p-ADIC PERIODS : A SURVEY

by

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RÉSUMÉ : Soient K un corps de caractéristique 0, complet pour une valuation discrète, à corps résiduel parfait de caractéristique $p > 0$ et $\overline{K}$ une clôture algébrique de K. On passe en revue

a) les principaux types de représentations p-adiques de $\mathrm{Gal}(\overline{K}/K)$: représentations de Hodge-Tate, de de Rham, cristallines, semi-stables, potentiellement semi-stables;

b) les principaux théorèmes et/ou conjectures reliant les différentes cohomologies p-adiques des variétés propres et lisses sur K (en particulier la cohomologie de de Rham d'une telle variété X et la cohomologie étale p-adique de $X \otimes \overline{K}$).

Chemin faisant, on esquisse la construction de Hyodo-Kato de la cohomologie cristalline à pôles logarithmiques des variétés propres et lisses sur K admettant un modèle sur l'anneau des entiers de K ayant réduction semi-stable.

ABSTRACT : Let K be a field of characteristic 0, complete with respect to a discrete valuation, with perfect residue field of characteristic $p > 0$; let $\overline{K}$ be an algebraic closure of K. In this survey, we discuss

a) the main kinds of p-adic representations of $\mathrm{Gal}(\overline{K}/K)$: Hodge-Tate, de Rham, crystalline, semi-stable, potentially semi-stable;

b) the main theorems and/or conjectures linking the different kinds of p-adic cohomologies of proper and smooth varieties over K.

Meanwhile, we sketch Hyodo-Kato's construction of crystalline cohomology with log poles for those varieties which admit a model over the integers with semi-stable reduction.

Mots-clés : Périodes p-adiques, représentations p-adiques, cohomologie cristalline, cohomologie étale, cohomologie de de Rham, théorèmes de comparaison, réduction semi-stable, monodromie.

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In these notes, K is a field, complete with respect to a discrete valuation, whose residue field k is perfect of characteristic $p > 0$; unless otherwise stated, K is assumed to be of characteristic 0. We choose an algebraic closure $\overline{K}$ of K and put $G = \mathrm{Gal}(\overline{K}/K)$. The action of G and the valuation extend to the completion C of $\overline{K}$ with respect to the usual topology. For any subfield L of C, we denote by $\mathcal{O}_L$ the ring of the integers of L. We denote by K_0 the fraction field of $W = W(k)$ and by σ the Frobenius automorphism of k ($\sigma x = x^p$, if $x \in k$), which acts also on W and K_0.

0. Introduction

Let X be a proper and smooth K-scheme. Very roughly speaking, the theory of p-adic periods is concerned with the problem of comparing the p-adic étale cohomology of $X_{\overline{K}} = X \otimes \overline{K}$, endowed with its natural action of G, with the de Rham or Hodge cohomology of X/K, endowed with certain additional structures. The story goes back to Tate, who conjectured [Se67] the existence of a canonical, G-equivariant decomposition (*"Hodge-Tate decomposition"*)

$$\oplus_{0 \leq i \leq m} C(-i) \otimes_K H^{m-i}(X, \Omega^i_{X/K}) \simeq C \otimes_{\mathbb{Q}_p} H^m(X_{\overline{K}}, \mathbb{Q}_p).$$

Tate [Ta67] proved his conjecture when X has good reduction and $m = 1$. Then several special cases where proved (see [Bo80] for the proof of Raynaud for $m = 1$ in the bad reduction case, and [Fo82b], [BK86], [FM87], [Hy88a]); now it has been proved in general by Faltings [Fa88a].

When X has good reduction, the p-adic and de Rham cohomologies of X in degree 1 are related in a somewhat indirect way by the p-divisible group Γ of the Néron model of the Albanese variety of X. Indeed, by a theorem of Tate [Ta67], Γ up to isogeny is determined by the p-adic representation $H^1(X_{\overline{K}}, \mathbb{Q}_p)$, which is the dual of $T_p(\Gamma) \otimes \mathbb{Q}$; on the other hand, by a theorem of Grothendieck ([Gr70], see also [Gr74], [Me72], [MM74]), Γ up to isogeny is determined by $H^1_{DR}(X/K)$, endowed with

i) its Hodge filtration $H^0(X, \Omega^1_{X/K})$,

ii) its K_0-structure, together with its σ-linear Frobenius automorphism, given by the Dieudonné module of $\Gamma \otimes k$.

Grothendieck (loc. cit.) raised the question of giving an algebraic construction (*"mysterious functor"*) by which one could recover one object from the other, without using the "crutch" of the p-divisible groupe Γ, and asked for a generalization in higher degree. Such a construction was given in [Fo77], and a generalization valid for X of good reduction and all degrees m, was proposed in [Fo79] and [Fo82a] as the (C_{cris}) conjecture. After some special cases ([FM87], [Ka86]), this conjecture was proved by Faltings [Fa88b] and is discussed in § 3. It involves a certain ring B_{cris}, whose definition is recalled in § 1, and the K_0-structure on $H^m_{DR}(X/K)$ (together with its σ-semi-linear Frobenius automorphism) given by the crystalline cohomology of the special fiber of a smooth model of X over $\mathcal{O}_K$.

When X is no longer assumed to have good reduction, it is still possible to compare $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ (as a representation of G) and $H^m_{DR}(X/K)$ (with its Hodge filtration) by means of a bigger ring B_{DR} : this is the (C_{DR}) conjecture [Fo82a], which has also been proved by Faltings [Fa88b], see § 5.1. In this case, the latter group can be recovered from the former one, but not conversely.

Between the case of good reduction and the general case lies the semi-stable reduction case, namely the case where X admits a proper and flat model $\mathcal{X}$ over $\mathcal{O}_K$ with semi-stable reduction. In this case, a new cohomology theory, due to Hyodo and Kato ([Hy88b], [HK89]), gives on $H^m_{DR}(X/K)$ a K_0-structure endowed with a σ-linear automorphism (as in the good reduction case) and a nilpotent endomorphism N, playing the role of the logarithm of the monodromy. Following a suggestion of Jannsen (see also [Ja88]), a variant of the conjecture (C_{cris}), the conjecture (C_{st}) was made ([Fo87], [Fo89b]) and proved in a special case by Kato [Ka88b]. It involves a certain ring B_{st} intermediate between B_{cris} and B_{DR}, and enables one to build $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ (as a representation of G) from $H^m_{DR}(X/K)$ (with its additonal structure), and vice-versa. This is discussed in § 4 after some glances at the (better known) complex and ℓ-adic situations (see also [Il89]).

It may be too optimistic to believe that any proper and smooth K-scheme X has *"potentially semi-stable reduction"*, i.e. that if we replace K by a suitable finite extension, then we get a scheme which has semi-stable reduction. Nevertheless, it seems reasonable to expect that the cohomology behaves as if it was true. This leads to the p-adic monodromy conjecture discussed in § 5, section 2, to the effect that there should always exist on $H^m_{DR}(X)$ enough additonal structure to recover $H^m(X_{\overline{K}}, \mathbb{Q}_p)$ from it. In particular, one should be able to define the monodromy operator, a nilpotent endomorphism of the K-vector space $H^m_{DR}(X/K)$, without any restriction on X.

