The fiber $H_0 = H^*(Y, \omega_Y^\cdot)$ underlies in fact a much richer structure. In order to explain this, fix a parameter t on S, a universal cover $\widetilde{S}^* \longrightarrow S^*$, and a determination of $\log t$ on S^*. Denote by $X_{\overline{\eta}}$ the pull-back of X^* by $\widetilde{S}^* \longrightarrow S^*$ ($X_{\overline{\eta}}$ plays the role of a "generic geometric fiber"). Then, by Steenbrink [St76a], there is a canonical isomorphism

(4.1.5) $$\psi_t : H_0 \xrightarrow{\sim} H^*(X_{\overline{\eta}}, \mathbb{C})$$

(deriving in fact from a finer isomorphism $\psi_t : \omega_Y^\cdot \xrightarrow{\sim} R\psi_{\overline{\eta}}(\mathbb{C})$, where $R\psi_{\overline{\eta}}(\mathbb{C})$ is the complex of vanishing cycles, as in (SGA7 XIV)). This isomorphism depends on the choices made : if $(u, \log u)$ is another choice), then one has

(4.1.6) $$\psi_t = \psi_u \exp(2\pi i N \log((u/t)(0))).$$

Moreover, there is defined on H_0 a mixed Hodge structure (depending on the choices), with Hodge filtration given by $Fil^i H_0 = H^*(Y, \omega_Y^{\geq i})$, which can be considered as the *"limit"* of the (pure) Hodge structures of the fibers of f nearby. In particular, the Hodge numbers $h_0^{pq} := \dim H^q(Y, \omega^p)$ verify $h_0^{pq} = \dim H^q(X_t, \Omega^p)$ $(t \neq 0)$ (more precisely, the Hodge to de Rham spectral sequence

$$E_1^{pq} = R^q f_* \omega_{X/S}^p \Longrightarrow R^* f_* \omega_{X/S}^\cdot$$

degenerates at E_1, and the E_1 terms are vector bundles on S, whose formation commutes with base change). Furthermore, the nilpotent endomorphism N of H_0 is of type $(-1,-1)$, which gives strong bounds on its **exponent of nilpotence** : on $H^n(Y, \omega_Y^\cdot)$ we have

(4.1.7) $$N^{a+1} | H^n(Y, \omega_Y^\cdot) = 0,$$

where a is the length of the largest interval without zeroes in the sequence of Hodge numbers $(h^{n,0}, h^{n-1,1}, \ldots, h^{0,n})$ of X_t.

For all this, see [St76a], [St76b], and [SZ85] for a correct treatment of the weight filtration over $\mathbb{Q}$.

4.2. Inputs from the ℓ-adic theory

In this section, **we do not assume the complete field K to be of characteristic zero.** We put $A = \mathcal{O}_K$ and denote by I the inertia subgroup, given by the exact sequence

$$1 \longrightarrow I \longrightarrow G \longrightarrow \mathrm{Gal}(\overline{k}/k) \longrightarrow 1,$$

where $\overline{k}$ is the residue field of $\overline{K}$.

Let $f : X \longrightarrow S = \operatorname{Spec} A$ be proper and flat, with semi-stable reduction, which means that X is regular, f generically smooth, and the special fiber $Y \subset X$ is a reduced divisor with normal crossings (in other words, X is étale locally isomorphic to $\operatorname{Spec} A[t_1, \ldots, t_n]/(t_1 \cdots t_r - \pi)$, where π is a uniformizing parameter). Consider the ℓ-adic cohomology of the generic geometric fiber

$$H := H^*(X_{\overline{K}}, \mathbb{Q}_\ell)$$

as a representation of the Galois group $G = \operatorname{Gal}(\overline{K}/K)$. A close analysis of the spectral sequence of vanishing cycles ((SGA 7 I), completed by [RZ82]) shows that the elements $g \in I$ act unipotently (**geometric local monodromy theorem**). Therefore, the action of I factors through the ℓ-tame quotient $t_\ell : I \longrightarrow \mathbb{Z}_\ell(1)$ (whose kernel is a profinite group of order prime to ℓ). More precisely, there exists a unique nilpotent map

