$, since $\text{Ext}^1(\mathcal{O}_X, \mathcal{O}_X) = H^1(X, \mathcal{O}_X) = 0$, $\mathscr{E} \simeq \mathcal{O}_X^2$ and we are finished. Suppose thus that $d \geq 1$. Since $-d + 2 \leq d$ there exists a non-zero morphism

$$u : \mathcal{O}_X(-d+2) \xrightarrow{\neq 0} \mathcal{O}_X(d).$$

Pulling back the exact sequence (5) via u one obtains an exact sequence

$$0 \longrightarrow \mathcal{O}_X(-d) \longrightarrow \mathscr{E}' \longrightarrow \mathcal{O}_X(-d+2) \longrightarrow 0$$

with a morphism $\mathscr{E}' \to \mathscr{E}$ that is generically an isomorphism. Twisting this exact sequence by $\mathcal{O}_X(d-1)$ one obtains

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathscr{E}'(d-1) \longrightarrow \mathcal{O}_X(1) \longrightarrow 0.$$

By hypothesis,

$$H^0(X, \mathscr{E}'(d-1)) \neq 0$$

and thus there exists a non-zero morphism $\mathcal{O}_X(1-d) \to \mathscr{E}'$. Composed with $\mathscr{E}' \to \mathscr{E}$ this gives a non-zero morphism $\mathcal{O}_X(1-d) \to \mathscr{E}$. This contradicts the maximality of $\deg \mathscr{L}$ (the schematical closure of the image of $\mathcal{O}_X(1-d) \to \mathscr{E}$ has degree $\geq 1-d$). $\square$

6.3.4. *Modifications of vector bundles associated to p-divisible groups: Hodge-de-Rham periods.* We still suppose F is algebraically closed. Let $L|E$ be the completion of the maximal unramified extension of E with residue field $\overline{\mathbb{F}}_q := \overline{\mathbb{F}_q}^F$. We thus have

$$L = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)[\tfrac{1}{\pi}]$$

equipped with a Frobenius σ that lifts $x \mapsto x^q$. Let

$$\varphi\text{-Mod}_L$$

be the associated category of isocrystals, that is to say couples (D,φ) where D is a finite dimensional L-vector space and φ a σ-linear automorphism of D. There is a functor

$$\begin{array}{rcl} \varphi\text{-Mod}_L & \longrightarrow & \text{Bun}_X \\ (D,\varphi) & \longmapsto & \mathscr{E}(D,\varphi) := \widetilde{M(D,\varphi)} \end{array}$$

where $M(D,\varphi)$ is the P-graded module

$$M(D,\varphi) = \bigoplus_{d\geq 0} \big(D \otimes_L \mathrm{B}\big)^{\varphi=\pi^d}.$$

In fact one checks that

$$\mathscr{E}(D,\varphi) \simeq \bigoplus_{\lambda\in\mathbb{Q}} \mathcal{O}_X(-\lambda)^{m_\lambda}$$

where m_λ is the multiplicity of the slope λ in the Dieudonné-Manin decomposition of (D,φ). One has the following concrete description: if $U \subset X$ is a non-empty open subset, $U = D^+(t)$ with $t \in P_d$ for some $d \geq 0$, then

$$\Gamma(U, \mathscr{E}(D,\varphi)) = \big(D \otimes_L \mathrm{B}[\tfrac{1}{t}]\big)^{\varphi=\mathrm{Id}}.$$

Remark 6.14. *With respect to the motivation given in section 4.1 for the introduction of the curve, one sees that the vector bundle $\mathscr{E}(D,\varphi)$ should be understood as being the vector bundle on "$X = Y/\varphi^{\mathbb{Z}}$" associated to the φ-equivariant vector bundle on "Y" whose global sections are* $D \otimes_L \mathrm{B}$.

Let $\infty \in |X|$ be a closed point and $C|E$ the associated residue field. Note $\mathrm{B}_{dR}^+ = \widehat{\mathcal{O}}_{X,\infty}$ with uniformizing element t. One checks there is a canonical identification

$$\widehat{\mathscr{E}(D,\varphi)_\infty} = D \otimes_L \mathrm{B}_{dR}^+.$$

To any lattice $\Lambda \subset D \otimes_L \mathrm{B}_{dR}^+$ there is associated an effective modification $\mathscr{E}(D,\varphi,\Lambda)$ of $\mathscr{E}(D,\varphi)$,

$$0 \longrightarrow \mathscr{E}(D,\varphi,\Lambda) \longrightarrow \mathscr{E}(D,\varphi) \longrightarrow i_{\infty*}\big(D \otimes \mathrm{B}_{dR}^+/\Lambda\big) \longrightarrow 0.$$

Such lattices Λ that satisfy $t.D \otimes \mathrm{B}_{dR}^+ \subset \Lambda \subset D \otimes \mathrm{B}_{dR}^+$ are in bijection with sub-C-vector spaces

$$\mathrm{Fil}\, D_C \subset D_C := D \otimes_L C.$$

Thus, to any sub vector space $\mathrm{Fil}\, D_C \subset D_C$ there is associated a "minuscule" modification

$$0 \longrightarrow \mathscr{E}(D,\varphi,\mathrm{Fil}\, D_C) \longrightarrow \mathscr{E}(D,\varphi) \longrightarrow i_{\infty*}\big(D_C/\mathrm{Fil}\, D_C\big) \longrightarrow 0.$$

One has

$$H^0(X, \mathscr{E}(D,\varphi,\mathrm{Fil}\, D_C)) = \mathrm{Fil}\big(D \otimes_L \mathrm{B}[\tfrac{1}{t}]\big)^{\varphi=Id}.$$

By definition, a π-divisible $\mathcal{O}_E$-module over an $\mathcal{O}_E$-scheme (or formal scheme) S is a p-divisible group H over S equipped with an action of $\mathcal{O}_E$ such that the induced action on $\mathrm{Lie}\, H$ is the canonical one deduced from the structural morphism $S \to \mathrm{Spec}(\mathcal{O}_E)$.

If H is a π-divisible $\mathcal{O}_E$-module over $\overline{\mathbb{F}}_q$ one can define its covariant $\mathcal{O}$-Dieudonné module $\mathbb{D}_{\mathcal{O}}(H)$. This is a free $\mathcal{O}_L = W_{\mathcal{O}_E}(\overline{\mathbb{F}}_q)$-module of rank

$$\mathrm{ht}_{\mathcal{O}}(H) := \frac{\mathrm{ht}(H)}{[E:\mathbb{Q}_p]}$$

equipped with a σ-linear morphism F and a σ^{-1}-linear one V_π satisfying $FV_\pi = \pi$ and $V_\pi F = \pi$. If $\mathbb{D}(H)$ is the covariant Dieudonné module of the underlying p-divisible group one has a decomposition given by the action of the maximal unramified extension of $\mathbb{Q}_p$ in E

$$\mathbb{D}(H) = \bigoplus_{\tau:\mathbb{F}_q\hookrightarrow\overline{\mathbb{F}}_q} \mathbb{D}(H)_\tau.$$

If τ_0 is the canonical embedding then by definition

$$\mathbb{D}_{\mathcal{O}}(H) = \mathbb{D}(H)_{\tau_0}.$$

Moreover if $F:\mathbb{D}(H)\to\mathbb{D}(H)$ is the usual Frobenius and $q=p^r$ then $F:\mathbb{D}_{\mathcal{O}}(H)\to\mathbb{D}_{\mathcal{O}}(H)$ is given by $(F^r)_{|\mathbb{D}(H)_{\tau_0}}$. From now we note φ for F acting on $\mathbb{D}_{\mathcal{O}}(H_0)$.

For H a π-divisible $\mathcal{O}_E$-module over $\mathcal{O}_C$ there is associated a universal $\mathcal{O}_E$-vector extension (see appendix B of [9])

$$0\longrightarrow V_{\mathcal{O}}(H)\longrightarrow E_{\mathcal{O}}(H)\longrightarrow H\longrightarrow 0.$$

One has $V_{\mathcal{O}}(H)=\omega_{H^\vee}$ where $H^\vee$ is the strict dual of H as defined by Faltings ([7]), the usual Cartier dual when $E=\mathbb{Q}_p$. The Lie algebra of the preceding gives an exact sequence

$$0\longrightarrow \omega_{H^\vee}\longrightarrow \mathrm{Lie}\, E_{\mathcal{O}}(H)\longrightarrow \mathrm{Lie}\, H\longrightarrow 0.$$

Suppose now given a π-divisible $\mathcal{O}_E$-module H_0 over $\overline{\mathbb{F}}_q$ and a quasi-isogeny

$$\rho: H_0\otimes_{\overline{\mathbb{F}}_q}\mathcal{O}_C/p\mathcal{O}_C\longrightarrow H\otimes_{\mathcal{O}_C}\mathcal{O}_C/p\mathcal{O}_C.$$

Thanks to the crystalline nature of the universal $\mathcal{O}_E$-vector extension ρ induces an isomorphism

$$\mathbb{D}_{\mathcal{O}}(H_0)\otimes_{\mathcal{O}_L} C \xrightarrow{\sim} \mathrm{Lie}\, E_{\mathcal{O}}(H)\big[\tfrac{1}{\pi}\big].$$

Via this isomorphism we thus get a Hodge Filtration

$$\omega_{H^\vee}\big[\tfrac{1}{\pi}\big]\simeq \mathrm{Fil}\,\mathbb{D}_{\mathcal{O}}(H_0)_C\subset \mathbb{D}_{\mathcal{O}}(H)_C.$$

There is then a period morphism

$$V_\pi(H)\longrightarrow \mathbb{D}_{\mathcal{O}}(H_0)\otimes_{\mathcal{O}_L}\mathrm{B}$$

inducing an isomorphism

$$V_\pi(H)\xrightarrow{\sim}\mathrm{Fil}\,(\mathbb{D}_{\mathcal{O}}(H_0)\otimes_{\mathcal{O}_L}\mathrm{B})^{\varphi=\pi}.$$

Here $V_\pi(H)=V_p(H)$ but we prefer to use the notation $V_\pi(H)$ since most of what we say can be adapted in equal characteristic when $E=\mathbb{F}_q((\pi))$, for example in the context of Drinfeld modules. This period morphism is such that the induced morphism

$$V_\pi(H)\otimes_E\mathrm{B}\longrightarrow \mathbb{D}_{\mathcal{O}}(H_0)\otimes_{\mathcal{O}_L}\mathrm{B}$$

is injective with cokernel killed by t where here $t\in\mathrm{B}^{\varphi=\pi}$ is a non zero period of a Lubin-Tate group attached to E over $\mathcal{O}_C$. It induces a morphism

$$V_p(H)\otimes_E\mathcal{O}_X\longrightarrow \mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0)\big[\tfrac{1}{\pi}\big],\pi^{-1}\varphi,\mathrm{Fil}\,\mathbb{D}_O(H_0)_C\big).$$

Since

$$V_\pi(H)\otimes_E\mathrm{B}_e\xrightarrow{\sim}\big(D\otimes_E\mathrm{B}[\tfrac{1}{t}]\big)^{\varphi=Id}$$

where $\mathrm{B}_e=H^0(X\setminus\{\infty\},\mathcal{O}_X)$, the preceding morphism is an isomorphism outside ∞. Since both vector bundles are of degree 0 this is an isomorphism. We thus obtain the following theorem.

Theorem **6.15.** *If H is a π-divisible $\mathcal{O}_E$-module over $\mathcal{O}_C$, H_0 a π-divisible $\mathcal{O}_E$-module over $\overline{\mathbb{F}}_q$ and $\rho: H_0\otimes\mathcal{O}_C/p\mathcal{O}_C\to H\otimes\mathcal{O}_C/p\mathcal{O}_C$ a quasi-isogeny then*

$$\mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0)\big[\tfrac{1}{\pi}\big],\pi^{-1}\varphi,\omega_{H^\vee}\big[\tfrac{1}{p}\big]\big)\simeq V_\pi(H)\otimes_E\mathcal{O}_X.$$

Let H_0 be fixed over $\overline{\mathbb{F}}_q$ and let $\widehat{\mathcal{M}}$ be its deformation space by quasi-isogenies as defined by Rapoport and Zink ([27]), a $\mathrm{Spf}(\mathcal{O}_L)$-formal scheme. We note $\mathcal{M}$ for its generic fiber as a Berkovich analytic space over L. In fact, we won't use the analytic space structure on $\mathcal{M}$, but only the C-points $\mathcal{M}(C) = \widehat{\mathcal{M}}(\mathcal{O}_C)$. Let $\mathcal{F}^{dR}$ be the Grassmanian of subspaces of $\mathbb{D}_{\mathcal{O}}(H)\big[\frac{1}{\pi}\big]$ of codimension $\dim H_0$, seen as an L-analytic space. There is then a period morphism

$$\pi^{dR} : \mathcal{M} \longrightarrow \mathcal{F}^{dR}$$

associating to a deformation its associated Hodge filtration. This morphism is étale and its image $\mathcal{F}^{dR,a}$, the admissible locus, is thus open. To each point $z \in \mathscr{F}^{dR}(C)$ is associated a filtration $\mathrm{Fil}_z\mathbb{D}_{\mathcal{O}}(H_0)_C$ and a vector bundle $\mathscr{E}(z)$ on X that is a modification of $\mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0)\big[\frac{1}{\pi}\big], \pi^{-1}\varphi\big)$. Now the preceding theorem says the following.

Theorem **6.16.** *If $z \in \mathcal{F}^{dR,a}(C)$ then $\mathscr{E}(z)$ is a trivial vector bundle.*

Remark 6.17. *In fact a theorem of Faltings (*[8]*), translated in the language of vector bundle on our curve, says that a point $z \in \mathcal{F}^{dR}(C)$ is in the admissible locus if and only if $\dim_E H^0(X, \mathscr{E}(z)) = rk(\mathscr{E}(z))$. Using the classification theorem 6.9 this amounts to saying that $\mathscr{E}(z)$ is trivial or equivalently semi-stable of slope* 0*. Thus, once theorem 6.9 is proved, we have a characterization of $\mathcal{F}^{dR,a}$ in terms of semi-stability as this is the case for the weakly admissible locus $\mathcal{F}^{dR,wa}$ (*[27] *chap.I,* [5]*) (in the preceding, if we allow ourselves to vary the curve X, we can make a variation of the complete algebraically closed field $C|E$).*

We will use the following theorem that gives us the image of the period morphism for Lubin-Tate spaces.