The definitions and basic properties of the rings B_{DR}, B_{cris} and B_{st} are recalled in § 1. They give rise to several functors from the category of p-adic representations of G to certain categories of filtered modules (with additional operators) and lead to a classification of these representations into main types : crystalline, semi-stable, de Rham; this is discussed in § 2.

Finally, in § 6, we consider the case where k is finite and ask some questions on the comparison of the various ℓ-adic representations (with possibly $\ell = p$) of the Weil-Deligne group of K arising from the ℓ-adic étale cohomology, for $\ell \neq p$, and from the structure alluded to above for $\ell = p$.

Between January and May 1988, a seminar on p-adic periods was held at the IHES in Bures-sur-Yvette. Hopefully a written version of this seminar [Bures], including some recent results, should be available soon. The present notes, which intend to be no more than an introduction to both this seminar and Faltings's work on this subject, contain no proofs, not even sketches of proofs.

Contents

§ 1. The rings of periods

1.1. The ring R

1.1.1. Let R be the commutative ring which is the projective limit of

$$\mathcal{O}_{\overline{K}}/p \xleftarrow{\sigma} \mathcal{O}_{\overline{K}}/p \xleftarrow{\sigma} \cdots \mathcal{O}_{\overline{K}}/p \xleftarrow{\sigma} \mathcal{O}_{\overline{K}}/p \xleftarrow{\sigma} \cdots,$$

where $\sigma a = a^p$.

Let $x = (x_m)_{m \in \mathbb{N}} \in R$; for each $m \in \mathbb{N}$, choose a lifting $\widehat{x}_m$ of x_m in $\mathcal{O}_C$. If $n \in \mathbb{N}$, the sequence $\{(\widehat{x}_{n+m})^{p^m} | m \in \mathbb{N}\}$ converges in $\mathcal{O}_C$ to an element $x^{(n)}$ independent of the choice of the lifting; the map

$$x \longmapsto (x^{(n)})_{n \in \mathbb{N}}$$

is a bijection between R and the set of sequences $(x^{(n)})_{n \in \mathbb{N}}$ of elements of $\mathcal{O}_C$ satisfying $(x^{(n+1)})^p = x^{(n)}$ for $n \in \mathbb{N}$. In what follows, we use this bijection to identify these two sets; we then have

$$(xy)^{(n)} = x^{(n)} y^{(n)} \quad \text{and} \quad (x+y)^{(n)} = \lim_{m \mapsto \infty} (x^{(n+m)} + y^{(n+m)})^{p^m}.$$

1.1.2. The ring R is a valuation ring whose residue field can be identified to the residue field $\overline{k}$ of $\overline{K}$:

- If v denotes a valuation of $\overline{K}$ and also its unique extension to C, the map $x \longmapsto v_R(x) = v(x^{(0)})$ is a valuation of R;
- the identification of $\overline{k}$ to a subfield of R is given by the map

$$a \longmapsto ([a^{p^{-n}}])_{n \in \mathbb{N}}$$

(where $[b] \in \mathcal{O}_{\overline{K}}$ is the Teichmüller representative of $b \in \overline{k}$).

1.1.3. The group G acts in an obvious way on R and on its fraction field FrR, which is algebraically closed. More precisely, let's choose a generator of *"the multiplicative Tate module of G_m"*, that is an element $\varepsilon = (\varepsilon^{(n)})_{n \in \mathbb{N}} \in R$ satisfying $\varepsilon^{(0)} = 1$ and $\varepsilon^{(1)} \neq 1$. The field of formal power series $E = k((\varepsilon - 1))$ is a subfield of FrR, its separable closure F in FrR is separably closed and FrR may be identified to its completion for the $(\varepsilon - 1)$-adic topology; moreover, if $H = \mathrm{Gal}(\overline{K}/K(\mu_{p^\infty}))$, E and F are stable under G, E is fixed under H and the natural map $H \longrightarrow \mathrm{Gal}(F/E)$ is an isomorphism ([FW79], [Wi83]).

1.2. The field B_{DR} and the ring B_{HT}

1.2.1. The ring $W(R)$ of Witt vectors with coefficients in R is a W-algebra that is an integral domain. The map

$$\theta : W(R) \longrightarrow \mathcal{O}_C,$$

which sends $(a_0, a_1, \ldots, a_n, \ldots)$ to $\sum_{n\in\mathbb{N}} p^n a_n^{(n)}$ is a surjective homomorphism of W-algebras, and the kernel is a principal ideal.

1.2.2. If $W_K(R) = K \otimes_W W(R)$, θ extends in an unique way to a surjective homomorphism of K-algebras, that we still denote by

$$\theta : W_K(R) \longrightarrow C.$$

The kernel of this map is again principal ([Fo82a], if π is a uniformising parameter of K and if $x \in R$ is such that $x^{(0)} = \pi$, $1 \otimes [x] - \pi \otimes 1$ is a generator[1]). We denote by B_{DR}^+ the completion of $W_K(R)$ with respect to the $(\mathrm{Ker}\,\theta)$-adic topology and by B_{DR} its fraction field. Therefore B_{DR} is a discrete valuation field whose residue field is C; its ring of integers is B_{DR}^+ and contains $W_K(R)$ as a dense subring.

As the map θ is G-equivariant, the natural action of G on $W_K(R)$ extends to B_{DR}.

1.2.3. The series

$$\log([\varepsilon]) = \sum_{n\geq 1} (-1)^{n-1} \cdot ([\varepsilon] - 1)^n / n$$

(where ε is as in 1.1.3) converges in B_{DR}^+ to a non-zero element t. This enables us to identify $\mathbb{Z}_p(1)$ to the sub-$\mathbb{Z}_p$-module of B_{DR}^+ generated by t. It is not hard to prove [Fo82a] that t is a generator of the maximal ideal of B_{DR}^+.

1.2.4. There is a natural filtration on B_{DR}, indexed by $\mathbb{Z}$, given by the (positive and negative) powers of the maximal ideal of B_{DR}^+ : we have

$$Fil^i B_{DR} = t^i \cdot B_{DR}^+, \quad \text{for any } i \in \mathbb{Z}.$$

1.2.5. We denote by B_{HT} (HT stands for Hodge-Tate) the associated graded algebra. For each $i \in \mathbb{Z}$, $gr^i B_{HT} = Fil^i B_{DR} / Fil^{i+1} B_{DR}$ is a one-dimensional C-vector space, spanned by the image of t^i, hence

$$B_{HT} = \oplus_{i\in\mathbb{Z}} C(i),$$

where $C(i) = C \otimes \mathbb{Z}_p(i)$ is the usual Tate's twist.

1.2.6. As an abstract field B_{DR} is isomorphic to the field $C((t))$, because the projection of B_{DR}^+ onto C as a section; it is important to

[1] For any $y \in R$, we denote by $[y] = (y, 0, 0, \ldots, 0, \ldots) \in W(R)$ its Teichmüller representative.

note that there is no canonical section, even **no section which is Galois equivariant**. Nevertheless, the restriction of any section to $\overline{K}$ is unique and this enables us to view B_{DR}^+ and B_{DR} as $\overline{K}$-algebras.

1.3. The ring B_{cris}

1.3.1. We call A_{cris} the ring (often called $W^{DP}(R)$, cf. [Fo83a], [Fo83b]) which is the p-adic completion of the divided power envelope of $W(R)$ with respect to the kernel of θ. In down-to-earth terms, if we identify $W(R)$ to a subring of $W_{K_0}(R) = K_0 \otimes_W W(R) = W(R)[1/p]$ and if a is a generator of $\ker\theta$, A_{cris} is the p-adic completion of the sub-$W(R)$-algebra (or, equivalently, the sub-$W(R)$-module) A_{cris}^f of $W_{K_0}(R)$ generated by the $\gamma_m(a) := a^m/m!$, for $m \in \mathbb{N}$. This is a W-algebra which is an integral domain and we put

$$B_{cris}^+ = K_0 \otimes_W A_{cris} = A_{cris}[1/p].$$

1.3.2. The map

$$\varphi : W(R) \longrightarrow W(R),$$

which sends $(a_0, a_1, \ldots, a_n, \ldots)$ to $(a_0^p, a_1^p, \ldots a_n^p, \ldots)$ is an automorphism of the ring $W(R)$, which is σ-semi-linear and can uniquely be extended to an automorphism of $W_{K_0}(R)$. Because A_{cris}^f is stable under φ, φ extends by continuity to A_{cris} and also to B_{cris}^+.

In the same way (the ring A_{cris} is stable under the natural action of G on $W_{K_0}(R)$), the action of G extends to A_{cris} and to B_{cris}^+.

1.3.3. The composite map

$$A_{cris}^f \longrightarrow W_{K_0}(R) \longrightarrow W_K(R) \longrightarrow B_{DR}^+$$

extends by continuity[2] to a map from A_{cris} to B_{DR}^+ which is injective and compatible with the action of G. We use it to identify A_{cris} and B_{cris}^+ to subrings of B_{DR}^+.

One has $t \in A_{cris}$ and $t^{p-1} \in pA_{cris}$ (where t is as in 1.2.3). We put

$$B_{cris} = B_{cris}^+[1/t] = A_{cris}[1/t].$$

This is a subring of B_{DR} stable under G and, because $\varphi t = pt$, the action of φ on B_{cris}^+ extends to B_{cris} (we have $\varphi(t^{-1}) = p^{-1}t^{-1}$).

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[2] One as to be a little bit careful about topologies : on R, we take the topology defined by the valuation, on $W(R)$ (which is $R^{\mathbb{N}}$ as a set) the product topology, on $W_K(R) = K \otimes_W W(R)$ the tensor product topology, on $W_K(R)/(\mathrm{Ker}\,\theta)^i$ the induced topology and on $B_{DR}^+ = \lim \cdot \mathrm{proj} \cdot W_K(R)/(\mathrm{Ker}\,\theta)^i$ the projective limit topology.

1.4. The ring B_{st}

1.4.1. For each commutative group Γ, let $V_{(p)}(\Gamma) = \mathrm{Hom}(\mathbb{Z}[1/p], \Gamma)$. With multiplicative notations, one can identify $V_{(p)}(\Gamma)$ to the set of the $(x^{(n)})_{n\in\mathbb{Z}}$, with $x^{(n)} \in \Gamma$ satisfying $(x^{(n+1)})^p = x^{(n)}$. We have a canonical short exact sequence

$$0 \longrightarrow T_p(\Gamma) \longrightarrow V_{(p)}(\Gamma) \longrightarrow \Gamma,$$

where $T_p(\Gamma)$ is the $\mathbb{Z}_p$-module $\mathrm{Hom}(\mathbb{Q}_p/\mathbb{Z}_p, \Gamma)$. If Γ is p-divisible, the map $V_{(p)}(\Gamma) \longrightarrow \Gamma$ is onto; if moreover Γ is p-torsion free, this map is an isomorphism.

1.4.2. For any ring A, let $A^\times$ the group of its units. If $R^{\times,+}$ (resp. $\mathcal{O}_C^{\times,+}$) denotes the subgroup of $R^\times$ (resp. $\mathcal{O}_C^\times$) consisting of units congruent to one mod the maximal ideal, we have obvious identifications

$$\mathcal{O}_C^\times = \overline{k}^\times \times \mathcal{O}_C^{\times,+}, \quad V_{(p)}(\mathcal{O}_C^\times) = R^\times = \overline{k}^\times \times R^{\times,+}, \quad V_{(p)}(C^\times) = (FrR)^\times.$$

If v is a choosen valuation of $\overline{K}$ and if v_R is the corresponding valuation of FrR (cf. n° 1.1.2), we get a commutative diagram

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & R^\times & \longrightarrow & \mathcal{O}_C^\times & \longrightarrow & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathbb{Z}_p(1) & \longrightarrow & (Fr\ R)^\times & \longrightarrow & C^\times & \longrightarrow & 0 \\
 & & & & \downarrow{\scriptstyle v_R} & & \downarrow{\scriptstyle v} & & \\
 & & & & \mathbb{Q} & = & \mathbb{Q} & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

whose rows and columns are exact.

1.4.3. Let $x = 1 + y \in R^{\times,+}$. For $n \gg 0, [y]^n/n \in A^f_{cris}$, and the sequence $[y]^n/n$ tends p-adically to 0. Therefore the series

$$\sum_{n>0}(-1)^{n-1}[y]^n/n$$

converges to an element $\lambda(x) \in B^+_{cris}$. The map

$$\lambda : R^{\times,+} \longrightarrow B^+_{cris}$$

is an injective homomorphism, that we extend to $R^\times$ by setting $\lambda(x) = 0$, if $x \in \overline{k}^\times$.

1.4.4. We can define B^+_{st} (resp. B_{st}) as the solution to a universal problem : this is the initial object of the category of pairs (S, λ_S), where S is a B^+_{cris}-algebra (resp. a B_{cris}-algebra) and $\lambda_S : (FrR)^\times \longrightarrow S$ is an homomorphism which extends λ (i.e. such that $\lambda_S(x)$ is the image of $\lambda(x)$ if $x \in R^\times$). Concretely

$$B^+_{st} = Sym((FrR)^\times) \otimes_{Sym(R^\times)} B_{cris}$$

and

$$B_{st} = Sym((FrR)^\times) \otimes_{Sym(R^\times)} B_{cris}.$$

Even more concretely, if $\lambda = \lambda_{B^+_{st}}$ and if b is any element of $(FrR)^\times$ which does not belong to $R^\times$, $B^+_{st} = B^+_{cris}[\lambda(b)]$ (isomorphic to the ring of polynomials in one variable with coefficients in B^+_{cris}) and $B_{st} = B^+_{st}[1/t] = B_{cris}[\lambda(b)]$.