Theorem **6.18** (Laffaille [24], Gross-Hopkins [17])**.** *Let H_0 be a one dimensional formal π-divisible $\mathcal{O}_E$-module of $\mathcal{O}_E$-height n and $\mathcal{F}^{dR} = \mathbb{P}^{n-1}$ the associated Grassmanian. Then one has*

$$\mathcal{F}^{dR} = \mathcal{F}^{dR,a}.$$

Remark 6.19. *The preceding theorem says that for Lubin-Tate spaces, the weakly admissible locus coincides with the admissible one (one has always $\mathcal{F}^{dR,a} \subset \mathcal{F}^{dR,wa}$). We will need this theorem for all points of the period domain $\mathbb{P}^{n-1}$, not only classical ones associated to finite extensions of L.*

Translated in terms of vector bundles on the curve the preceding theorem gives the following.

Theorem **6.20.** *Let*

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_X\big(\tfrac{1}{n}\big) \longrightarrow \mathscr{F} \longrightarrow 0$$

be a degree one modification of $\mathcal{O}_X\big(\frac{1}{n}\big)$, that is to say $\mathscr{F}$ is a torsion coherent sheaf on X of length 1 *(i.e. of the form $i_{x*}k(x)$ for some $x \in |X|$). Then $\mathscr{E}$ is a trivial vector bundle, $\mathscr{E} \simeq \mathcal{O}_X^n$.*

Remark 6.21. *We will use theorem 6.20 to prove the classification theorem 6.9. Reciprocally, it is not difficult to see that the classification theorem 6.9 implies theorem 6.20. In fact, suppose*

$$\mathscr{E} \simeq \oplus_{i\in I}\mathcal{O}_X(\lambda_i)$$

is a degree one modification of $\mathcal{O}_X\big(\frac{1}{n}\big)$. Write $\lambda_i = \frac{d_i}{h_i}$ with $(d_i, h_i) = 1$. Since $rk(\mathscr{E}) = n$ one has $h_i \leq n$. But

$$Hom\big(\mathcal{O}_X(\lambda_i), \mathcal{O}_X\big(\tfrac{1}{n}\big)\big) \neq 0$$

implies $\lambda_i \leq \frac{1}{n}$. Thus, for $i \in I$, either $\lambda_i \leq 0$ or $\lambda_i = \frac{1}{n}$. Using $\deg \mathscr{E} = 0$ one concludes that for all i, $\lambda_i = 0$.

6.3.5. *Modifications of vector bundles associated to p-divisible groups: Hodge-Tate periods.* Let V be a finite dimensional E-vector space and W a finite dimensional $C = k(\infty)$-vector space. Consider extensions of coherent sheaves on X

$$0 \longrightarrow V \otimes_E \mathcal{O}_X \longrightarrow \mathscr{H} \longrightarrow i_{\infty*}W \longrightarrow 0.$$

Those extensions are rigid since $\mathrm{Hom}(i_{\infty*}W, V \otimes_E \mathcal{O}_X) = 0$. Consider the category of triples (V, W, ξ) where V and W are vector spaces and ξ is an extension as before. Morphisms in this category are linear morphisms of vector spaces inducing morphisms of extensions. Fix a Lubin-Tate group over $\mathcal{O}_E$ and let $E\{1\}$ be its rational Tate module over $\mathcal{O}_C$, a one dimensional E-vector space. One has

$$E\{1\} \subset \mathrm{B}^{\varphi=\pi} = H^0(X, \mathcal{O}_X(1)).$$

There is a canonical extension

$$0 \longrightarrow \mathcal{O}_X\{1\} \longrightarrow \mathcal{O}_X(1) \longrightarrow i_{\infty *}C \longrightarrow 0.$$

where $\mathcal{O}_X\{1\} = \mathcal{O}_X \otimes_E E\{1\}$. If V and W are as before and

$$u : W \longrightarrow V\{-1\} \otimes_E C =: V_C\{-1\}$$

is C-linear there is an induced extension

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V \otimes_E \mathcal{O}_X & \longrightarrow & \mathscr{H}(V,W,u) & \longrightarrow & i_{\infty *}W & \longrightarrow & 0 \\ & & \| & & \downarrow & & \downarrow{\scriptstyle i_{\infty *}u} & & \\ 0 & \longrightarrow & V \otimes_E \mathcal{O}_X & \longrightarrow & V\{-1\} \otimes_E \mathcal{O}_X(1) & \longrightarrow & i_{\infty *}V_C\{-1\} & \longrightarrow & 0 \end{array}$$

where the upper extension is obtained by pullback from the lower one via $i_{\infty *}u$ and we used the formula $V\{-1\} \otimes_E i_{\infty *}C = i_{\infty *}V_C\{-1\}$. It is easily seen that this induces a category equivalence between triplets (V, W, u) and the preceding category of extensions:

$$\mathrm{Hom}_C(W, V_C\{-1\}) \xrightarrow{\sim} \mathrm{Ext}^1(i_{\infty *}W, V \otimes_E \mathcal{O}_X)$$

canonically in W and V. The coherent sheaf $\mathscr{H}(V, W, u)$ is a vector bundle if and only if u is injective. In this case, $V \otimes_E \mathcal{O}_X$ is a "minuscule" modification of the vector bundle $\mathscr{H}(V, W, u)$.

There is another period morphism associated to p-divisible groups: Hodge Tate periods. Let H be a π-divisible $\mathcal{O}_E$-module over $\mathcal{O}_C$. There is then a Hodge-Tate morphism, an E-linear morphism

$$\alpha_H : V_\pi(H) \longrightarrow \omega_{H^\vee}\big[\tfrac{1}{\pi}\big].$$

It is defined in the following way. An element of $T_\pi(H)$ can be interpreted as a morphism of π-divisible $\mathcal{O}_E$-modules

$$f : E/\mathcal{O}_E \longrightarrow H.$$

Using the duality of [7] it gives a morphism

$$f^\vee : H^\vee \longrightarrow \mathcal{LT}_{\mathcal{O}_C}.$$

where $\mathcal{LT}$ is a fixed Lubin-Tate group over $\mathcal{O}_E$. Then, having fixed a generator α of $\omega_{\mathcal{LT}}$, one has

$$\alpha_H(f) = (f^\vee)^*\alpha.$$

Consider

$$\beta_H = {}^t(\alpha_{H^\vee} \otimes 1) : \mathrm{Lie}\, H\big[\tfrac{1}{\pi}\big] \longrightarrow V_\pi(H)_C\{-1\}$$

where

$$\alpha_{H^\vee} \otimes 1 : V_\pi(H)^*\{1\} \otimes_E C \longrightarrow \omega_H\big[\tfrac{1}{\pi}\big]$$

using the formula

$$V_\pi(H)^*\{1\} = V_\pi(H^\vee)$$

and β_H is the transpose of $\alpha_{H^\vee} \otimes 1$. All of this fits into an Hodge-Tate exact sequence of C-vector spaces ([11] chap.5 for $E = \mathbb{Q}_p$)

$$0 \longrightarrow \mathrm{Lie}\, H\big[\tfrac{1}{\pi}\big]\{1\} \xrightarrow{\beta_H\{1\}} V_\pi(H) \otimes_E C \xrightarrow{\alpha_H} \omega_{H^\vee}\big[\tfrac{1}{\pi}\big] \longrightarrow 0.$$

Suppose now (H_0, ρ) is as in the preceding section.

Theorem **6.22.** *One has a canonical isomorphism*

$$\mathscr{H}\big(V_\pi(H), \mathit{Lie}\, H\big[\tfrac{1}{\pi}\big], \beta_H\big) \simeq \mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0)\big[\tfrac{1}{\pi}\big], \pi^{-1}\varphi\big).$$

Sketch of proof. To prove the preceding theorem is suffices to construct a morphism f giving a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & V_\pi(H)\otimes_E \mathcal{O}_X & \longrightarrow & \mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0)\big[\tfrac{1}{\pi}\big],\pi^{-1}\varphi\big) & \longrightarrow & i_{\infty *}\mathrm{Lie}\, H\big[\tfrac{1}{\pi}\big] & \longrightarrow & 0 \\ & & \big\downarrow{\scriptstyle Id} & & \big\downarrow{\scriptstyle f} & & \big\downarrow{\scriptstyle i_{\infty *}\beta_H} & & \\ 0 & \longrightarrow & V_\pi(H)\otimes_E \mathcal{O}_X & \longrightarrow & V_\pi(H)\{-1\}\otimes_E \mathcal{O}_X(1) & \longrightarrow & i_{\infty *}V_\pi(H)_C\{-1\} & \longrightarrow & 0. \end{array}$$

By duality and shifting, to give f is the same as to give its transpose twisted by $\mathcal{O}_X(1)$

$$f^\vee(1): V_\pi(H^\vee)\otimes_E \mathcal{O}_X \longrightarrow \mathscr{E}\big(\mathbb{D}_O(H_0)\big[\tfrac{1}{\pi}\big],\pi^{-1}\varphi\big)^\vee(1) = \mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0^\vee),\pi^{-1}\varphi\big).$$

One checks that taking $f^\vee(1)$ equal to the period morphism for $H^\vee$

$$V_\pi(H^\vee)\longrightarrow \mathbb{D}_{\mathcal{O}}(H_0^\vee)\otimes_{\mathcal{O}_L}\mathrm{B}$$

makes the preceding diagram commutative. □

Let's come back to Rapoport-Zink spaces. Let $\mathcal{M}$ be as in the preceding section. Let $n=\mathrm{ht}_{\mathcal{O}}H_0$ and for $K\subset \mathrm{GL}_n(\mathcal{O}_E)$ an open subgroup let $\mathcal{M}_K\to\mathcal{M}$ be the étale finite covering given by level K-structure on the universal deformation. Set

$$\text{“}\mathcal{M}_\infty = \varprojlim_K \mathcal{M}_K\text{“}.$$

There are different ways to give a meaning to this as a generalized rigid anaytic space (see [9] for the case of Lubin-Tate and Drinfeld spaces) but we don't need it for our purpose. The only thing we need is the points

$$\mathcal{M}_\infty(C) = \{(H,\rho,\eta\}/\sim$$

where $(H,\rho)\in\mathcal{M}(C)=\widehat{\mathcal{M}}(\mathcal{O}_C)$ is as before and

$$\eta:\mathcal{O}_E^n \xrightarrow{\ \sim\ } T_\pi(H).$$

Let $\mathcal{F}^{HT}$ be the Grassmanian of subspaces of E^n of dimension $\dim H_0$ as an analytic space over L. The Hodge-Tate map induces a morphism, at least at the level of the C-points,

$$\begin{array}{rcl} \pi^{HT}:\mathcal{M}_\infty & \longrightarrow & \mathcal{F}^{HT} \\ (H,\rho,\eta) & \longmapsto & \big(\mathrm{Lie}\, H\big[\tfrac{1}{\pi}\big]\{1\}\xhookrightarrow{u} C^n\big) \end{array}$$

where

$$u:\mathrm{Lie}\, H\big[\tfrac{1}{\pi}\big]\xrightarrow{\beta_H\{1\}} V_\pi(H)_C \xrightarrow{\eta_H^{-1}\otimes 1} C^n.$$

To each point $z\in\mathscr{F}^{HT}(C)$ there is associated a vector bundle $\mathscr{H}(z)$ on X. The preceding thus gives the following.

Theorem **6.23.** *If $z\in\mathcal{F}^{HT}$ is in the image of the Hodge-Tate map $\pi^{HT}:\mathcal{M}_\infty(C)\to\mathscr{F}^{HT}(C)$ then $\mathscr{H}(z)\simeq\mathscr{E}\big(\mathbb{D}_{\mathcal{O}}(H_0)\big[\tfrac{1}{\pi}\big],\pi^{-1}\varphi\big)$.*

Consider now the case when $\dim H_0^\vee=1$, the dual Lubin-Tate case. Then, $\mathcal{F}^{HT}=\mathbb{P}^{n-1}$ as an analytic space over L. It is stratified in the following way. For $i\in\{0,\cdots,n-1\}$ let

$$(\mathbb{P}^{n-1})^{(i)} = \{x\in\mathbb{P}^{n-1}\mid \dim_E E^n\cap \mathrm{Fil}_x k(x)^n = i\}.$$

This is a locally closed subset of the Berkovich space $\mathbb{P}^{n-1}$ (but it has no analytic structure for $i>0$). The open stratum

$$(\mathbb{P}^{n-1})^{(0)} = \Omega^{n-1}$$

is Drinfeld space. For each $i>0$, $(\mathbb{P}^{n-1})^{(i)}$ is fibered over the Grassmanian Gr^i of i-dimensional subspaces of E^n (seen as a naïve analytic space)

$$\begin{array}{rcl} (\mathbb{P}^{n-1})^{(i)} & \longrightarrow & \mathrm{Gr}^i \\ x & \longmapsto & E^n\cap\mathrm{Fil}_x k(x)^n. \end{array}$$

with fibers Drinfeld spaces Ω^{n-1-i}. The π-divisible $\mathcal{O}$-modules H_0 over $\overline{\mathbb{F}}_q$ of $\mathcal{O}$-height n and dimension $n-1$ are classified by the height of their étale part

$$\mathrm{ht}_{\mathcal{O}}(H_0^{\mathrm{et}}) \in \{0, \cdots, n-1\}.$$

Let $H_0^{(i)}$ over $\overline{\mathbb{F}}_q$ be such that $\mathrm{ht}_{\mathcal{O}}(H_0^{\mathrm{et}}) = i$. Let $\mathcal{M}^{(i)}$ be the corresponding Rapoport-Zink space of deformations by quasi-isogenies of $H_0^{(i)}$ and $\mathcal{M}_\infty^{(i)}$ the space "with infinite level". When $i=0$, this is essentially the Rapoport-Zink space associated to Lubin-Tate space (the only difference is that the Hecke action is twisted by the automorphism $g \mapsto {}^t g^{-1}$ of $\mathrm{GL}_n(E)$) and for $i > 0$ this can be easily linked to a lower dimensional Lubin-Tate space of deformations of the dual of the connected component of $H_0^{(i)}$. One then has

$$\pi^{HT} : \mathcal{M}_\infty^{(i)} \longrightarrow (\mathbb{P}^{n-1})^{(i)}.$$

Theorem **6.24.** *For all $i \in \{0, \cdots, n-1\}$, the Hodge-Tate period map*

$$\pi^{HT} : \mathcal{M}_\infty^{(i)} \longrightarrow (\mathbb{P}^{n-1})^{(i)}$$

is surjective, that is to say

$$\pi^{HT} : \coprod_{i=0,\cdots,n-1} \mathcal{M}_\infty^{(i)} \longrightarrow \mathbb{P}^{n-1}$$

is surjective.