1.4.5. The natural action of G on B^+_{cris} and B_{cris} extends, by functoriality to B^+_{st} and B_{st}. In B_{cris}, we have $\varphi(\lambda(x)) = p \cdot \lambda(x)$. Therefore, there is a unique endomorphism φ of B_{st} which extends the Frobenius on B_{cris} and is such that $\varphi(\lambda(x)) = p{\cdot}\lambda(x)$, for any $x \in (FrR)^\times$. We have $\varphi(B^+_{st}) \subset B^+_{st}$.

1.4.6. We have canonical isomorphisms $(FrR)^\times/R^\times \simeq C^\times/\mathcal{O}^\times_C \simeq \overline{K}^\times/\mathcal{O}^\times_{\overline{K}}$ $(\simeq \mathbb{Q})$ and the map

$$\kappa : B_{st} \otimes_{\mathbb{Q}} (\overline{K}^\times/\mathcal{O}^\times_{\overline{K}}) \longrightarrow \Omega^1_{B_{st}/B_{cris}}$$

which sends $b \otimes \widetilde{x}$ to $b \cdot d(\lambda(x))$ (where $\widetilde{x}$ is the image of $x \in (FrR)^\times$) is an isomorphism. Let's choose an isomorphism

$$v : \overline{K}^\times/\mathcal{O}^\times_{\overline{K}} \longrightarrow \mathbb{Q},$$

(this amounts to choosing a valuation on $\overline{K}$ and we denote this valuation by the same letter). This gives us, by extending the scalars, an isomorphism $\widehat{v} : B_{st} \otimes_{\mathbb{Q}} (\overline{K}^{\times}/\mathcal{O}_{\overline{K}}^{\times}) \longrightarrow B_{st}$, hence a derivation

$$N := \widehat{v} \circ \kappa^{-1} \circ d : B_{st} \longrightarrow B_{st}$$

(which depends on the choice of v), whose kernel is B_{cris} and that we call a **monodromy operator.** On B_{st}, we have

$$N \circ \varphi = p\varphi \circ N.$$

A natural choice for v is v_0 (normalised by $v_0(p) = p$) and we call **"canonical"** the corresponding monodromy operator.

1.5. The p-adic logarithms and the embedding of B_{st} into B_{DR}

1.5.1. The embedding of B_{cris} into B_{DR} extends to an embedding of B_{st} into B_{DR}, but this embedding depends on the choice of an **extended usual** p-adic logarithm, that is of a G-equivariant homomorphism,

$$\log : \overline{K}^{\times} \longrightarrow \overline{K},$$

whose restriction to $\mathcal{O}_{\overline{K}}^{\times}$ is the usual map (i.e. 0 on $\overline{k}^{\times}$ and the usual series on $\mathcal{O}_{\overline{K}}^{\times,+}$).

1.5.2. Before we construct this embedding, let's recall that $V_{(p)}(\overline{K}^{\times}) \subset (FrR)^{\times}$ and observe that we have a natural homomorphism

$$\mathrm{LOG} : V_{(p)}(\overline{K}^{\times}) \longrightarrow Fil^1 B_{DR},$$

which is given by

$$\mathrm{LOG}(x) = \sum_{n>0} (-1)^{n-1} \cdot \left(\frac{[x]}{x^{(0)}} - 1\right)^n / n$$

(we have $[x] \in W(R) \subset B_{DR}^+$ and $x^{(0)} \in \overline{K}^{\times} \subset (B_{DR}^+)^{\times}$, hence $[x]/x^{(0)} \in B_{DR}^+$ and $\theta(\frac{[x]}{x^{(0)}} - 1) = 0$, therefore the serie converges).

Now, using the extended usual log, we can define an other homomorphism

$$\lambda_{DR} : V_{(p)}(\overline{K}^{\times}) \longrightarrow B_{DR}^+,$$

by $\lambda_{DR}(x) = \mathrm{LOG}(x) + \log(x^{(0)})$.

It is easy to check that, if $x \in V_{(p)}(\mathcal{O}_{\overline{K}}^{\times})(\subset R^{\times})$, then $\lambda_{DR}(x) = \lambda(x)$. Then, there exists a unique homomorphism of B_{cris}-algebras

$$\iota : B_{st} \longrightarrow B_{DR}$$

such that $\lambda_{DR}(x) = \iota(\lambda(x))$, for any $x \in V_{(p)}(\overline{K}^{\times})$. Moreover, the map ι is injective [Fo89a].

The sub-K-algebra of B_{DR} spanned by the image of ι is independent of the choice of the extended usual log.

1.5.3. To choose an extended usual p-adic logarithm amounts to choose $\log(p)$ which can be any element of K. A natural choice is "Iwasawa's logarithm" defined by $\log(p) = 0$, and we call **"canonical"** the corresponding embedding $\iota : B_{st} \longrightarrow B_{DR}$.

An other natural way to make a choice for an extended usual p-adic logarithm consists in choosing **a uniformising parameter** π of K : indeed, there is one and only one extended usual p-adic logarithm such that $\log(\pi) = 0$; we denote it by $\log_\pi$ and call $\iota_\pi : B_{st} \longrightarrow B_{DR}$ the corresponding embedding.

1.5.4. Remarks : i) From the point of view of periods of integrals, this is the map LOG which plays a role in the p-adic world analogous to the role of the complex logarithm. If $a \in \overline{K}^{\times}$, and if we choose $x \in V_{(p)}(\overline{K}^{\times})$ such that $x^{(0)} = a$, we "define" LOG(a) as being LOG$(x) \in B_{DR}$; this is well defined mod $\mathbb{Z}_p(1)$ which is the p-adic analog of $\mathbb{Z}(1)$.

ii) Let L be a finite extension of K contained in $\overline{K}$. There is an obvious map from the B_{DR} relative to K to the one relative to L and this map is an isomorphism. Similarly, one can identify the B_{HT} (resp. A_{cris}, B^+_{cris}, B_{cris}, B^+_{st}, B_{st}) relative to K to the one relative to L).

§ 2. p-adic representations

We choose once and for all a valuation v on $\overline{K}$ (resp. an extended usual p-adic logarithm log) and we call N the corresponding monodromy operator on B_{st} (resp. we use the corresponding embedding $\iota : B_{st} \longrightarrow B_{DR}$ to identify B_{st} to a sub-ring of B_{DR}).