The statement of this theorem is at the level of the points of the associated Berkovich topological spaces. It says that the associated maps are surjective at the level of the points with values in complete algebraically closed extensions of E. The proof of the preceding is easily reduced to the case when H_0 is formal, that is to say the Lubin-Tate case. One thus has to prove that

$$\pi^{HT} : \mathcal{M}_\infty^{(0)} \longrightarrow \Omega$$

is surjective: *up to varying the complete algebraically closed field $C|E$, any point in the Drinfeld space $\Omega(C)$ is the Hodge-Tate period of the dual of a Lubin-Tate group over $\mathcal{O}_C$.* The proof relies on elementary manipulations between Lubin-Tate and Drinfeld spaces and the following computation of the admissible locus for Drinfeld moduli spaces (see chapter II of [9] for more details).

Theorem **6.25** (Drinfeld [6])**.** *Let D be a division algebra central over E with invariant $1/n$. Let $\mathcal{M}$ be the analytic Rapoport-Zink space of deformations by quasi-isogenies of special formal $\mathcal{O}_D$-modules of $\mathcal{O}_E$-height n^2. Then, the image of*

$$\pi^{dR} : \mathcal{M} \longrightarrow \mathbb{P}^{n-1}$$

is Drinfeld's space Ω.

Remark 6.26. *Of course, Drinfeld's theorem is more precise giving an explicit description of the formal scheme $\widehat{\mathcal{M}}$ and proving that if $\mathcal{M}^{[i]}$ is the open/closed subset where the $\mathcal{O}_E$-height of the universal quasi-isogeny is ni then for any $i \in \mathbb{Z}$*

$$\pi^{dR} : \mathcal{M}^{[i]} \xrightarrow{\sim} \Omega.$$

In fact, to apply Drinfeld's result one need to compare its period morphism defined in terms of Cartier theory and the morphism π^{dR} defined in crystalline terms. This is done in [27] *5.19. Finally, one will notice that in* [24] *Laffaille gives another proof of theorem 6.25.*

As a consequence of theorems 6.23 and 6.24 we obtain the following result that is the one we will use to prove theorem 6.9.

Theorem **6.27.** *Let $\mathscr{E}$ be a vector bundle on X having a degree one modification that is a trivial vector bundle of rank n,*

$$0 \longrightarrow \mathcal{O}_X^n \longrightarrow \mathscr{E} \longrightarrow i_{x*}k(x) \longrightarrow 0$$

for some $x \in |X|$. Then, there exists an integer $i \in \{0, \cdots, n-1\}$ such that

$$\mathscr{E} \simeq \mathcal{O}_X^i \oplus \mathcal{O}_X\big(\tfrac{1}{n-i}\big).$$

Remark 6.28. *As in remark 6.21, one checks that the classification theorem implies theorem 6.27.*

6.3.6. *End of the proof of the classification theorem.* We will now use theorems 6.20 and 6.27 to prove the classification theorem 6.9. For this we use the dévissage given by proposition 6.13. We treat the case of rank 2 vector bundles, the general case being analogous but longer and more technical. According to proposition 6.13 we have to prove that if $\mathscr{E}$ is a rank 2 vector bundle that is an extension

$$0 \longrightarrow \mathcal{O}_X(-1) \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_X(1) \longrightarrow 0 \tag{6}$$

then $H^0(X, \mathscr{E}) \neq 0$. Let us choose $t \in H^0(X, \mathcal{O}_X(2)) \setminus \{0\}$. It furnishes a degree 2 modification

$$0 \longrightarrow \mathcal{O}_X(-1) \xrightarrow{\times t} \mathcal{O}_X(1) \longrightarrow \mathscr{F} \longrightarrow 0$$

where $\mathscr{F}$ is a torsion coherent sheaf of length 2. We can push forward the exact sequence (6)

$$\begin{array}{ccccccccc}
 & & 0 & & & & & & \\
 & & \downarrow & & & & & & \\
0 & \longrightarrow & \mathcal{O}_X(-1) & \longrightarrow & \mathscr{E} & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \times t} & & \downarrow & & \| & & \\
0 & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & \mathscr{E}' & \longrightarrow & \mathcal{O}_X(1) & \longrightarrow & 0 \\
 & & \downarrow & & & & & & \\
 & & \mathscr{F} & & & & & & \\
 & & \downarrow & & & & & & \\
 & & 0 & & & & & &
\end{array}$$

Since $\mathrm{Ext}^1(\mathcal{O}_X(1), \mathcal{O}_X(1)) = 0$ one has $\mathscr{E}' \simeq \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$. We thus obtain a degree 2 modification

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \longrightarrow \mathscr{F} \longrightarrow 0.$$

Consider a dévissage of $\mathscr{F}$

$$0 \longrightarrow i_{y*}k(y) \longrightarrow \mathscr{F} \longrightarrow i_{x*}k(x) \longrightarrow 0$$

for some closed points $x, y \in |X|$. Consider the composite surjection

$$\mathcal{O}_X(1) \oplus \mathcal{O}_X(1) \longrightarrow \mathscr{F} \longrightarrow i_{x*}k(x),$$

write it

$$(a, b) \mapsto u(a) + v(b)$$

for two morphisms $u, v \in \mathrm{Hom}(\mathcal{O}_X(1), i_{x*}k(x))$ and let $\mathscr{E}' \subset \mathcal{O}_X(1) \oplus \mathcal{O}_X(1)$ be the kernel of this morphism. One has $(u, v) \neq (0, 0)$. Moreover, if $u = 0$ or $v = 0$ then

$$\mathscr{E}' \simeq \mathcal{O}_X \oplus \mathcal{O}_X(1).$$

Suppose now that $u \neq 0$ and $v \neq 0$. Then, $u_{|\mathscr{E}'}$ is surjective giving rise to an exact sequence

$$0 \longrightarrow \underbrace{\ker u \oplus \ker v}_{\simeq \mathcal{O}_X \oplus \mathcal{O}_X} \longrightarrow \mathscr{E}' \xrightarrow{u_{|\mathscr{E}'}} i_{x*}k(x) \longrightarrow 0.$$

Applying theorem 6.27 we deduce that either $\mathscr{E}' \simeq \mathcal{O}_X \oplus \mathcal{O}_X(1)$ or $\mathscr{E}' \simeq \mathcal{O}_X\left(\frac{1}{2}\right)$. There is now a degree 1 modification

$$0 \longrightarrow \mathscr{E} \longrightarrow \mathscr{E}' \longrightarrow i_{y*}k(y) \longrightarrow 0.$$

If $\mathscr{E}' \simeq \mathcal{O}_X \oplus \mathcal{O}_X(1)$ the surjection $\mathscr{E}' \to i_{y*}k(y)$ is given by

$$(a, b) \mapsto u(a) + v(b)$$

for some $u \in \mathrm{Hom}(\mathcal{O}_X, i_{y*}k(y))$ and $v \in \mathrm{Hom}(\mathcal{O}_X(1), i_{y*}k(y))$. One has

$$0 \oplus \ker v \subset \mathscr{E}$$

but if $v \neq 0$, $\ker v \simeq \mathcal{O}_X$, and if $v = 0$ then $\ker v = \mathcal{O}_X(1)$. In both cases

$$H^0(X, 0 \oplus \ker v) \neq 0 \Longrightarrow H^0(X, \mathscr{E}) \neq 0.$$

Suppose now $\mathscr{E}' \simeq \mathcal{O}_X\left(\frac{1}{2}\right)$. We can then apply theorem 6.20 to conclude that $\mathscr{E} \simeq \mathcal{O}_X \oplus \mathcal{O}_X$ and thus $H^0(X, \mathscr{E}) \neq 0$. □

6.4. **Galois descent of vector bundles.** Now F is not necessarily algebraically closed. Let $\overline{F}$ be an algebraic closure of F. There is an action of G_F on $X_{\widehat{\overline{F}}}$. Define

$$\mathrm{Bun}_{X_{\widehat{\overline{F}}}}^{G_F}$$

to be the category of G_F-equivariant vector bundles on $X_{\widehat{\overline{F}}}$ together with a continuity condition on the action of G_F (we don't enter into the details).

Theorem **6.29.** *If* $\alpha : X_{\widehat{\overline{F}}} \longrightarrow X_F$*, the functor*

$$\alpha^* : Bun_{X_F} \longrightarrow Bun_{X_{\widehat{\overline{F}}}}$$

is an equivalence.

For rank 1 vector bundles this theorem is nothing else than theorem 6.4.

Example 6.30. *Let* $Rep_E(G_F)$ *be the category of continuous representations of* G_F *in finite dimensional* E*-vector spaces. The functor*

$$\begin{array}{rcl} Rep_E(G_F) & \longrightarrow & Bun_{X_{\widehat{\overline{F}}}}^{G_F} \\ V & \longmapsto & V \otimes_E \mathcal{O}_{X_{\widehat{\overline{F}}}} \end{array}$$

induces an equivalence between $Rep_E(G_F)$ *and the subcategory of* $Bun_{X_{\widehat{\overline{F}}}}^{G_F}$ *formed by equivariant vector bundles whose underlying vector bundle is trivial. An inverse is given by the global section functor* $H^0(X_{\widehat{\overline{F}}}, -)$*. Thus, via theorem 6.29 the category* $Rep_E(G_F)$ *embeds in* Bun_{X_F}*. According to theorem 6.9 this coincides with* G_F*-equivariant vector bundles semi-stable of slope* 0*.*

Remark 6.31. *With the notations of the preceding example if* $\mathscr{E} \in Bun_{X_F}$ *corresponds to* $V \otimes_E \mathcal{O}_{X_{\widehat{\overline{F}}}}$ *that is to say* $\alpha^*\mathscr{E} \simeq V \otimes_E \mathcal{O}_{\widehat{\overline{F}}}$ *then*

$$\begin{array}{rcl} H^1(X_F, \mathscr{E}) & \simeq & Ext^1_{Fib^{G_F}_{X_{\widehat{\overline{F}}}}}\left(\mathcal{O}_{\widehat{\overline{F}}}, V \otimes_E \mathcal{O}_{\widehat{\overline{F}}}\right) \\ & \simeq & H^1(G_F, V) \end{array}$$

since $H^1(X_{\widehat{\overline{F}}}, \mathcal{O}_{\widehat{\overline{F}}}) = 0$*. In particular,* $H^1(X_F, \mathcal{O}_{X_F}) \simeq Hom(G_F, E)$ *which is non zero in general when* F *is not algebraically closed.*

Skectch of proof of theorem 6.29. To prove the preceding theorem we use the following fundamental property of Harder-Narasimhan filtrations that is a consequence of their canonicity (see the proof of prop. 6.12 for example).

Proposition **6.32.** *Let* $\Gamma \subset Aut(X)$ *be a subgroup and* $\mathscr{E}$ *be a* Γ*-equivariant vector bundle on* X*. Then the Harder-Narasimhan filtration of* $\mathscr{E}$ *is* Γ*-invariant, that is to say is a filtration in the category of* Γ*-equivariant vector bundles.*

Let us fix $t \in \mathrm{B}_F^{\varphi=\pi} \setminus \{0\}$ and note $\{\infty\} = V^+(t)$, $\infty \in |X_F|$

$$\mathrm{B}_{F,e} = \mathrm{B}_F[\tfrac{1}{t}]^{\varphi=Id} = \Gamma(X_F \setminus \{\infty\}, \mathcal{O}_{X_F}), \quad \mathrm{B}^+_{F,dR} = \widehat{\mathcal{O}}_{X_F,\infty}.$$

$$\mathrm{B}_{\widehat{\overline{F}},e} = \mathrm{B}_{\widehat{\overline{F}}}[\tfrac{1}{t}]^{\varphi=Id} = \Gamma(X_{\widehat{\overline{F}}} \setminus \{\infty\}, \mathcal{O}_{X_{\widehat{\overline{F}}}}), \quad \mathrm{B}^+_{\widehat{\overline{F}},dR} = \widehat{\mathcal{O}}_{X_{\widehat{\overline{F}}},\infty}.$$

The category $\mathrm{Bun}_{X_{\widehat{\overline{F}}}}^{G_F}$ is equivalent to the following category of B-pairs ([3]) (M, W, u) where:

- $M \in \mathrm{Rep}_{\mathrm{B}_{\widehat{\overline{F}},e}}(G_F)$ is a semi-linear continuous representation of G_F in a free $\mathrm{B}_{\widehat{\overline{F}},e}$-module,
- $M \in \mathrm{Rep}_{\mathrm{B}^+_{\widehat{\overline{F}},dR}}(G_F)$ is a semi-linear continuous representation of G_F in a free $\mathrm{B}^+_{\widehat{\overline{F}},dR}$-module,

- $u : M \otimes_{\mathrm{B}_{\widehat{\overline{F}},e}} \mathrm{B}_{\widehat{\overline{F}},dR} \xrightarrow{\sim} W[\frac{1}{t}]$.

According to theorem 2.28, to prove theorem 6.29 it suffices to prove that if $\mathrm{Mod}^{proj}_{\mathrm{B}_{F,e}}$ is the category of projective $\mathrm{B}_{F,e}$-modules of finite type then

$$-\otimes_{\mathrm{B}_{F,e}} : \mathrm{Mod}^{proj}_{\mathrm{B}_{F,e}} \xrightarrow{\sim} \mathrm{Rep}_{\mathrm{B}_{\widehat{\overline{F}},e}}(G_F).$$

Here in the definition of a $\mathrm{B}_{\widehat{\overline{F}},e}$-representation M we impose that the G_F-action on $M \otimes \mathrm{B}_{\widehat{\overline{F}},dR}$ stabilizes a $\mathrm{B}^+_{\widehat{\overline{F}},dR}$-lattice (the continuity condition does not imply this) that is to say M comes from an equivariant vector bundle. Full faithfullness of the preceding functor is an easy consequence of the equality

$$\mathrm{B}_{F,e} = \big(\mathrm{B}_{\widehat{\overline{F}},e}\big)^{G_F}.$$

We now treat the essential surjectivity. An easy Galois descent argument tells us we can replace the field E that was fixed by a finite extension of it (for this we may have to replace F by a finite extension of it so that the residue field of the finite extension of E is contained in F but this is harmless by Hilbert 90). Let M be a $\mathrm{B}_{\widehat{\overline{F}},e}$-representation of G_F. Choose an equivariant vector bundle $\mathscr{E}$ on $X_{\widehat{\overline{F}}}$ such that

$$M = \Gamma(X_{\widehat{\overline{F}}} \setminus \{\infty\}, \mathscr{E}).$$

Applying the classification theorem 6.9 and proposition 6.32 we see $\mathscr{E}$ is a successive extension of equivariant vector bundles whose underlying bundle is isomorphic to $\mathcal{O}_{X_{\widehat{\overline{F}}}}(\lambda)^n$ for some $\lambda \in \mathbb{Q}$ and $n \in \mathbb{N}$. Up to making a finite extension of E one can suppose moreover that all such slopes λ are integers, $\lambda \in \mathbb{Z}$. Now, if $\lambda \in \mathbb{Z}$, an equivariant vector bundle whose underlying vector bundle is of the form $\mathcal{O}_{X_{\widehat{\overline{F}}}}(\lambda)^n$ is isomorphic to

$$V \otimes_E \mathcal{O}_{X_{\widehat{\overline{F}}}}(\lambda)$$

for a continuous representation V of G_F in a finite dimensional E-vector space ($\mathcal{O}_{X_{\widehat{\overline{F}}}(\lambda)} = \alpha^*\mathcal{O}_{X_F}(\lambda)$ has a canonical G_F-equivariant structure). Let us remark that $\mathcal{O}_{X_F}(\lambda)$ become trivial on $X_F \setminus \{\infty\}$. From this one deduces that M has a filtration whose graded pieces are of the form

$$V \otimes_E \mathrm{B}_{\widehat{\overline{F}},e}$$

for some finite dimensional E-representation of G_F. We now use the following result that generalizes theorem 5.7. Its proof is identical to the proof of theorem 5.7.