2.1. Basic properties of the B's

2.1.1. Let's start by summarizing some important properties of the rings just constructed from the point of view of p-adic representations :

i) **Galois continuous cohomology of the $C(i)$'s** ([Ta67]) : one has $C^G = H^0(G, C) = K$ and $H^1(G, C)$ is a one dimensional K-vector space; for any integer $i \neq 0$, $H^0(G, C(i)) = H^1(G, C(i)) = 0$.

ii) **Properties of linear disjunction** ([Fo82a] and [Fo89a]) : The natural maps

$$K \otimes_{K_0} B_{cris} \longrightarrow B_{DR} \text{ and } K \otimes_{K_0} B_{st} \longrightarrow B_{DR}$$

are injective; so are the maps

$$\overline{K} \otimes_{K_0^{ur}} B_{cris} \longrightarrow B_{DR} \text{ and } \overline{K} \otimes_{K_0^{ur}} B_{st} \longrightarrow B_{DR}$$

(where K_0^{ur} denote the maximal unramified extension of K_0 contained in $\overline{K}$).

iii) **Relation between B_{cris} and B_{st}** (obvious) : the sequence

$$0 \longrightarrow B_{cris} \longrightarrow B_{st} \xrightarrow{N} B_{st} \longrightarrow 0$$

is exact.

iv) **The fundamental exact sequence** ([Fo82a], [FM87]) : For each $i \in \mathbb{Z}$, let $Fil^i B_{cris} = Fil^i B_{DR} \cap B_{cris}$; the sequence

$$0 \longrightarrow \mathbb{Q}_p \longrightarrow Fil^0 B_{cris} \xrightarrow{\varphi - 1} B_{cris} \longrightarrow 0$$

is exact.

2.1.2. Remark : One can check that the image $B_{st,\overline{K}}$ of $\overline{K} \otimes_{K_0^{ur}} B_{st}$ in B_{DR} is independent of the choice of the log map which was used to define the embedding; the same thing is true for the $\overline{K}$-linear monodromy operator on $B_{st,\overline{K}}$ deduced from the initial one by scalar extension (which, on the contrary, depends on the choice of the valuation).

2.2. The different kinds of p-adic representations

We denote by $\mathbf{Rep}_{\mathbb{Q}_p}(G)$ the category of **p-adic representations of** G, that is the category whose objects are finite-dimensional $\mathbb{Q}_p$-vector spaces equipped with a linear and continuous action of G, and morphisms are $\mathbb{Q}_p$-linear G-equivariant maps.

2.2.1. It is easy to deduce from 2.1 that $(B_{HT})^G = (B_{DR})^G = K$ and that $(B_{cris})^G = (B_{st})^G = K_0$. For any p-adic representation V of G, and for $X \in \{HT, DR, st, cris\}$, define

$$D_X(V) = (B_X \otimes_{\mathbb{Q}_p} V)^G.$$

If $K_X = (B_X)^G$ (either K or K_0, depending on X), it is easy (using 2.1 again) to prove that the obvious map

$$\alpha_X : B_X \otimes_{K_X} D_X(V) \longrightarrow B_X \otimes_{\mathbb{Q}_p} V$$

is always injective and that $\dim_{K_X} D_X(V) \leq \dim_{\mathbb{Q}_p} V$, with the equality if and only if α_X is bijective.

2.2.2. The natural graduation on B_{HT} induces a structure of graded K-vector space on $D_{HT}(V)$; the natural filtration on B_{DR} induces a structure of filtered K-vector space on $D_{DR}(V)$ and there is an obvious homomorphism of graded K-vector spaces

$$gr^{\cdot} D_{DR}(V) \longrightarrow D_{HT}(V)$$

which is always injective.

The K_0-vector space $D_{cris}(V)$ and $D_{st}(V)$ are endowed with a natural action of φ, which is bijective and σ-semi-linear. On $D_{st}(V)$, there is also a nilpotent K_0-linear action of the monodromy N, satisfying $N\varphi = p\varphi N$ and $D_{cris}(V)$ is nothing else than the kernel of N on $D_{st}(V)$. The obvious maps

$$K \otimes_{K_0} D_{cris}(V) \longrightarrow D_{DR}(V) \quad \text{and} \quad K \otimes_{K_0} D_{st}(V) \longrightarrow D_{DR}(V)$$

are injective.

2.2.3. We say that the p-adic representation V is **Hodge-Tate** (resp. **de Rham, semi-stable, crystalline**) if the corresponding α_X is an isomorphism, or, equivalently, if $\dim_{K_X} D_X(V) = \dim_{\mathbb{Q}_p} V$. We denote by $\mathbf{Rep}_X(G)$ the full subcategory of $\mathbf{Rep}_{\mathbb{Q}_p}(G)$ consisting of such representations. It is a sub-Tannakian category ([Sa72], [DM82]), i.e. it is stable under sub-objects, quotients, direct sums, tensor products, contragredients (and, in an obvious way, the restriction of D_X to this category is a tensor functor).

- If V is de Rham, then V is Hodge-Tate and $D_{HT}(V) = gr^{\cdot} D_{DR}(V)$;

- if V is semi-stable, then V is de Rham and $D_{DR}(V) = K \otimes_{K_0} D_{st}(V)$;

- if V is crystalline, then V is semi-stable and $D_{st}(V) = D_{cris}(V)$ (with $N = 0$).

2.2.4. It is an easy consequence of the remark (ii) of 1.5.4 that if V is Hodge-Tate, de Rham, semi-stable or crystalline as a representation of $G = \mathrm{Gal}(\overline{K}/K)$, then for any finite extension L of K contained in $\overline{K}$, V remains so as a representation of $G_L = \mathrm{Gal}(\overline{K}/L)$. As B_{HT} and B_{DR} are $\overline{K}$-algebras, if there is a finite extension $L \subset \overline{K}$ of K such that V is Hodge-Tate or de Rham as a representation of G_L, then V is already Hodge-Tate or de Rham.

We say that a p-adic representation is **potentially semi-stable** (resp. **potentially crystalline**) if there exists a finite extension L contained in $\overline{K}$ such that V is semi-stable (resp. crystalline) as a representation of G_L. Of course,

$$\text{pot. crystalline} \Longrightarrow \text{pot. semi-stable} \Longrightarrow \text{de Rham}.$$

Moreover, a representation is crystalline if and only if it is simultaneously semi-stable and potentially crystalline.

2.3. Examples :

2.3.1. Let $\chi : G \longrightarrow \mathbb{Z}_p^\times$ be the character giving the action of G on $\mathbb{Z}_p(1)$ and let V be a one dimensional representation of G acting through a character η. Then ([Se89], III A7) V is Hodge-Tate if and only if there exists $i \in \mathbb{Z}$ (necessarily unique) such that the restriction of $\eta\chi^{-i}$ to the inertia subgroup I_K of G is finite; if so, V is potentially crystalline and

$$V \text{ crystalline} \Longleftrightarrow \eta\chi^{-i}|_{I_K} = 1 \text{ [F}_0\text{79]}.$$

2.3.2. Let $i \in \mathbb{Z}$ and

$$0 \longrightarrow \mathbb{Q}_p(i) \longrightarrow V \longrightarrow \mathbb{Q}_p \longrightarrow 0$$

a non trivial extension. Then ([BK89] when the field is finite, see also [Pe88]) :

a) if $i \geq 2$, the representation is crystalline;

b) if $i = 1$, the representation is semi-stable; Kummer's theory identifies $\mathrm{Ext}^1(\mathbb{Q}_p, \mathbb{Q}_p(1)) = H^1(G, \mathbb{Q}_p(1))$ to

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\varprojlim K^\times/(K^\times)^{p^n});$$

the representation is crystalline if and only if the class of V lies inside the image of $\mathbb{Q}_p \otimes \mathcal{O}_K^\times$;

c) if $i = 0$, V Hodge-Tate $\Longleftrightarrow$ V crystalline $\Longleftrightarrow$ V unramified;

d) if $i < 0$, V is Hodge-Tate and is not de Rham.