Theorem **6.33.** *For V a continuous finite dimensional E-representation of G_F one has*

$$H^1(G_F, V \otimes_E \mathrm{B}_{\widehat{\overline{F}},e}) = 0$$
$$H^0\big(G_F, V \otimes_E \mathrm{B}_{\widehat{\overline{F}},e}\big) \neq 0$$

The vanishing assertion in the preceding theorem tells us that in fact M is a direct sum of representations of the form $V \otimes_E \mathrm{B}_{\widehat{\overline{F}},e}$. We can thus suppose $M = V \otimes_E \mathrm{B}_{\widehat{\overline{F}},e}$. We now proceed by induction on the rank of M. Choose $x \in M^{G_F} \setminus \{0\}$ and let $N \subset M$ be the saturation of the submodule $\mathrm{B}_{\widehat{\overline{F}},e}.x$ that is to say

$$N/\mathrm{B}_{\widehat{\overline{F}},e}.x = (M/\mathrm{B}_{\widehat{\overline{F}},e}.x)_{tor}.$$

According to theorem 2.28 the $\mathrm{B}_{F,e}$-module $(N/\mathrm{B}_{\widehat{\overline{F}},e}.x)^{G_F}$ is of finite length and generates $N/\mathrm{B}_{\widehat{\overline{F}},e}.x$. Using the vanishing $H^1(G_F, \mathrm{B}_{\widehat{\overline{F}},e}) = 0$ one deduces that N^{G_F} is a torsion free finite type $\mathrm{B}_{F,e}$-module satisfying

$$N^{G_F} \otimes_{\mathrm{B}_{F,e}} \mathrm{B}_{\widehat{\overline{F}},e} = N.$$

Now, by induction we know that $(M/N)^{G_F}$ is a finite rank projective $\mathrm{B}_{F,e}$-module such that

$$(M/N)^{G_F} \otimes_{\mathrm{B}_{F,e}} \mathrm{B}_{\widehat{\overline{F}},e} = M/N.$$

Since $H^1(G_F, \mathrm{B}_{\widehat{\overline{F}},e}) = 0$ one has $H^1(G_F, N) = 0$. From this one deduces that M^{G_F} is a finite rank projective module satisfying

$$M^{G_F} \otimes_{\mathrm{B}_{F,e}} \mathrm{B}_{\widehat{\overline{F}},e} = M.$$

□

Here is an interesting corollary of theorem 6.29.

Corollary **6.34.** *Any G_F-equivariant vector bundle on $X_{\widehat{\overline{F}}}$ is a successive extension of G_F-equivariant line bundles.*

Example 6.35. *Let V be a finite dimensional E-representation of G_F. Then, even if V is irreducible, $V \otimes_E \mathcal{O}_{X_{\widehat{\overline{F}}}}$ is a successive extension of line bundles of the form $\chi \otimes_E \mathcal{O}_{X_{\widehat{\overline{F}}}}(d)$ where $\chi : G_F \to E^\times$ and $d \in \mathbb{Z}$.*

7. Vector bundles and φ-modules

7.1. The Robba ring and the bounded Robba ring.

We define a new ring

$$\mathscr{R}_F = \varinjlim_{\rho \to 0} \mathrm{B}_{]0,\rho]}$$

where $\mathrm{B}_{]0,\rho]}$ is the completion of B^b with respect to $(|.|_{\rho'})_{0<\rho'\leq\rho}$. Since

$$\varphi : \mathrm{B}_{]0,\rho]} \xrightarrow{\sim} \mathrm{B}_{]0,\rho^q]}$$

the ring $\mathscr{R}_F$ is equipped with a bijective Frobenius φ. In equal characteristic, when $E = \mathbb{F}_q((\pi))$, $\mathscr{R}_F = \mathcal{O}_{\mathbb{D}^*,0}$ the germs of holomorphic functions at 0 on $\mathbb{D}^*$.

Theorem **7.1** (Kedlaya [21] theo.2.9.6). *For all $\rho \in]0,1[$, the ring $\mathrm{B}_{]0,\rho]}$ is Bezout. Any closed ideal of $\mathrm{B}_{]0,\rho]}$ is principal.*

Proof. As in the proof of theorem 3.8, using theorem 3.8, the set of closed ideals of $\mathrm{B}_{]0,\rho]}$ is in bijection with $\mathrm{Div}^+(Y_{]0,\rho]})$ where $|Y_{]0,\rho]}| = \{\mathfrak{m} \in |Y| \mid 0 < \|\mathfrak{m}\| \leq \rho\}$. If F is algebraically closed, one can then write any $D \in \mathrm{Div}^+(Y_{]0,\rho]})$ as

$$D = \sum_{i\geq 0} [\mathfrak{m}_i]$$

with $\mathfrak{m}_i = (\pi - [a_i]) \in |Y_{]0,\rho]}|$ and $\lim_{i\to+\infty} \|\mathfrak{m}_i\| = 0$. The Weierstrass product

$$\prod_{i\geq 0} \Big(1 - \frac{[a_i]}{\pi}\Big)$$

is convergent in $\mathrm{B}_{]0,\rho]}$ and thus the divisor D is principal. For a general F this result remains true since one can prove that $H^1\big(G_F, \mathrm{B}^\times_{\widehat{\overline{F}},]0,\rho]}\big) = 0$ (this uses prop.7.10). This proves that any closed ideal of $\mathrm{B}_{]0,\rho]}$ is principal. It now remains to prove that for $f, g \in \mathrm{B}_{]0,\rho]}$ non zero satisfying $\mathrm{supp}(\mathrm{div}(f)) \cap \mathrm{supp}(\mathrm{div}(g)) = \emptyset$ the ideal generated by f and g is $\mathrm{B}_{]0,\rho]}$. This is a consequence of the following more general fact:

$$\mathrm{B}_{]0,\rho]}/(f) \xrightarrow{\sim} \prod_{\substack{\mathfrak{m}\in|Y| \\ \|\mathfrak{m}\|\leq\rho}} \mathrm{B}^+_{dR,\mathfrak{m}}/\mathrm{Fil}^{\mathrm{ord}_{\mathfrak{m}}(f)}\mathrm{B}^+_{dR,\mathfrak{m}}.$$

In fact, for $0 < \rho' \leq \rho$ one has

$$\mathrm{B}_{[\rho',\rho]}/(f) \xrightarrow{\sim} \prod_{\substack{\mathfrak{m}\in|Y| \\ \rho'\leq\|\mathfrak{m}\|\leq\rho}} \mathrm{B}^+_{dR,\mathfrak{m}}/\mathrm{Fil}^{\mathrm{ord}_{\mathfrak{m}}(f)}\mathrm{B}^+_{dR,\mathfrak{m}}.$$

The preceding isomorphism inserts into a projective system of exact sequences

$$0 \longrightarrow \mathrm{B}_{[\rho',\rho]} \xrightarrow{\times f} \mathrm{B}_{[\rho',\rho]} \longrightarrow \prod_{\substack{\mathfrak{m}\in|Y| \\ \rho'\leq\|\mathfrak{m}\|\leq\rho}} \mathrm{B}^+_{dR,\mathfrak{m}}/\mathrm{Fil}^{\mathrm{ord}_{\mathfrak{m}}(f)}\mathrm{B}^+_{dR,\mathfrak{m}} \longrightarrow 0.$$

when ρ' varies. Then, using remark 0.13.2.4 of [19] we have a Mittag Leffler type property and we can take the projective limit to obtain the result. □

Corollary **7.2.** *The ring $\mathscr{R}_F$ is Bezout.*

Define now for $\rho\in]0,1[$

$$\mathrm{B}^b_{]0,\rho]} = \big\{ f\in \mathrm{B}_{]0,\rho]} \mid \exists N\in\mathbb{Z},\ \sup_{0<\rho'\leq\rho} |\pi^N f|_{\rho'} < +\infty\big\}.$$

One can define the Newton polygon of an element of $\mathrm{B}_{]0,\rho]}$. This is defined only on an interval of $\mathbb{R}$ and has slopes in bewteen $-\log_q\rho$ and $+\infty$. The part with slopes in $]-\log_q\rho,+\infty]$ is the Legendre transform of the function $r\mapsto v_r(x)$ as in definition 1.11. As in the proof of theorem 3.9 the definition of the $-\log_q\rho$ slope part is a little bit more tricky. Anyway, using those Newton polygons, we have the following proposition that is of the same type as proposition 1.14.

Proposition **7.3.** *Any element of $\mathrm{B}^b_{]0,\rho]}$ is "meromorphic at 0", that is to say*

$$\mathrm{B}^b_{]0,\rho]} = \Big\{ \sum_{n\gg-\infty} [x_n]\pi^n \in W_{\mathcal{O}_E}(F)\big[\tfrac{1}{\pi}\big] \mid \lim_{n\to+\infty} |x_n|\rho^n = 0\Big\}.$$

Define now

$$\mathscr{E}^\dagger_F = \lim_{\substack{\longrightarrow \\ \rho\to 0}} \mathrm{B}^b_{]0,\rho]}.$$

One has $\mathscr{E}^\dagger_F \subset \mathscr{E}_F = W_{\mathcal{O}_E}(F)\big[\frac{1}{\pi}\big]$ (see the beginning of section 1.2.1). The valuation v_π on $\mathscr{E}_F$ induces a valuation v_π on $\mathscr{E}^\dagger_F$. In equal characteristic we have $v_\pi = \mathrm{ord}_0$. One then verifies easily the following.

Proposition **7.4.** *The ring $\mathscr{E}^\dagger_F$ is a Henselian valued field with completion the value field $\mathscr{E}_F$.*

7.2. **Link with the "classical Robba rings".** Choose $\epsilon\in\mathfrak{m}_F\setminus\{0\}$ and consider $\pi_\epsilon := [\epsilon]_Q \in W_{\mathcal{O}_E}(\mathcal{O}_F)$ as in section 2.4. Fix a perfect subfield $k\subset\mathcal{O}_F$ containing $\mathbb{F}_q$. Define the closed subfield

$$F_\epsilon = k((\epsilon)) \subset F.$$

The ring

$$\begin{aligned} \mathcal{O}_{\mathscr{E}_{F_\epsilon}} &= \widehat{W_{\mathcal{O}_E}(k)[\![u]\!]}[\tfrac{1}{u}] \\ &= \Big\{ \sum_{n\in\mathbb{Z}} a_n u^n \mid a_n\in W_{\mathcal{O}_E}(k),\ \lim_{n\to-\infty} a_n = 0\Big\} \end{aligned}$$

is a Cohen ring for F_ϵ, that is to say a π-adic valuation ring with

$$\mathcal{O}_{\mathscr{E}_{F_\epsilon}}/\pi\mathcal{O}_{\mathscr{E}_{F_\epsilon}} = F_\epsilon.$$

Its fraction field is $\mathscr{E}_{F_\epsilon} = \mathcal{O}_{\mathscr{E}_{F_\epsilon}}\big[\frac{1}{\pi}\big]$. This complete valued field has a henselian approximation, the henselian valued field $\mathscr{E}^\dagger_{F_\epsilon}$ with ring of integers

$$\mathcal{O}_{\mathscr{E}^\dagger_{F_\epsilon}} = \Big\{ \sum_{n\in\mathbb{Z}} a_n u^n \mid a_n\in W_{\mathcal{O}_E}(k),\ \exists\rho\in]0,1[\ \lim_{n\to-\infty} |a_n|\rho^n = 0\Big\}.$$

Consider now the Robba ring

$$\mathscr{R}_{F_\epsilon} = \lim_{\substack{\rho\to 1 \\ <}} \mathcal{O}\big(\mathbb{D}_{[\rho,1[}\big)$$

where $\mathcal{O}\big(\mathbb{D}_{[\rho,1[}\big)$ is the ring of rigid analytic functions *of the variable* u on the annulus $\{\rho\leq|u|<1\}$. One then has

$$\mathscr{E}^\dagger_{F_\epsilon} = \mathscr{R}^b_{F_\epsilon}$$

the sub ring of analytic functions on some $\mathbb{D}_{[\rho,1[}$ that are bounded. Via this rigid analytic description of $\mathscr{E}^\dagger_{F_\epsilon}$, the valuation v_π on it is such that for $f \in \mathscr{E}^\dagger_{F_\epsilon}$ seen as an element of $\mathscr{R}_{F_\epsilon}$

$$q^{-v_\pi(f)} = \lim_{\rho\to 1} |f|_\rho = |f|_1$$

where $|.|_\rho$ is the Gauss supremum norm on the annulus $\{|u| = \rho\}$. We equip those rings with the Frobenius φ given by

$$\varphi(u) = Q(u).$$

Proposition **7.5.** *The correspondence $u \mapsto \pi_\epsilon$ induces embeddings compatible with the Frobenius and the valuations*

$$\begin{array}{ccc} \mathscr{E}_{F_\epsilon} & \subset & \mathscr{E}_F \\ \cup & & \cup \\ \mathscr{E}^\dagger_{F_\epsilon} & \subset & \mathscr{E}^\dagger_F \\ \cap & & \cap \\ \mathscr{R}_{F_\epsilon} & \subset & \mathscr{R}_F. \end{array}$$

Proof. The injection $\mathcal{O}_{\mathscr{E}_{F_\epsilon}} \subset \mathcal{O}_{\mathscr{E}_F}$ is the natural injection between Cohen rings induced by the extension $F|F_\epsilon$. Since $\pi_\epsilon = \varphi\big(\Pi^-(u_\epsilon)\big)$, the Newton polygon of π_ϵ is $+\infty$ on $]-\infty, 0[$, takes the value $v(\epsilon)$ at 0 and has slopes $\big(\frac{\lambda}{q^n}\big)_{n\geq 0}$ with multiplicities 1 on $[0, +\infty[$ where

$$\lambda = \frac{q-1}{q} v(\epsilon).$$

In particular, for $\rho > 0$ satisfying $\rho \leq |\epsilon|^{\frac{q}{q-1}}$ one has

$$|\pi_\epsilon|_\rho = |\epsilon|.$$

Thus, if $a \in W_{\mathcal{O}_E}(k)_\mathbb{Q}$ and $n \in \mathbb{Z}$ then for $\rho = q^{-r} \in \big]0, |\epsilon|^{\frac{q}{q-1}}\big]$ one has

$$|a\pi_\epsilon|_\rho = |a|^r.|\epsilon|^n.$$

Thus, if $f(u) = \sum_{n\in\mathbb{Z}} a_n u^n \in \mathscr{R}_{F_\epsilon}$ then for $\rho = q^{-r} \in \big]0, |\epsilon|^{\frac{q}{q-1}}\big]$

$$|a_n \pi_\epsilon^n|_\rho \leq \big(|a_n|.(|\epsilon|^{1/r})^n\big)^r$$

where one has to be careful that on the left hand side of this expression $|.|_\rho$ stands for the Gauss norm on B^b with respect to the "formal variable π" and the right hand side the Gauss norm $|.|_{|\epsilon|^{1/r}}$ is taken with respect to the formal variable u. From this one deduces that if f is holomorphic on the annulus $\{|u| = |\epsilon|^{1/r}\}$ then the series $f(\pi_\epsilon) := \sum_{n\in\mathbb{Z}} a_n \pi_\epsilon^n$ converges in B_ρ. Since the condition $\rho \to 0$ is equivalent to $|\epsilon|^{1/r} \to 1$ one deduces a morphism

$$\begin{array}{ccc} \mathscr{R}_{F_\epsilon} & \longrightarrow & \mathscr{R}_F \\ f & \longmapsto & f(\pi_\epsilon) \end{array}$$

such that for all $\rho = q^{-r}$ sufficiently small,

$$|f(\pi_\epsilon)|_\rho \leq |f|^r_{|\epsilon|^{1/r}}.$$

A look at Newton polygons of elements of $\mathscr{R}_{F_\epsilon}$ tells us that that if $f \in \mathscr{E}^\dagger_{F_\epsilon}$ then for $r > 0$ sufficiently small, there exists $\alpha, \beta \in \mathbb{R}$ such that

$$v_r(f) = \alpha r + \beta.$$

This implies that for $r \gg 0$

$$|f|^r_{|\epsilon|^{1/r}} \leq A.\mathrm{B}^r$$

for some constants $A, \mathrm{B} \in \mathbb{R}_+$. From this one deduces easily that $f(\pi_\epsilon) \in \mathscr{E}^\dagger_F$. □

7.3. Harder-Narasimhan filtration of φ-modules over $\mathscr{E}^\dagger$.

7.3.1. *An analytic Dieudonné-Manin theorem.* Let $\varphi\text{-Mod}_{\mathscr{E}_F^\dagger}$ be the category of finite dimensional $\mathscr{E}_F^\dagger$-vector spaces equipped with a semi-linear automorphism. For $(D,\varphi)\in\varphi\text{-Mod}_{\mathscr{E}_F^\dagger}$ set

$$\deg(D,\varphi) = -v_\pi(\det\varphi)$$

which is well defined independently of the choice of a base of D since v_π is φ-invariant. Since the category $\varphi\text{-Mod}_{\mathscr{E}_F^\dagger}$ is abelian, there are Harder-Narasimhan filtrations in $\varphi\text{-Mod}_{\mathscr{E}_F^\dagger}$ for the slope function $\mu = \frac{deg}{rk}$. Let us remark that we also have such filtrations for the opposite slope function $-\mu$ that is to say up to replacing φ by φ^{-1}.

In the next theorem, if we replace $\mathscr{E}_F^\dagger$ by $\mathscr{E}_F = \widehat{\mathscr{E}_F^\dagger} = W_{O_E}(F)\big[\frac{1}{\pi}\big]$ we obtain the Dieudonné-Manin classification theorem. This theorem tells us that this classification extends to the Henselian case of $\mathscr{E}_F^\dagger$ that is to say the scalar extension induces an equivalence

$$\varphi\text{-Mod}_{\mathscr{E}_F^\dagger} \xrightarrow{\sim} \varphi\text{-Mod}_{\mathscr{E}_F}.$$

If F is algebraically closed, for each $\lambda\in\mathbb{Q}$ we note $\mathscr{E}_F^\dagger(\lambda)$ the standard isoclinic isocrystal with Dieudonné-Manin slope λ. One has $\mu(\mathscr{E}_F^\dagger(\lambda)) = -\lambda$.

Theorem **7.6.**

1. *A φ-module $(D,\varphi)\in\varphi\text{-}Mod_{\mathscr{E}_F^\dagger}$ is semi-stable of slope $-\lambda = \frac{d}{h}$ if and only if there is a $\mathcal{O}_{\mathscr{E}_F^\dagger}$-lattice $\Lambda\subset M$ such that $\varphi^h(\Lambda) = \pi^d\Lambda$.*
2. *If F is algebraically closed then semi-stable objects of slope λ in $\varphi\text{-}Mod_{\mathscr{E}_F^\dagger}$ are the ones isomorphic to a finite direct sum of $\mathscr{E}_F^\dagger(-\lambda)$.*
3. *The category of semi-stable φ-modules of slope 0 is equivalent to the category of E-local systems on $Spec(F)_{ét}$. In concrete terms, after the choice of an algebraic closure $\overline{F}$ of F*
$$\begin{array}{ccc} \varphi\text{-}Mod^{ss,0}_{\mathscr{E}_F^\dagger} & \xrightarrow{\sim} & Rep_E(G_F) \\ (D,\varphi) & \longmapsto & \big(D\otimes_{\mathscr{E}_F^\dagger}\mathscr{E}^\dagger_{\widehat{\overline{F}}}\big)^{\varphi=Id}. \end{array}$$
4. *The Harder-Narasimhan filtrations of (D,φ) (associated to the slope function μ) and (D,φ^{-1}) (associated to the slope function $-\mu$) are opposite filtrations that define a canonical splitting of the Harder-Narasimhan filtration. There is a decomposition*
$$\varphi\text{-}Mod_{\mathscr{E}_F^\dagger} = \bigoplus_{\lambda\in\mathbb{Q}}^{\perp}\varphi\text{-}Mod^{ss,\lambda}_{\mathscr{E}_F^\dagger}$$
that is orthogonal in the sense that if A, resp. B, is semi-stable of slope λ, resp. μ, with $\lambda\neq\mu$ then $Hom(A,\mathrm{B}) = 0$. If F is algebraically closed the category $\varphi\text{-}Mod_{\mathscr{E}_F^\dagger}$ is semi-simple.

The main point we wanted to stress in this section is point (4) of the preceding theorem. In fact, if one replaces $\mathscr{E}_F^\dagger$ by $\mathscr{R}_F$ we will see in the following section that φ-modules of $\mathscr{R}_F$ have Harder-Narasimhan filtrations for the slope function μ. But, although φ is bijective on $\mathscr{R}_F$, there are no Harder-Narasimhan filtrations for the opposite slope function $-\mu$ that is to say for φ^{-1}-modules (we have to use prop. 7.11). In as sens, this is why there is no canonical splitting of the Harder-Narasimhan filtration for φ-modules over $\mathscr{R}_F$.

Sketch of proof of theorem 7.6. The non-algebraically closed case is deduced from the algebraically closed one thanks to the following Galois descent result (see [4] III.3.1 that applies for any F thanks to the vanishing result 2.16).

Proposition **7.7** (Cherbonnier-Colmez)**.** *The scalar extension functor is an equivalence between finite dimensional $\mathscr{E}_F^\dagger$-vector spaces and finite dimensional $\mathscr{E}^\dagger_{\widehat{\overline{F}}}$ vector spaces equipped with a continuous semi-linear action of $Gal(\overline{F}|F)$.*

We now suppose F is algebraically closed. We note $\psi=\varphi^{-1}$ that is more suited than φ for what we want to do. One then checks the proof of the theorem is reduced to the following statements:

(1) If $a\in\mathbb{Z}$ and $b\in\mathbb{N}_{\geq 1}$, for any $(D,\varphi)\in\varphi\text{-Mod}_{\mathscr{E}_F^\dagger}$ admitting an $\mathcal{O}_{\mathscr{E}_F^\dagger}$-lattice Λ such that $\psi^b(\Lambda)=\pi^a\Lambda$ one has $D^{\psi^b=\pi^a}=(D\otimes\mathscr{E}_F)^{\psi^b=\pi^a}$.

(2) If $a\in\mathbb{Z}$ and $b\in\mathbb{N}_{\geq 1}$, $Id-\pi^a\psi^b:\mathscr{E}_F^\dagger\to\mathscr{E}_F^\dagger$ is surjective.

In fact, the first point implies that any (D,φ) has a decreasing filtration $(\text{Fil}^\lambda D)_{\lambda\in\mathbb{Q}}$ satisfying $\text{Gr}^\lambda D\simeq\mathscr{E}_F^\dagger(-\lambda)$. The second point shows that for $\mu<\lambda$ one has $\text{Ext}^1(\mathscr{E}_F^\dagger(\lambda),\mathscr{E}_F^\dagger(\mu))=0$ and thus the preceding filtration is split.

For point (1), up to replacing ψ by a power, that is to say E by a finite unramified extension, and twisting we can suppose $a=0$ and $b=1$. For $\rho\in]0,1[$ let

$$\begin{aligned} A_\rho &= \big\{x\in \mathrm{B}^b_{]0,\rho]}\mid \forall\rho'\in]0,\rho],\ |x|_{\rho'}\leq 1\big\}\\ &= \Big\{\sum_{n\geq 0}[x_n]\pi^n\in \mathrm{B}^b_{]0,\rho]}\cap\mathcal{O}_{\mathscr{E}_F^\dagger}\mid\forall n,\ |x_n|\rho^n\leq 1\Big\}.\end{aligned}$$

Then A_ρ is stable under ψ,

$$A_\rho\big[\tfrac{1}{\pi}\big]=\mathrm{B}^b_{]0,\rho]}$$

and

$$\varinjlim_{\rho\to 0}A_\rho=\big\{x\in\mathcal{O}_{\mathscr{E}_F^\dagger}\mid x \bmod \pi\in\mathcal{O}_F\big\}.$$

Now, $\Lambda/\pi\Lambda$ is an F-vector space equipped with a Frob_q^{-1}-linear endomorphism $\overline{\psi}$ the reduction of ψ. One can find a basis of this vector space in which the matrix of $\overline{\psi}$ has coefficients in $\mathcal{O}_F$. Lifting such a basis we obtain a basis of Λ in which the matrix of ψ has coefficients in $\varinjlim_{\rho\to 0}A_\rho$. Let $C\in M_h(A_\rho)$, ρ sufficiently small, be the matrix of ψ in such a basis. For $k\geq 0$ and $x=\sum_{i\geq 0}[x_i]\pi^i\in\mathcal{O}_{\mathscr{E}_F}$ set

$$|x|_{k,\rho}=\sup_{0\leq i\leq k}|x_i|\rho^i.$$

One has for $x,y\in\mathcal{O}_{\mathscr{E}_F}$

$$|xy|_{k,\rho}\leq|x|_{k,\rho}|y|_{k,\rho}.$$

Now for $x=(x_1,\dots,x_h)\in\mathcal{O}_{\mathscr{E}_F}^h$ set

$$\|x\|_{k,\rho}=\sup_{1\leq j\leq h}|x_j|_{k,\rho}.$$

If $x\in\mathcal{O}_{\mathscr{E}_F}^h$ satisfies

$$C\psi(x)=x$$

by iterating, we obtain for all n

$$C\psi(C)\cdots\psi^{n-1}(C).\psi^n(x)=x.$$

But since C has coefficients in A_ρ, for all $i\geq 0$, $\psi^i(C)$ has coefficients in A_ρ and one deduces that for all $k,n\geq 0$

$$\|x\|_{k,\rho}\leq\|\psi^n(x)\|_{k,\rho}.$$

But for $y\in\mathcal{O}_{\mathscr{E}_F}$,

$$\lim_{n\to+\infty}|\psi^n(y)|_{k,\rho}\leq 1.$$

From this one deduces that for all k, $\|x\|_{k,\rho}\leq 1$ and thus $x\in(\mathrm{B}^b_{]0,\rho']})^h$ as soon as $\rho'<\rho$. This proves point (1).

For point (2), $\mathrm{B}^b_{]0,\rho]}$ is complete with respect to $(|.|_{\rho'})_{0\leq\rho'\leq\rho}$ where $|.|_0=q^{-v_\pi}$. Moreover one checks that the operator $\pi^a\psi^b$ is topologically nilpotent with respect to those norms and thus $Id-\pi^a\psi^b$ is bijective on $\mathrm{B}^b_{]0,\rho]}$. □

7.3.2. *The non-perfect case: Kedlaya's flat descent.* Let $\epsilon \in \mathfrak{m}_F$ non zero and $F_\epsilon = k((\epsilon))$ as in section 7.2. The Frobenius φ of $\mathscr{E}^\dagger_{F_\epsilon}$ is not bijective like in the preceding sub-section. Let $\varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}}$ be the category of couples (D,φ) where D is a finite dimensional $\mathscr{E}^\dagger_{F_\epsilon}$-vector space and φ a semi-linear automorphism, that is to say the linearization of φ is an isomorphism $\Phi : D^{(\varphi)} \xrightarrow{\sim} D$. We define in the same way $\varphi\text{-Mod}_{\mathscr{E}_{F_\epsilon}}$. As before, setting $\deg(D,\varphi) = -v_\pi(\det\varphi)$, there is a degree function on those abelian categories of φ-modules.