2.3.3. Let f be an elliptic modular form which is a newform of level N; let V be the representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ associated by Deligne to f. V is de Rham ([Fa88b], [Fa88c]). It is crystalline if p doesn't divide the level (op.cit.). When p divides the level, it is expected to be always potentially semi-stable, that is to become semi-stable after restriction to a suitable open subgroup of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

2.4. Constructing semi-stable p-adic representations [Fo89b]

2.4.1. Let's define a **filtered (φ, N)-module over K** as being a K_0-vector space D equipped with

i) a σ-semi-linear injective map $\varphi : D \longrightarrow D$;

ii) a K-linear map $N : D \longrightarrow D$ satisfying $N\varphi = p\varphi N$;

(iii) a decreasing filtration $(Fil^i D_K)_{i\in\mathbb{Z}}$ on $D_K := K \otimes_{K_0} D$ by sub-K-vector spaces satisfying $\cup Fil^i D_K = D_K$ and $\cap Fil^i D_K = 0$.

The filtered (φ, N)-modules over K form, in an obvious way, an additive ($\mathbb{Q}_p$-linear) category $\mathbf{MF}_K(\varphi, N)$.

2.4.2. There is a natural tensor product on $\mathbf{MF}_K(\varphi, N)$ [3], hence it makes sense also to speak about $\Lambda^r D$ if D is an object of this category and $r \in \mathbb{N}$.

For any object D of $\mathbf{MF}_K(\varphi, N)$ whose underlying K_0-vector space is of dimension 1, we define $t_H(D)$ as the biggest integer i such that $Fil^i D_K \neq 0$; if d is a non-zero element of D and if $\varphi d = \lambda d$, we define $t_N(D)$ as the p-adic valuation of λ (this is independent of the choice of d). For any object D of $\mathbf{MF}_K(\varphi, N)$ whose underlying K_0-vector space is finite-dimensional, we define

$$t_H(D) = t_H(\Lambda^{max} D) \quad \text{and} \quad t_N(D) = t_N(\Lambda^{max} D).$$

A **sub-object** D' of an object D of $\mathbf{MF}_K(\varphi, N)$ is a sub K_0-vector space stable under φ and N; we endow it with the filtration of $D'_K = K \otimes_{K_0} D' \subset D_K$ induced by the filtration of D_K.

We say that an object D of $\mathbf{MF}_K(\varphi, N)$, which is finite-dimensional as a K_0-vector space, is **weakly admissible** if it satisfies :

i) $t_H(D) = t_N(D)$;

ii) for any sub-object D' of D, $t_H(D') \leq t_N(D')$.

We denote by $\mathbf{MF}^f_K(\varphi, N)$ the full sub-category of $\mathbf{MF}_K(\varphi, N)$ whose objects are those which are weakly admissible. It is an abelian category (a sub-object in $\mathbf{MF}^f_K(\varphi, N)$ of an object D of this category is a sub-object D' of D in $\mathbf{MF}_K(\varphi, N)$ such that $t_H(D') = t_N(D')$). It is likely that this category is stable under tensor product[4], hence is a Tannakian category.

2.4.3. If V is a semi-stable p-adic representation of G, $D_{st}(V)$ has a natural structure of filtered (φ, N)-module over K (the action of φ and N are induced by the corresponding operators on B_{st} and the filtration is induced by the filtration of B_{DR} and the fact that $K \otimes_{K_0} D_{st}(V) = D_{DR}(V)$). One can prove that $D_{st}(V)$ is actually an object of $\mathbf{MF}^f_K(\varphi, N)$.

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[3] On $D' \otimes D'' := D' \otimes_{K_0} D''$, we define $\varphi(d' \otimes d'') = \varphi d' \otimes \varphi d''$, $N(d' \otimes d'') = Nd' \otimes d'' + d' \otimes Nd''$, and the usual tensor product filtration on $(D' \otimes D'')_K = D'_K \otimes_K D''_K$.

[4] It has been prove by Laffaille [La80] in the case $K = K_0$ and extended by Faltings (private communication) to an arbitrary K that the sub-category of $\mathbf{MF}_K(\varphi, N)$ on which N is 0 is stable under tensor product. It seems more than likely that Faltings's proof extends to the general case.

Moreover, the functor

$$D_{st} : \mathbf{Rep}_{st}(G) \longrightarrow \mathbf{MF}^{f}_{K}(\varphi, N)$$

if fully faithful and induces a tensor equivalence between $\mathbf{Rep}_{st}(G)$ and its essential image, the category $\mathbf{MF}^{ad}_{K}(\varphi, N)$ of **admissible filtered** (φ, N)**-modules over** K, which is a full sub-category[5] of $\mathbf{MF}^{f}_{K}(\varphi, N)$, stable under sub-object, quotient, $\oplus, \otimes, ^*$.

A quasi-inverse functor V_{st} of D_{st} is given by

$$V_{st}(D) = \{x \in B_{st} \otimes D | Nx = 0,\ \varphi x = x \ and\ 1 \otimes x \in Fil^0(B_{DR} \otimes D_K)\}.$$

2.4.4. When there is no monodromy, we speak about filtered φ-modules instead of filtered (φ, N)-modules. We denote by $\mathbf{MF}_K$ (resp. $\mathbf{MF}^{f}_{K}$, $\mathbf{MF}^{ad}_{K}$) the full sub-category of $\mathbf{MF}_K(\varphi, N)$ (resp. $\mathbf{MF}^{f}_{K}(\varphi, N)$, $\mathbf{MF}^{ad}_{K}(\varphi, N)$) on which $N = 0$. Of course, D_{cris} induces a tensor-equivalence between the categories $\mathbf{Rep}_{cris}(G)$ and $\mathbf{MF}^{ad}_{K}$, a quasi-inverse of D_{cris} being given by

$$V_{cris}(D) = \{x \in B_{cris} \otimes D | \varphi x = x \text{ and } 1 \otimes x \in Fil^0(B_{DR} \otimes D_K)\}.$$

2.5. Potentially semi-stable representations

2.5.1. For any p-adic representation V, let

$$D_{pst}(V) := \lim.\text{ind}.(B_{st} \otimes_{\mathbb{Q}_p} V)^{G_L},$$

for L running through the finite extensions of K contained in $\overline{K}$. This is a finite-dimensional K_0^{ur}-vector space which has a natural structure of **discrete** (φ, N, G)**-module**, i.e. there is a Frobenius endomorphism φ, semi-linear with respect to the usual Frobenius on K_0^{ur}, a monodromy operator N, K_0^{ur}-linear and satisfying $N\varphi = p\varphi N$, and a discrete action of G (i.e. the fixator of each point is open in G), semi-linear with respect to the natural action of G on K_0^{ur}, commuting with φ and N.