Theorem **7.8** (Kedlaya [22])**.** *A φ-module (D,φ) over $\mathscr{E}^\dagger_{F_\epsilon}$, resp. $\mathscr{E}_{F_\epsilon}$, is semi-stable of slope $-\lambda = \frac{d}{h}$ if and only if there exists an $\mathcal{O}_{\mathscr{E}^\dagger_{F_\epsilon}}$-lattice, resp. $\mathcal{O}_{\mathscr{E}_{F_\epsilon}}$-lattice, $\Lambda \subset D$ satisfying $\Phi^h(\Lambda) = \pi^d\Lambda$.*

Outline of the proof. Let us recall how this theorem is deduced from Dieudonné-Manin by Kedlaya using a faithfully flat descent technique. We treat the case of $\varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}}$, the other case being identical. We can suppose F is algebraically closed. We consider the scalar extension functor

$$- \otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F : \varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}} \longrightarrow \varphi\text{-Mod}_{\mathscr{E}_F}.$$

Via this scalar extension, the Harder-Narasimhan slope functions correspond:

$$\mu\big(D \otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F, \varphi\otimes\varphi\big) = \mu(D,\varphi).$$

The first step is to prove that $(D,\varphi) \in \varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}}$ is semi-stable of slope λ if and only its scalar extension to $\mathscr{E}_F$ is semi-stable of slope λ. One direction is clear: if $(D\otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F, \varphi\otimes\varphi)$ is semi-stable of slope λ then (D,φ) is semi-stable of slope λ. In the other direction, let us consider the diagram of rings equipped with Frobenius

$$\mathscr{E}^\dagger_{F_\epsilon} \longrightarrow \mathscr{E}_F \underset{i_2}{\overset{i_1}{\rightrightarrows}} \mathscr{E}_F \underset{\mathscr{E}^\dagger_{F_\epsilon}}{\otimes} \mathscr{E}_F$$

where the Frobenius on $\mathscr{E}_F \underset{\mathscr{E}^\dagger_{F_\epsilon}}{\otimes} \mathscr{E}_F$ is $\varphi\otimes\varphi$, $i_1(x) = x\otimes 1$ and $i_2(x) = 1\otimes x$. This induces a diagram of categories of φ-modules

$$\varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}} \longrightarrow \varphi\text{-Mod}_{\mathscr{E}_F} \underset{i_{2*}}{\overset{i_{1*}}{\rightrightarrows}} \varphi\text{-Mod}_{\mathscr{E}_F \underset{\mathscr{E}^\dagger_{F_\epsilon}}{\otimes} \mathscr{E}_F}.$$

Now, faithfully flat descent tells us that for $A \in \varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}}$, the sub-objects of A are in bijection with the sub-objects B of $A \otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F$ satisfying $i_{1*}\mathrm{B} = i_{2*}\mathrm{B}$. Suppose now $A \in \varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}}$ is semi-stable and $A' := A\otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F$ is not. Let

$$0 \subsetneq A'_1 \subsetneq \cdots \subsetneq A'_r = A'$$

be the Harder-Narasimhan filtration of A'. The Dieudonné-Manin theorem gives us the complete structure of the graded pieces of this filtration in $\varphi\text{-Mod}_{\mathscr{E}_F}$. Let us prove by descending induction on $j \geq 1$ that

$$i_{1*}A'_1 \subset i_{2*}A'_j.$$

In fact, if $j > 1$ and $i_{1*}A'_1 \subset i_{2*}A'_j$ then one can look at the composite morphism

$$i_{1*}A'_1 \hookrightarrow i_{2*}A'_j \longrightarrow i_{2*}A'_j/A'_{j-1}.$$

Dieudonné-Manin tells us that this morphism is given by a finite collection of elements in

$$\big(\mathscr{E}_F \otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F\big)^{\varphi^h = \pi^d}$$

where $h \in \mathbb{N}_{\geq 1}$, $d \in \mathbb{Z}$ and $\frac{d}{h}$ is the Dieudonné-Manin slope of A'_1 minus the one of A'_j/A'_{j-1} which is thus strictly negative (recall the Harder-Narasimhan slope is the opposite of the Dieudonné-Manin one). We thus have $d < 0$ and lemma 7.9 tells us this is 0. We conclude $i_{1*}A'_1 \subset i_{2*}A'_{j-1}$

and obtain by induction that $i_{1*}A'_1 \subset i_{2*}A'_j$. By symmetry we thus have

$$i_{1*}A'_1 = i_{2*}A'_1$$

and A'_1 descends to a sub-object of A which contradicts the semi-stability of A. We thus have proved that $A \in \varphi\text{-Mod}_{\mathscr{E}^\dagger_{F_\epsilon}}$ is semi-stable if and only if $A \otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F$ is semi-stable.

Theorem 7.8 is easily reduced to the slope 0 case. Thus, let (D, φ) be semi-stable of slope 0. Let $\Lambda \subset D$ be a lattice. Then,

$$\Lambda' = \sum_{k \geq 0} \mathcal{O}_{\mathscr{E}^\dagger_{F_\epsilon}} \varphi^k(\Lambda) \subset D$$

is a lattice since after scalar extension to $\mathscr{E}_F$, $(D \otimes_{\mathscr{E}^\dagger_{F_\epsilon}} \mathscr{E}_F, \varphi \otimes \varphi)$ is isoclinic with slope 0. This lattice is stable under φ, but since (D, φ) has slope 0, automatically

$$\mathcal{O}_{\mathscr{E}^\dagger_{F_\epsilon}} \varphi(\Lambda') = \Lambda'.$$

□

Lemma **7.9.** *The ring* $\mathcal{O}_{\mathscr{E}_F} \underset{\mathcal{O}_{\mathscr{E}^\dagger_{F_\epsilon}}}{\otimes} \mathcal{O}_{\mathscr{E}_F}$ *is* π*-adically separated.*

7.4. **The Harder-Narasimhan filtration of φ-modules over $\mathscr{R}_F$.** Since the ring $\mathscr{R}_F$ is Bezout, for an $\mathscr{R}_F$-module M the following are equivalent:

- M is free of finite rank,
- M is torsion free of finite type,
- M is projective of finite type.

Moreover, if $\text{Frac}(\mathscr{R}_F)$ is the fraction field of $\mathscr{R}_F$ and $\text{Vect}_{\text{Frac}(\mathscr{R}_F)}$ is the associated category of finite dimensional vector spaces, the functor $- \otimes_{\mathscr{R}_F} \text{Frac}(\mathscr{R}_F)$ is a generic fiber functor in the sense that for a free $\mathscr{R}_F$-module of finite type M it induces a bijection

$$\big\{\text{direct factor sub modules of } M\big\} \xrightarrow{\sim} \{\text{sub-Frac}(\mathscr{R}_F)\text{-vector spaces of } M \otimes_{\mathscr{R}_F} \text{Frac}(\mathscr{R}_F).$$

with inverse the map $W \mapsto W \cap M$, "the schematical closure of W in M". Let $\varphi\text{-Mod}_{\mathscr{R}_F}$ be the category of finite rank free $\mathscr{R}_F$-modules M equipped with a φ-linear isomorphism $\varphi : M \xrightarrow{\sim} M$. There are two additive functions on the exact category $\varphi\text{-Mod}_{\mathscr{R}_F}$

$$\deg, \text{rk} : \varphi\text{-Mod}_{\mathscr{R}_F} \longrightarrow \mathbb{Z}$$

where the rk is the rank and the degree is defined using the following proposition that is deduced from Newton polygons considerations.

Proposition **7.10.** *One has the equality* $\big(\text{B}_{]0,\rho]}\big)^\times = \big(\text{B}^b_{]0,\rho]}\big)^\times$ *and thus*

$$\mathscr{R}_F^\times = (\mathscr{E}_F^\dagger)^\times.$$

Of course the valuation v_π on $\mathscr{E}_F^\dagger$ is invariant under φ. This allows us to define

$$\deg(M, \varphi) = -v_\pi(\det \varphi).$$

As in [10], to have Harder-Narasimhan filtrations in the exact category $\varphi\text{-Mod}_{\mathscr{R}_F}$ we now need to prove that any isomorphism that is "an isomorphism in generic fiber", that is to say after tensoring with $\text{Frac}(\mathscr{R}_F)$,

$$f : (M, \varphi) \longrightarrow (M', \varphi')$$

induces the inequality

$$\deg(M, \varphi) \leq \deg(M', \varphi')$$

with equality if and only if f is an isomorphism. This is achived by the following proposition.

Proposition **7.11.** *Let* $x \in \mathscr{R}_F$ *non zero such that* $\varphi(x)/x \in \mathscr{E}_F^\dagger$*. Then*

$$v_\pi(\varphi(x)/x) \leq 0$$

with equality if and only if $x \in \mathscr{R}_F^\times$*.*

Proof. For $x \in \mathscr{E}_F^\dagger$ one has
$$v_\pi(x) = \lim_{r\to+\infty} \frac{v_r(x)}{r}.$$
But, for $r \gg 0$,
$$v_r(\varphi(x)/x) = qv_{r/q}(x) - v_r(x).$$
But if $x \in \mathrm{B}_{]0,\rho]}$, for $q^{-r} \in]0,\rho[$, the number $\frac{v_r(x)}{r}$ is the intersection with the x-axis of the line with slope r that is tangent to Newt(x). From this graphic interpretation, one deduces that as soon as r_0 is such that the intersection of the tangent line to Newt(x) of slope r_0 with $Newt(x)$ is in the upper half plane then for $r \geq r_0$, $r \mapsto \frac{v_r(x)}{r}$ is a decreasing function and it is bounded if and only if Newt(x)(t) $= +\infty$ for $t \ll 0$. □

We thus have a good notion of Harder-Narasimhan filtrations in the exact category $\varphi\text{-Mod}_{\mathscr{R}_F}$. We note $\mu = \deg/\mathrm{rk}$ the associated slope function.

7.5. **Classification of φ-modules over $\mathscr{R}_F$: Kedlaya's theorem.** Suppose F is algebraically closed. For each slope $\lambda \in \mathbb{Q}$ there is associated an object
$$\mathscr{R}_F(\lambda) \in \varphi\text{-Mod}_{\mathscr{R}_F}$$
satisfying
$$\mu\big(\mathscr{R}_F(\lambda)\big) = -\lambda.$$
This is the image of the simple isocrystal with Dieudonné-Manin slope λ via the scalar extension functor
$$\varphi\text{-Mod}_{\mathscr{E}_F^\dagger} \longrightarrow \varphi\text{-Mod}_{\mathscr{R}_F}.$$
Next theorem tells us that this functor is essentially surjective (but not full).

Theorem **7.12** (Kedlaya [21])**.** *Suppose F is algebraically closed.*

1. *The semi-stable objects of slope λ in φ-Mod$_{\mathscr{R}_F}$ are the direct sums of $\mathscr{R}_F(-\lambda)$.*
2. *The Harder-Narasimhan filtration of a φ-module over $\mathscr{R}_F$ is split.*
3. *There is a bijection*
$$\begin{aligned} \{\lambda_1 \geq \cdots \geq \lambda_n \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{Q}\} &\xrightarrow{\sim} \varphi\text{-}Mod_{\mathscr{R}_F}/\sim \\ (\lambda_1,\dots,\lambda_n) &\longmapsto \Big[\bigoplus_{i=1}^n \mathscr{R}_F(-\lambda_i)\Big]. \end{aligned}$$

In particular for each slope $\lambda \in \mathbb{Q}$ scalar extension induces an equivalence
$$\varphi\text{-Mod}^{ss,\lambda}_{\mathscr{E}_F^\dagger} \xrightarrow{\sim} \varphi\text{-Mod}^{ss,\lambda}_{\mathscr{R}_F}$$
and $(M,\varphi) \in \varphi\text{-Mod}_{\mathscr{R}_F}$ is semi-stable of slope $\lambda = \frac{d}{h}$ if and only if there is a free of the same rank as M $\mathcal{O}_{\mathscr{E}_F^\dagger}$-sub-module $\Lambda \subset M$ generating M and satisfying $\varphi^h(\Lambda) = \pi^{-d}\Lambda$.

7.6. **Application: classification of φ-modules over** B**.** As a consequence of theorem 3.9 one obtains the following.

Theorem **7.13.** *The algebra* B *is a Frechet-Stein algebra in the sense of Schneider-Teitelbaum (*[28]*).*

Recall ([28]) there is a notion of coherent sheaf on the Frechet-Stein algebra B. A coherent sheaf on B is a collection of modules $(M_I)_I$ where I goes through the set of compact intervals in $]0,1[$ and M_I is a B_I-module together with isomorphisms
$$M_I \otimes_{\mathrm{B}_I} \mathrm{B}_J \xrightarrow{\sim} M_J$$
for $J \subset I$, satisfying the evident compatibility relations for three intervals $K \subset J \subset I$. This is an abelian category. There is a global section functor
$$\Gamma : (M_I)_I \longmapsto \varprojlim_I M_I$$

from coherent sheaves to B-modules. It is fully faithful exact and identifies the category of coherent sheaves with an abelian subcategory of the category of B-modules. This functor has a left adjoint

$$M \longmapsto (M \otimes_{\mathrm{B}} \mathrm{B}_I)_I.$$

On the essential image of Γ this induces an equivalence with coherent sheaves. By definition, a coherent sheaf $(M_I)_I$ on B is a vector bundle if for all I, M_I is a free B_I-module of finite rank.

Proposition **7.14.** *The global sections functor* Γ *induces an equivalence of categories between vector bundles on* B *and finite type projective* B*-modules.*

The proof is similar to the one of proposition 2.1.15 of [23]. More precisely, the main difficulty is to prove that the global sections M of a coherent sheaf $(M_I)_I$ such that for some integer r all M_I are generated by r elements is a finite type B-module. For this one writes $]0,1[= F_1 \cup F_2$ where F_1 and F_2 are locally finite infinite *disjoint* unions of compact intervals. Then, one constructs for $i = 1, 2$ by approximations techniques global sections $f_{i,1}, \cdots, f_{i,r} \in M$ that generate each M_I for I a connected component of F_i. The sum of those sections furnishes a morphism $\mathrm{B}^{2r} \to M$ that induces a surjection $\mathrm{B}_I^{2r} \to M_I$ for all I a connected component of F_i, $i = 1, 2$. Thanks to lemma 7.15 this induces surjections $\mathrm{B}_I^{2r} \to M_I$ for any compact interval of $]0,1[$.

The following lemma is an easy consequence of theorem 3.9.

Lemma **7.15.** *For a finite collection of compact intervals* $I_1, \cdots, I_n \subset]0,1[$ *with union* I *the morphism* $\coprod_{k=1,\cdots,n} \mathit{Spec}(\mathrm{B}_{I_k}) \to \mathit{Spec}(\mathrm{B}_I)$ *is an fpqc covering.*

Let us come back to φ-modules. Let $\varphi\text{-Mod}_{\mathrm{B}}$ be the category of finite type projective B-modules M equipped with a semi-linear isomorphism $\varphi : M \xrightarrow{\sim} M$. For $\rho \in]0,1[$ the ring $\mathrm{B}_{]0,\rho]}$ is equipped with the endomorphism φ^{-1} satisfying $\varphi^{-1}\big(\mathrm{B}_{]0,\rho]}\big) = \mathrm{B}_{]0,\rho^{1/q}]}$ and thus

$$\mathrm{B} = \bigcap_{n\geq 0} \varphi^{-n}\big(\mathrm{B}_{]0,\rho]}\big).$$

Note $\varphi^{-1}\text{-mod}_{\mathrm{B}_{]0,\rho]}}$ the category of finite rank free $\mathrm{B}_{]0,\rho]}$-modules M equipped with a semi-linear isomorphism $\varphi : M \to M$ (by a semi-linear isomorphism we mean a semi-linear morphism whose linearisation is an isomorphism). Of course,

$$\varinjlim_{\rho\to 0} \varphi^{-1}\text{-mod}_{\mathrm{B}_{]0,\rho]}} \xrightarrow{\sim} \varphi^{-1}\text{-mod}_{\mathscr{R}_F} = \varphi\text{-Mod}_{\mathscr{R}_F}.$$

If $(M, \varphi^{-1}) \in \varphi^{-1}\text{-mod}_{\mathrm{B}_{]0,\rho]}}$ then the collection of modules $\big(\varphi^{-n} M\big)_{n\geq 0}$ defines a vector bundle on B whose global sections is

$$\bigcap_{n\geq 0} \varphi^{-n}(M).$$

Using proposition 7.14 one obtains the following.