Moreover $(\overline{K} \otimes_{K_0^{ur}} D_{pst}(V))^G$ can be identified with

$$D_{st,\overline{K}}(V) := (B_{st,\overline{K}} \otimes_{\mathbb{Q}_p} V)^G$$

and is a sub-K-vector space of $D_{DR}(V)$ whose dimension is equal to the dimension of $D_{pst}(V)$ over K_0^{ur}. Let's remark that one can view $D_{st,\overline{K}}(V)$

5 It is likely, but not certain, that $\mathbf{MF}^{ad}_{K}(\varphi, N) = \mathbf{MF}^{f}_{K}(\varphi, N)$; there are partial results in this direction [FL82].

as a filtered K-vector space equipped with a monodromy operator N that is a nilpotent K-linear endomorphism.

2.5.2. For any p-adic representation V, $\dim_{K_0^{ur}} D_{pst}(V) \leq \dim_{\mathbb{Q}_p}(V)$. We have

V pot semi-stable $\Leftrightarrow \dim_{K_0^{ur}} D_{pst}(V) = \dim_{\mathbb{Q}_p}(V) \Leftrightarrow \dim_K D_{st,\overline{K}}(V) = \dim_{\mathbb{Q}_p} V \Leftrightarrow V$ is de Rham and $D_{st,\overline{K}}(V) \to D_{DR}(V)$ is an isomorphism.

The functor D_{pst} induces a tensor equivalence between the category of potentially semi-stable p-adic representations and a suitable full subcategory $\mathbf{MPF}_K^{ad}(\varphi, N, G)$ of the category $\mathbf{MPF}_K(\varphi, N, G)$ of **filtered** (φ, N, G)**-modules over** K, i.e. of discrete (φ, N, G)-modules D equipped with a filtration of $(\overline{K} \otimes_{K_0^{ur}} D)^G$. One can describe a quasi-inverse V_{pst} of D_{pst} in the same spirit as was described the quasi-inverse V_{st} of D_{st}.

§ 3. The case of good reduction : $\mathcal{C}_{cris}$

3.1. Crystalline cohomology

3.1.1. To any k-scheme Y are associated in a functorial way W-modules denoted $H^i_{cris}(Y/W)$ (or $H^i(Y/W)$), the crystalline cohomology groups of Y. For Y proper and smooth, these are of finite type, and $Y \longmapsto H^*(Y/W)$ is a good Weil cohomology, i.e. satisfies a formalism of Poincaré duality, Künneth formulas, cycle class maps; for Y projective and smooth, one has $rk_W H^i(Y/W) = \dim H^i(Y \otimes \overline{k}, \mathbb{Q}_\ell)$ $(\ell \neq p)$ by a result of Katz-Messing ([KM74]).

Crystalline cohomology was invented by Grothendieck [Gr66]. An outline is given in [Gr68]. The main program was carried out by Berthelot [Be74]. As an introduction to the theory, the reader may consult the notes of Berthelot-Ogus [BO78] or the summaries [Il75] and [Il76]. Cohomology classes of smooth cycles are constructed in [Be74], the case of singular cycles is discussed in [GM87] and [Gr85].

3.1.2. Recall that K_0 is the fraction field of W, so that K/K_0 is a finite totally ramified extension. Let $e = [K : K_0]$. Let Y/k be proper and smooth, and suppose we have $\mathcal{X}/\mathcal{O}_K$ proper and smooth, lifting Y. One basic property of crystalline cohomology is that, according to Berthelot ([Be74]), if $e \leq p-1$, one has a canonical isomorphism

$$\mathcal{O}_K \otimes_W H^*(Y/W) \simeq H^*_{DR}(\mathcal{X}/\mathcal{O}_K). \tag{3.1.2.1}$$

This result doesn't extend to an arbitrary e, but, by a theorem of Berthelot and Ogus [BO78], one has a canonical isomorphism
(3.1.2.2)
$K \otimes_W H^*(Y/W) \simeq K \otimes_{\mathcal{O}_K} H^*_{DR}(\mathcal{X}/\mathcal{O}_K)$ $(= H^*_{DR}(X)$ if $X = \mathcal{X} \otimes_{\mathcal{O}_K} K)$,

which, for $e \leq p-1$, is deduced from the previous one by scalar extension.

3.1.3. Let Y/k as above. By functoriality, the (absolute) endomorphism of Y induces a σ-linear endomorphism φ of $H^*(Y/W)$ (where σ is the Frobenius automorphism of W). Moreover the unique extension of φ to a σ-linear endomorphism of $K_0 \otimes_W H^*(Y/W)$ is bijective. By definition, this makes $H^*(Y/W)$ into an **F-crystal** in the sense of Grothendieck. In particular, one can speak of the **slopes** of φ on $H^i(Y/W)$, (roughly speaking the p-adic valuations of its eigenvalues) which are rational numbers between 0 and i. The study of this structure has been the focus of considerable activity during the past twenty years. One central result is the proof of the **Katz conjecture** on the relative positions of the Newton and Hodge polygons of Y ([Ma72], [Ma73], [BO78], [Ny81]). A deep insight into this structure is furnished by the theory of the de Rham-Witt complex ([Il79b], [IR83], see [Il79a] and [Il83] for a survey).

3.2. A theorem of Faltings

3.2.1. Let X be a proper and smooth K-scheme. Assume that X has good reduction, i.e. that there exists a proper and smooth $\mathcal{O}_K$-scheme $\mathcal{X}$ whose generic fibre $\mathcal{X}_K = \mathcal{X} \otimes K$ is isomorphic to X.

Then, if we choose such an $\mathcal{X}$, the de Rham cohomology of X, $H^*_{DR}(X/K)$, comes equipped with the structure of a filtered φ-module (cf. n° 2.4.1), that is, we have :

(a) **the Hodge filtration** $Fil^{\cdot} H^*_{DR}(X/K)$, which is the abutment of the (degenerating at E_1) Hodge to de Rham spectral sequence

$$E_1^{ij} = H^j(X, \Omega^i_{X/K}) \Longrightarrow H^*_{DR}(X/K);$$

(b) a K_0-structure $H^*_{cris}(X) := K_0 \otimes_W H^*(Y/W)$, where Y is the special fiber $\mathcal{X} \otimes k$ (thanks to the isomorphism 3.1.2.2);

(c) the σ-linear automorphism φ of $H^*_{cris}(X)$.

It has been shown by Gillet-Messing [GM87], as a consequence of their theory of cycle maps and the compatibility of (3.1.2.2) with duality, that this structure of filtered φ-module depends only on X, not on the choice of the model $\mathcal{X}$.