Proposition **7.16.** *The scalar extension functor induces an equivalence*

$$\varphi\text{-}\mathit{Mod}_{\mathrm{B}} \xrightarrow{\sim} \varphi\text{-}\mathit{Mod}_{\mathscr{R}_F}.$$

Applying Kedlaya's theorem [21] one thus obtains:

Theorem **7.17.** *If* F *is algebraically closed there is a bijection*

$$\begin{array}{ccc} \{\lambda_1 \geq \cdots \geq \lambda_n \mid n \in \mathbb{N},\ \lambda_i \in \mathbb{Q}\} & \xrightarrow{\sim} & \varphi\text{-}\mathit{Mod}_{\mathrm{B}}/\sim \\ (\lambda_1, \ldots, \lambda_n) & \longmapsto & \Big[\bigoplus_{i=1}^n \mathrm{B}(-\lambda_i)\Big]. \end{array}$$

For $(M, \varphi) \in \varphi\text{-Mod}_{\mathrm{B}}$ define

$$\mathscr{E}(M,\varphi) = \widetilde{\bigoplus_{d\geq 0} M^{\varphi=\pi^d}},$$

a quasi-coherent sheaf on the curve X. Using theorem 7.17 together with the classification of vector bundles theorem 6.9 one obtains the following theorem.

Theorem **7.18.** *If F is algebraically closed there is an equivalence of exact categories*

$$\begin{array}{rcl} \varphi\text{-}Mod_{\mathrm{B}} & \xrightarrow{\sim} & Bun_X \\ (M,\varphi) & \longmapsto & \mathscr{E}(M,\varphi). \end{array}$$

Via this equivalence one has

$$\begin{array}{rcl} H^0\big(X,\mathscr{E}(M,\varphi)\big) & = & M^{\varphi=Id} \\ H^1\big(X,\mathscr{E}(M,\varphi)\big) & = & \mathrm{coker}\big(M \xrightarrow{Id-\varphi} M\big). \end{array}$$

Remark 7.19. *In equal characteristic, when $E=\mathbb{F}_q((\pi))$, $Y=\mathbb{D}_F^*$ and the classification of φ-vector bundles on Y is due to Hartl and Pink ([20]). Via theorem 7.18 this is the same as the classification of φ-modules over* B*. We explained the proof of the classification theorem 6.9 only when $E|\mathbb{Q}_p$. However, the same proof works when $E=\mathbb{F}_q((\pi))$ using periods of π-divisible $\mathcal{O}_E$-modules. In this case, theorem 7.18 is thus still valid.*

Sadly, there is no direct short proof of theorem 7.18 that would allow us to recover Kedlaya or Hartl Pink classification theorem from the classification of vector bundles on the curve.

However, one of the first steps in their proof is that if $(M,\varphi)\in\varphi\text{-Mod}_{\mathscr{R}_F}$ then $M^{\varphi=\pi^d}\neq 0$ for $d\gg 0$. As a consequence, any φ-module over $\mathscr{R}_F$ (and thus B) is an iterated extension of rank 1 modules. Those are easy to classify and thus any φ-module over B is a successive extension of $\mathrm{B}(\lambda)$ with $\lambda\in\mathbb{Z}$. Taking this granted plus the fact that for $\lambda\in\mathbb{Z}$, $H^1(\mathrm{B}(\lambda+d))$ (the cokernel of $Id-\varphi$) is zero for $d\gg 0$, one deduces that for any $(M,\varphi)\in\varphi\text{-Mod}_{\mathrm{B}}$, $\mathscr{E}(M,\varphi)$ is a vector bundle. Then, if one knows explicitly that for all $d\in\mathbb{Z}$ and $i=0,1$, $H^i(\mathrm{B}(d))\xrightarrow{\sim} H^i(X,\mathcal{O}_X(d))$ (this is easy for $i=0$, and is deduced from the fundamental exact sequence plus computations found in the work of Kedlaya and Hartl Pink for $i=1$) one can deduce a proof that the functor $(M,\varphi)\mapsto\mathscr{E}(M,\varphi)$ is fully faithful and thus the classification of vector bundles on the curve gives back Kedlaya and Hartl Pink theorem.

7.7. **Classification of φ-modules over** B^+**.** Recall from section 1.2.1 that for $\rho\in]0,1[$, $\mathrm{B}^+_\rho=\mathrm{B}^+_{[\rho,1[}$ and that

$$\mathrm{B}^+=\bigcap_{\rho>0}\mathrm{B}^+_\rho=\bigcap_{n\geq 0}\varphi^n\big(\mathrm{B}^+_{\rho_0}\big)$$

for any ρ_0. Moreover, for any $x\in\mathrm{B}_{[\rho,1[}$ there is defined a Newton polygon Newt(x) and

$$\begin{array}{rcl} \mathrm{B}^+_\rho & = & \big\{x\in\mathrm{B}_{[\rho,1[}\mid \mathrm{Newt(x)}\geq 0\big\} \\ \mathrm{B}^+ & = & \big\{x\in\mathrm{B}\mid \mathrm{Newt(x)}\geq 0\big\} \end{array}$$

In fact, by concavity of the Gauss valuation $r\mapsto v_r(x)$, for any $x\in\mathrm{B}_{[\rho,1[}$ the limit

$$|x|_1:=\lim_{\rho\to 1}|x|_\rho$$

exists in $[0,+\infty]$ and equals $q^{-v_0(x)}$ for $x\in\mathrm{B}^b$ and

$$\mathrm{B}^+_\rho=\big\{x\in\mathrm{B}_{[0,\rho[}\mid |x|_1\leq 1\big\}.$$

Let us note $|x|_1:=q^{-v_0(x)}$. One has

$$v_0(x)=\lim_{+\infty}\mathrm{Newt(x)}$$

and on B^+_ρ, v_0 is a valuation extending the valuation previously defined on B^b.

One has to be careful that, contrary to B, the Frechet algebra B^+ is not Frechet-Stein since the rings B^+_ρ are not noetherian and it is not clear whether $\varphi:\mathrm{B}^+_\rho\to\mathrm{B}^+_\rho$ (that is to say the inclusion $\mathrm{B}^+_{\rho^q}\subset\mathrm{B}^+_\rho$) is flat or not.

Note $\varphi\text{-Mod}_{\mathrm{B}^+}$ and $\varphi\text{-Mod}_{\mathrm{B}^+_\rho}$ for the associated categories of finite rank *free* modules equipped with a semi-linear isomorphism. The category $\varphi\text{-Mod}_{\mathrm{B}^+_\rho}$ does not depend on ρ. There is a scalar extension functor

$$\varphi\text{-Mod}_{\mathrm{B}^+} \longrightarrow \varphi\text{-Mod}_{\mathrm{B}}$$

and, using proposition 7.14, a functor

$$\begin{array}{ccc} \varphi\text{-Mod}_{\mathrm{B}^+_\rho} & \longrightarrow & \varphi\text{-Mod}_{\mathrm{B}} \\ (M,\varphi) & \longmapsto & \bigcap_{n\geq 0} \varphi^n\big(M\otimes_{\mathrm{B}^+_\rho}\mathrm{B}_{[\rho,1[}\big). \end{array}$$

Proposition **7.20.** *The functors φ-Mod$_{\mathrm{B}^+} \to \varphi$-Mod$_{\mathrm{B}}$ and φ-Mod$_{\mathrm{B}^+_\rho} \to \varphi$-Mod$_{\mathrm{B}}$ are fully faithfull.*

Proof. Let's treat the case of $\varphi\text{-Mod}_{\mathrm{B}^+}$, the case of $\varphi\text{-Mod}_{\mathrm{B}^+_\rho}$ being identical. Using internal Hom's this is reduced to proving that for $(M,\varphi)\in\varphi\text{-Mod}_{\mathrm{B}^+}$ one has

$$M^{\varphi=Id} \xrightarrow{\sim} \big(M\otimes_{\mathrm{B}^+}\mathrm{B}\big)^{\varphi=Id}.$$

Let us fix a basis of M and for $x=(x_1,\dots,x_n)\in \mathrm{B}^n\simeq M\otimes\mathrm{B}$ and $r>0$ set

$$W_r(x)=\inf_{1\leq i\leq n} v_r(x_i).$$

An element $a\in\mathrm{B}^+$ satisfies $v_r(a)\geq 0$ for $r\gg 0$. Let us fix $r_0\gg 0$ such that all the coefficients $(a_{ij})_{i,j}$ of the matrix of φ in the fixed basis of M satisfy $v_{r_0}(a_{i,j})\geq 0$. Then if $x\in M\otimes\mathrm{B}$ satisfies $\varphi(x)=x$ one has

$$qW_{\frac{r_0}{q}}(x)=W_{r_0}(\varphi(x))\geq W_{r_0}(x)$$

and thus for $k\geq 1$

$$W_{\frac{r_0}{q^n}}(x)\geq \frac{1}{q^n}W_{r_0}(x).$$

Taking the limit when $n\to+\infty$ one obtains $W_0(x)\geq 0$ that is to say $x\in(\mathrm{B}^+)^n\simeq M$. □

The preceding proposition together with theorem 7.17 then gives the following.

Theorem **7.21.** *Suppose F is algebraically closed. For $A\in\{\mathrm{B}^+,\mathrm{B}^+_\rho\}$ there is a bijection*

$$\begin{array}{ccc} \{\lambda_1\geq\dots\geq\lambda_n \mid n\in\mathbb{N},\ \lambda_i\in\mathbb{Q}\} & \xrightarrow{\sim} & \varphi\text{-}Mod_A/\sim \\ (\lambda_1,\dots,\lambda_n) & \longmapsto & \Big[\bigoplus_{i=1}^n A(-\lambda_i)\Big]. \end{array}$$

One deduces there are equivalences of categories

$$\varphi\text{-Mod}_{\mathrm{B}^+}\xrightarrow{\sim}\varphi\text{-Mod}_{\mathrm{B}^+_\rho}\xrightarrow{\sim}\varphi\text{-Mod}_{\mathrm{B}}\xrightarrow{\sim}\varphi\text{-Mod}_{\mathscr{R}_F}$$

where an inverse of the first equivalence is given by $M\mapsto \cap_{n\geq 0}\varphi^n(M)$.

7.8. **Another proof of the classification of φ-modules over B^+ and B^+_ρ.** We explain how to give a direct proof of theorem 7.21 without using Kedlaya's theorem 7.12. This proof is much simpler and in fact applies even if the field F is not algebraically closed:

Theorem **7.22.** *Theorem 7.21 remains true for any F with algebraically closed residue field.*

This relies on the introduction of a new ring called $\overline{\mathrm{B}}$. Set

$$\begin{aligned} \mathfrak{P} &= \big\{x\in\mathrm{B}^{b,+}\mid v_0(x)>0\big\} \\ &= \big\{\sum_{n\gg-\infty}[x_n]\pi^n \mid x_n\in\mathcal{O}_F,\ \exists C>0,\ \forall n, v(x_n)\geq C\big\} \end{aligned}$$

and

$$\overline{\mathrm{B}}=\mathrm{B}^{b,+}/\mathfrak{P}.$$

If k stands for the residue field of $\mathcal{O}_F$, there is a reduction morphism

$$\mathrm{B}^{b,+} \longrightarrow W_{\mathcal{O}_E}(k)_{\mathbb{Q}}.$$

Then, $\overline{\mathrm{B}}$ is a local ring with residue field $W_{\mathcal{O}_E}(k)_{\mathbb{Q}}$. Let's begin by classifying φ-modules over $\overline{\mathrm{B}}$.

Theorem **7.23.** *Any φ-module over $\overline{\mathrm{B}}$ is isomorphic to a direct sum of $\overline{\mathrm{B}}(\lambda)$, $\lambda \in \mathbb{Q}$.*

Sketch of proof. One first proves that for $\lambda, \mu \in \mathbb{Q}$, $\mathrm{Ext}^1(\overline{\mathrm{B}}(\lambda), \overline{\mathrm{B}}(\mu)) = 0$. For two φ-modules M and M' one has $\mathrm{Ext}^1(M, M') = H^1(M^\vee \otimes M')$ where for a φ-module M'', $H^1(M'') = \mathrm{coker}(Id - \varphi_{M''})$. Up to replacing φ by a power of itself, that is to say replacing E by an unramified extension, we are thus reduced to proving that for any $d \in \mathbb{Z}$,

$$Id - \pi^d \varphi : \overline{\mathrm{B}} \longrightarrow \overline{\mathrm{B}}$$

is surjective. For $d > 0$, this is a consequence of the fact that

$$Id - \pi^d \varphi : W_{\mathcal{O}_E}(\mathcal{O}_F) \longrightarrow W_{\mathcal{O}_E}(\mathcal{O}_F)$$

is surjective since $\pi^d \varphi$ is topologically nilpotent on $W_{\mathcal{O}_E}(\mathcal{O}_F)$ for the π-adic topology. For $d < 0$ this is deduced in the same way using $Id - \pi^{-d}\varphi^{-1}$. For $d = 0$, this is a consequence of the fact that $Id - \varphi$ is bijective on $W_{\mathcal{O}_E}(\mathfrak{m}_F)$ since φ is topologically nilpotent on $W_{\mathcal{O}_E}(\mathfrak{m}_F)$ for the $([a], \pi)$-adic topology for any $a \in \mathfrak{m}_F \setminus \{0\}$ (the topology induced by the Gauss norms $(|.|_\rho)_{\rho \in]0,1[}$) and since the residue field k of F is algebraically closed.