The hypothesis of good reduction implies that, for $\ell \neq p$, the action of the Galois group $G = \mathrm{Gal}(\overline{K}/K)$ on the ℓ-adic étale cohomology $H^*(X_{\overline{K}}, \mathbb{Q}_\ell)$ is unramified (i.e. the inertia acts trivially). However, the action of G on the p-adic étale cohomology $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ may be very complicated. Faltings proved the following theorem [Fa88b] :

3.2.2. THEOREM. — *There exists a functorial B_{cris}-linear isomorphism*

$$B_{cris} \otimes_{K_0} H^*_{cris}(X) \simeq B_{cris} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p) \qquad (3.2.2.1)$$

(with $H^*_{cris}(X)$ as above and B_{cris} as in 1.3), having the following properties :

(i) it is compatible with the action of G, where $g \in G$ acts on the left (resp. *right) hand side by $g \otimes 1$* (resp. $g \otimes g$),

(ii) it is compatible with Frobenius, acting by $\varphi \otimes \varphi$ (resp. $\varphi \otimes 1$) *on the left* (resp. *right) hand side,*

(iii) the isomorphism

$$B_{DR} \otimes_K H^*_{DR}(X/K) = B_{DR} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p),$$

deduced from (3.2.2.1) *by extension of scalars (thanks to* (b)*) is compatible with the natural filtration on both sides, where $Fil^\cdot$ on the left* (resp. *right) hand side is $Fil^\cdot \otimes Fil^\cdot$* (resp. $Fil^\cdot \otimes H^*(X_{\overline{K}}, \mathbb{Q}_p)$)*,*

(iv) (3.2.2.1) *is compatible with cup-products, Künneth isomorphisms, Poincaré duality and cycle class maps.*

3.2.3. Remark : With the terminology of 2.4.4, properties (i) to (iii) mean that $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ is a crystalline representation, that the φ-filtered module $H^*_{cris}(X)$ is admissible and that we have a canonical isomorphism $H^*_{cris}(X) = D_{cris}(H^*(X_{\overline{K}}, \mathbb{Q}_p))$ (or equivalently $H^*(X_{\overline{K}}, \mathbb{Q}_p) \simeq V_{cris}(H^*_{cris}(X))$). Let's recall that it means that we have canonical isomorphisms, compatible with the natural structures,

$$H^*_{cris}(X) \simeq (B_{cris} \otimes H^*(X_{\overline{K}}, \mathbb{Q}_p))^G,$$

$$H^*_{DR}(X/K) \simeq (B_{DR} \otimes H^*(X_{\overline{K}}, \mathbb{Q}_p))^G,$$

$$H^*(X_{\overline{K}}, \mathbb{Q}_p) \simeq \{x \in B_{cris} \otimes H^*_{cris}(X) | \varphi x = x,\ 1 \otimes x \in Fil^0(B_{DR} \otimes H^*_{DR}(X/K)\}.$$

Notice that this implies again the result of Gillet-Messing mentioned above (to the effect that the structure of filtered φ-module depends only on X).

§ 4. The semi-stable case : C_{st}

4.1. Inputs from the analytic theory

Let S be an open disc in $\mathbb{C}$, $0 \in S$, $S^* = S - \{0\}$, $f : X \to S$ a projective map of analytic spaces, with X smooth over $\mathbb{C}$, $f|_{S^*}$ smooth and the special fiber $Y = f^{-1}(0)$ a reduced divisor with normal crossings in X (so that f is locally given by $f(z_1, z_2, \ldots, z_n) = z_1 z_2 \cdots z_r$ where $z_1, \ldots, z_n$ are local coordinates). The complex cohomology of a general fiber $H^*(X_t, \mathbb{C})$, $t \in S^*$, is endowed with the **monodromy** automorphism T_t, given by the action of the positive generator of $\pi_1(S^*)$. It is a basic result that, in the situation we are considering, T_t is **unipotent** (had we allowed multiplicities in Y, then T_t would have been only quasi-unipotent).

This monodromy operator can be interpreted à la de Rham in the following way. Let $\omega_X^{\cdot} := \Omega_X^{\cdot}(\log Y)$ (resp. $\omega_S^{\cdot} := \Omega_S^{\cdot}(\log 0)$) be the absolute de Rham complex of X (resp. S) with log poles on Y (resp. 0). Let $\omega_{X/S}^{\cdot}$ be the relative de Rham complex of X/S with log poles on Y, defined as $\Lambda\omega_{X/S}^1$ where $\omega_{X/S}^1 = \omega_X^1/f^*\omega_S^1$, with the differential induced by that of $\omega_X^{\cdot}$. We have a short exact sequence of complexes on X :

(4.1.1) $$0 \longrightarrow \omega_S^1 \otimes \omega_{X/S}^{\cdot-1} \longrightarrow \omega_X^{\cdot} \longrightarrow \omega_{X/S}^{\cdot} \longrightarrow 0.$$

The coboundary of the long exact sequence deduced from (4.1.1) by applying R^*f_* gives a connection on $H := R^*f_*\omega_{X/S}^{\cdot}$ with log poles at 0,

(4.1.2) $$\nabla : H \longrightarrow \omega_S^1 \otimes H.$$

The restriction of ∇ to S^* is the standard Gauss-Manin connection on $R^*f_*\Omega_{X^*/S^*}^{\cdot}$ ($X^* := X - Y$), whose sheaf of horizontal sections is the local system $t \longmapsto H^*(X_t, \mathbb{C})$. The following results are due to Steenbrink [St76a] :

(a) H is a vector bundle on S, whose formation commutes with any base change; in particular, the fiber H_0 of H at 0 is $H^*(Y, \omega_Y^{\cdot})$, where $\omega_Y^{\cdot} := \omega_{X/S}^{\cdot} \otimes \mathcal{O}_Y$.

(b) Let $N : H_0 \longrightarrow H_0$ be the **residue** of ∇ at 0, deduced from (4.1.2) by tensoring with $\mathbb{C}_{\{0\}}$ and applying the residue isomorphism $\omega_S^1/\Omega_S^1 \xrightarrow{\sim} \mathbb{C}$; equivalently, if

(4.1.3) $$0 \longrightarrow \omega_Y^{\cdot-1} \longrightarrow \omega_X^{\cdot} \otimes \mathcal{O}_Y \longrightarrow \omega_Y^{\cdot} \longrightarrow 0$$

is the exact sequence deduced from (4.1.1) by applying $\mathbb{C}_{\{0\}} \otimes_{\mathcal{O}_S} -$ (and using the residue isomorphism), N is the coboundary of the long exact sequence obtained by applying $H^*(Y, -)$ (to (4.1.3). Then N is **nilpotent**.

It follows from (a) and (b) that (4.1.2) is the **canonical extension** (in the sense of Deligne-Manin [De70, II 5.2]) of the Gauss-Manin connection on $H|S^*$. Therefore, by (loc. cit., II 1.17), the monodromy operators T_t above are the fibers of an automorphism T of H, whose fiber T_0 at 0 is given by

(4.1.4) $$T_0 = \exp(-2\pi i N).$$