Let $(M, \varphi) \in \varphi\text{-Mod}_{\overline{\mathrm{B}}}$ and $M_k = M \otimes W_{\mathcal{O}_E}(k)_{\mathbb{Q}}$ be the associated isocrystal. Let λ be the smallest slope of (M_k, φ). It suffices now to prove that M has a sub φ-module isomorphic to $\overline{\mathrm{B}}(\lambda)$ whose underlying $\overline{\mathrm{B}}$-module is a direct factor. Up to raising φ to a power and twisting one is reduced to the case $\lambda = 0$. Then M_k has a sub-lattice Λ stable under φ. Let us remark that any element of $\mathrm{B}^{b,+}$ whose image in $W_{\mathcal{O}_E}(k)_{\mathbb{Q}}$ lies in $W_{\mathcal{O}_E}(k)$ is congruent modulo $\mathfrak{p}$ to an element of $W_{\mathcal{O}_E}(\mathcal{O}_F)$. Lifting the basis of Λ to a basis of M (recall $\overline{\mathrm{B}}$ is a local ring), one then checks there is a free φ-module N over $W_{\mathcal{O}_E}(\mathcal{O}_F)$ together with a morphism $N \to M$ inducing an isomorphism $N \otimes_{W_{\mathcal{O}_E}(\mathcal{O}_F)} \overline{\mathrm{B}} \xrightarrow{\sim} M$ and such that $N \otimes W_{\mathcal{O}_E}(k) \xrightarrow{\sim} \Lambda$. For such an N, there is an isomorphism

$$N^{\varphi = Id} \xrightarrow{\sim} \Lambda^{\varphi = Id}$$

such that $N^{\varphi = Id} \otimes_E W_{\mathcal{O}_E}(\mathcal{O}_F)$ is a direct factor in N. In fact, after fixing a basis of N, $N \simeq W_{\mathcal{O}_E}(\mathcal{O}_F)^n$ is complete with respect to the family of norms $(\|.\|_\rho)_{\rho \in]0,1[}$ where $\|(x_1, \ldots, x_n)\|_\rho = \sup_{1 \le i \le n} |x_i|_\rho$. Moreover the Frobenius of N is topologically nilpotent on $W_{\mathcal{O}_E}(\mathfrak{m}_F)^n$ for this set of norms. The result is deduced (for the direct factor assertion, one has to use that $W_{\mathcal{O}_E}(\mathfrak{m}_F)$ is contained in the Jacobson radical of $W_{\mathcal{O}_E}(\mathcal{O}_F)$ together with Nakayama lemma). □

To make the link between $\mathrm{B}^+, \mathrm{B}^+_\rho$ and $\overline{\mathrm{B}}$ we need the following.

Lemma **7.24.** *For any $a \in \mathfrak{m}_F \setminus \{0\}$, $\mathrm{B}^+ = [a]\mathrm{B}^+ + \mathrm{B}^{b,+}$ and $\mathrm{B}^+_\rho = [a]\mathrm{B}^+_\rho + \mathrm{B}^{b,+}$.*

In fact, for any $x = \sum_{n \gg -\infty} [x_n]\pi^n$ let us note $x^+ = \sum_{n \ge 0} [x_n]\pi^n$ and $x^- = \sum_{n<0} [x_n]\pi^n$. Then any $x \in \mathrm{B}^+$, resp. B^+_ρ, can be written as $\sum_{n \ge 0} x_n$ with $x_n \in \mathrm{B}^{b,+}$ going to zero when $n \to +\infty$. But one checks that if $x_n \underset{n \to +\infty}{\longrightarrow} 0$ then for $n \gg 0$, $x_n^- \in [a]\mathrm{B}^{b,+}$. This proves the lemma.

As a consequence of this lemma, if $r = v(a)$, $\{x \in \mathrm{B}^+ \mid v_0(x) \ge r\} = [a]\mathrm{B}^+$ and the same for B^+_ρ. Moreover, we deduce that the inclusion $\mathrm{B}^{b,+} \to \mathrm{B}^+$ induces an isomorphism

$$\overline{\mathrm{B}} \xrightarrow{\sim} \mathrm{B}^+/\{v_0 > 0\}$$

and the same for B^+_ρ. From this we deduce surjections $\mathrm{B}^+ \to \overline{\mathrm{B}}$ and $\mathrm{B}^+_\rho \to \overline{\mathrm{B}}$. Now, theorem 7.21 is a consequence of theorem 7.23 and the following.

Proposition **7.25.** *The reduction functor φ-Mod$_{\mathrm{B}^+} \to \varphi$-Mod$_{\overline{\mathrm{B}}}$ is fully faithful. The same holds for B^+_ρ.*

Sketch of proof. We treat the case of B^+, the case of B^+_ρ being identical. This is reduced to proving that for $(M,\varphi)\in\varphi\text{-Mod}_{\mathrm{B}^+}$, if $\overline{M}$ is the associated module over $\overline{\mathrm{B}}$, then $M^{\varphi=Id}\xrightarrow{\sim}\overline{M}^{\varphi=Id}$. If $\Lambda=\mathfrak{p}M$ this is reduced to proving that

$$Id-\varphi:\Lambda\xrightarrow{\sim}\Lambda.$$

Fix a basis of $M\simeq(\mathrm{B}^+)^n$ and note $A=(a_{ij})_{i,j}\in\mathrm{GL}_n(\mathrm{B}^+)$ the matrix of φ in this base. For each $r\geq 0$ and $m=(x_1,\cdots,x_n)\in M$ set $\|m\|_r=\inf_{1\leq i\leq n}v_r(x_i)$. Let $r_0>0$ be fixed. Then $\Lambda=\mathfrak{p}^n$ is complete with respect to the set of additive norms $(\|.\|)_{r>0}$. Note $\|A\|_r=\inf_{i,j}v_r(a_{i,j})$.

Fix an $r>0$. One first checks that for any $m\in M$ and $k\geq 1$

$$\|\varphi^k(m)\|_r\geq q^k\|m\|_{\frac{r}{q^k}}+\sum_{i=0}^{k-1}q^i\|A\|_{\frac{r}{q^i}}.$$

Now, according to inequality (3) of section 1.2.1 (the inequality is stated for $\mathrm{B}^{b,+}$ but extends by continuity to B^+), for any $r'\leq r$ one has

$$\|A\|_{r'}\geq\frac{r'}{r}\|A\|_r$$

and thus

$$\sum_{i=0}^{k-1}q^i\|A\|_{\frac{r}{q^k}}\geq\alpha k+\beta$$

for some constants $\alpha,\beta\in\mathbb{R}$. But now, if $m\in\Lambda$, $\lim\limits_{r'\to 0}\|m\|_{r'}=\|m\|_0>0$ and thus

$$\lim_{k\to+\infty}\|\varphi^k(m)\|_r=+\infty.$$

We deduce that φ is topologically nilpotent on Λ and thus $Id-\varphi$ is bijective on it. □

Remark 7.26. *The preceding proof does not use the fact that F is algebraically closed and thus theorem 7.21 remains true when F is any perfectoid field with algebraically closed residue field. On this point, there is a big difference between φ-modules over* B^+ *and the ones over* B. *In fact, one can prove that φ-modules over* B *satisfy Galois descent like vector bundles (th.6.29) for any perfectoid field F and thus for a general F theorem 7.17 is false.*

Remark 7.27. *As a consequence of the classification theorem and the first part of the proof of theorem 7.23, for any $M,M'\in\varphi\text{-Mod}_{\mathrm{B}^+}$, $Ext^1(M,M')=0$. This is not the case for φ-modules over* B. *The equivalence 7.18 is an equivalence of exact categories and for example $Ext^1_{\varphi\text{-}Mod_{\mathrm{B}}}(\mathrm{B},\mathrm{B}(1))\neq 0$. In fact, although the scalar extension functor $\varphi\text{-}Mod_{\mathrm{B}^+}\xrightarrow{\sim}\varphi\text{-}Mod_{\mathrm{B}}$ is exact, its inverse is not.*

Let us conclude with a geometric interpretation of the preceding result. Set $\Sigma=\mathrm{Spec}(\mathbb{Z}_p)$ and for an $\mathbb{F}_p$-scheme S note $F\text{-Isoc}_{S/\Sigma}$ for the category of F-isocrystals. If $S\hookrightarrow S'$ is a thickening then $F\text{-Isoc}_{S/\Sigma}\simeq F\text{-Isoc}_{S'/\Sigma}$. Let now $a\in\mathfrak{m}_F\setminus\{0\}$, $\rho=|a|$ and $S_\rho=\mathrm{Spec}(\mathcal{O}_F/\mathcal{O}_Fa)$. The category $F\text{-Isoc}_{S_\rho/\Sigma}$ does not depend on the choice of $\rho\in]0,1[$. The crystalline site $\mathrm{Cris}(S_\rho/\Sigma)$ has an initial object $A_{cris,\rho}$ such that $A_{cris,\rho}\big[\frac{1}{p}\big]=\mathrm{B}_{cris,\rho}$ (see sec. 1.2.1). We thus have an equivalence

$$F\text{-Isoc}_{S_\rho/\Sigma}\simeq\varphi\text{-Mod}_{\mathrm{B}^+_{cris,\rho}}.$$

But since $\mathrm{B}^+_{\rho^p}\subset\mathrm{B}^+_{cris,\rho}\subset\mathrm{B}^+_{\rho^{p-1}}$ we have an equivalence

$$\varphi\text{-Mod}_{\mathrm{B}^+_{cris,\rho}}\simeq\varphi\text{-Mod}_{\mathrm{B}^+_\rho}.$$

Moreover, one can think of $\varphi\text{-Mod}_{\mathrm{B}^+}$ as being the category of "convergent F-isocrystals on S_ρ". We thus have proved the following.

Theorem **7.28.** *Suppose F is a perfectoid field with algebraically closed residue field. Then any F-isocrystal, resp. convergent F-isocrystal, on S_ρ is isotrivial that is to say comes from the residue field of k after the choice of a splitting $\mathcal{O}_F \rightleftarrows k$*

References

[1] Y. André. Slope filtrations. *Confluentes Mathematici*, 1, 2009.

[2] Laurent Berger. Représentations p-adiques et équations différentielles. *Invent. Math.*, 148(2):219–284, 2002.

[3] Laurent Berger. Construction de (ϕ, Γ)-modules: représentations p-adiques et B-paires. *Algebra Number Theory*, 2(1):91–120, 2008.

[4] F. Cherbonnier and P. Colmez. Représentations p-adiques surconvergentes. *Invent. Math.*, 133(3):581–611, 1998.

[5] J.-F. Dat, S. Orlik, and M. Rapoport. *Period domains over finite and p-adic fields*, volume 183 of *Cambridge Tracts in Mathematics*. Cambridge University Press, Cambridge, 2010.

[6] Vladimir G. Drinfeld. Coverings of p-adic symmetric domains. *Functional Analysis and its Applications*, 10(2):29–40, 1976.

[7] G. Faltings. Group schemes with strict $\mathcal{O}$-action. *Mosc. Math. J.*, 2(2):249–279, 2002.

[8] G. Faltings. Coverings of p-adic period domains. *J. Reine Angew. Math.*, 643:111–139, 2010.

[9] L. Fargues. L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld et applications cohomologiques. In *L'isomorphisme entre les tours de Lubin-Tate et de Drinfeld*, Progress in math., 262, pages 1–325. Birkhäuser, 2008.

[10] L. Fargues. La filtration de Harder-Narasimhan des schémas en groupes finis et plats. *J. Reine Angew. Math.*, 645:1–39, 2010.

[11] L. Fargues. La filtration canonique des points de torsion des groupes p-divisibles. *Annales scientifiques de l'ENS*, 44(6):905–961, 2011.

[12] L. Fargues and J.-M. Fontaine. Courbes et fibrés vectoriels en théorie de Hodge p-adique. *Prépublication.*

[13] L. Fargues and J.-M. Fontaine. Factorization of analytic functions in mixed characteristic. *To appear in the proceedings of a conference in Sanya.*

[14] L. Fargues and J.-M. Fontaine. Vector bundles and p-adic galois representations. *AMS/IP Studies in Advanced Mathematics Volume 51, 2011.*

[15] J.-M. Fontaine. *Groupes p-divisibles sur les corps locaux.* Société Mathématique de France, Paris, 1977. Astérisque, No. 47-48.

[16] J.-M. Fontaine. Le corps des périodes p-adiques. *Astérisque*, (223):59–111, 1994. With an appendix by Pierre Colmez, Périodes p-adiques (Bures-sur-Yvette, 1988).

[17] B. Gross and M. Hopkins. Equivariant vector bundles on the Lubin-Tate moduli space. In *Topology and representation theory (Evanston, IL, 1992)*, volume 158 of *Contemp. Math.*, pages 23–88. Amer. Math. Soc., Providence, RI, 1994.

[18] A. Grothendieck. Sur la classification des fibrés holomorphes sur la sphère de Riemann. *Amer. J. Math.*, 79:121–138, 1957.

[19] A. Grothendieck. Éléments de géométrie algébrique. III. Étude cohomologique des faisceaux cohérents. I. *Inst. Hautes Études Sci. Publ. Math.*, (11):167, 1961.

[20] U. Hartl and R. Pink. Vector bundles with a Frobenius structure on the punctured unit disc. *Compos. Math.*, 140(3):689–716, 2004.

[21] K. Kedlaya. Slope filtrations revisited. *Doc. Math.*, 10:447–525, 2005.

[22] K. Kedlaya. Slope filtrations for relative Frobenius. *Astérisque*, (319):259–301, 2008. Représentations p-adiques de groupes p-adiques. I. Représentations galoisiennes et (ϕ, Γ)-modules.

[23] K. Kedlaya, J. Pottharst, and L. Xiao. Cohomology of arithmetic families of (phi,gamma)-modules. *arXiv:1203.5718v1.*

[24] G. Laffaille. Groupes p-divisibles et corps gauches. *Compositio Math.*, 56(2):221–232, 1985.

[25] M. Lazard. Les zéros des fonctions analytiques d'une variable sur un corps valué complet. *Inst. Hautes Études Sci. Publ. Math.*, (14):47–75, 1962.

[26] B. Poonen. Maximally complete fields. *Enseign. Math. (2)*, 39(1-2):87–106, 1993.

[27] M. Rapoport and T. Zink. *Period spaces for p-divisible groups.* Number 141 in Annals of Mathematics Studies. Princeton University Press, Princeton, NJ, 1996.

[28] P. Schneider and J. Teitelbaum. Algebras of p-adic distributions and admissible representations. *Invent. Math.*, 153(1):145–196, 2003.

[29] P. Scholze. Perfectoid spaces. *Preprint.*

[30] S. Sen. Continuous cohomology and p-adic Galois representations. *Invent. Math.*, 62(1):89–116, 1980/81.

[31] J. T. Tate. *p-divisible groups.*. In *Proc. Conf. Local Fields (Driebergen, 1966)*, pages 158–183. Springer, Berlin, 1967.

[32] J.-P. Wintenberger. Le corps des normes de certaines extensions infinies de corps locaux; applications. *Ann. Sci. École Norm. Sup. (4)*, 16(1):59–89, 1983.

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