(4.2.1) $$N : H(1) \longrightarrow H$$

such that $gx = \exp(Nt_\ell(g))x$ for $g \in I$, $x \in H$ (cf. [De80, 1.7]). Since N is unique, it commutes with the action of Galois. In particular, if k is the finite field $\mathbb{F}_q$, and $F \in \operatorname{Aut} H$ is the action of an element in G whose image in $\operatorname{Gal}(\overline{k}/k)$ is the geometric Frobenius $x \longmapsto x^{-q}$, we have

(4.2.2) $$NF = qFN.$$

In this case, the structure on H given by the action of G and the nilpotent endomorphism N is best described as a representation of the **Weil-Deligne group** $'W(\overline{K}/K)$ [De73, 8.4.1], see 5.1 below. The situation, unfortunately, is far from being as well understood as in the complex case. Let us just mention one basic open question.

4.2.3. Since N is nilpotent, there exists a unique finite, increasing filtration M on H satisfying $NM_iH \subset M_{i-2}H(-1)$ and such that N^i induces an isomorphism $gr_i^M H \xrightarrow{\sim} gr_{-i}^M H(-i)$; this filtration is called the **monodromy filtration**. In the complex case considered in 4.1, the analogous filtration M on H_0 coincides with the weight filtration of the mixed Hodge structure, (see [St76a, 5.9][6]; in particular, $gr_i^M H_0^n$ underlies a pure Hodge structure of weight $n+i$. Suppose now that $k = \mathbb{F}_q$. Is it true that $gr_i^M H^n$ is pure of weight $n+i$, i.e. the eigenvalues of an element F as above acting on $gr_i^M H^n$ have absolute values $q^{(n+i)/2}$ for any embedding of $\overline{\mathbb{Q}}_\ell$ into $\mathbb{C}$? By a fundamental result of Deligne [De80, 1.8.4], the answer is yes if X/A comes by base change from a proper and flat model over a

[6] the proof here has a gap; a correct proof was given by M. Saito [Sa86, 4.2.5].

smooth curve over $\mathbb{F}_q$. In mixed characteristic, the answer is still yes if $\dim Y \leq 2$ [RZ82 2.13], but is unknown in the general case. It is also unknown whether the characteristic polynomial $\det(1 - Ft, gr_i^M H^n)$ has integral coefficients, independant of ℓ.

4.3. Crystalline cohomology with log poles

From now on, we assume that K is of characteristic zero.

In [MTT86], modular forms under $\Gamma_0(p)$ of weight $k \geq 3$ are constructed for which the corresponding 2-dimensional p-adic representation is irreducible while the analogous ℓ-adic representation, $\ell \neq p$, has a nontrivial N. This excludes the possibility that, for $S = \operatorname{Spec} A$ and $\mathcal{X}/S$ as in 4.2, there could exist on $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ a p-adic analogue of the operator N of (4.2.1). Following a suggestion of Jannsen (private communication), it seemed reasonable to conjecture that such an operator N should exist, instead, on the de Rham cohomology $H^*_{DR}(\mathcal{X}_K/K)$, and that one should also have a K_0-structure and a Frobenius automorphism as in 3.4 (b), (c), satisfying a compatibility of type (4.2.2). This has recently been proven by Hyodo and Kato [Hy88b], [HK89]. Here is an outline of their construction.

Let $\mathcal{X}$ be regular, flat (but not necessarily proper) over S, with semi-stable reduction. Let $X = \mathcal{X} \otimes K$ and $Y = \mathcal{X} \otimes k$. Denote by $i : Y \hookrightarrow \mathcal{X}$, $j : X \hookrightarrow \mathcal{X}$, $v : Z = U \otimes k \hookrightarrow Y$ the inclusions, where U is the smooth locus of $\mathcal{X}$ over S. Let $n \geq 1$. Consider the de Rham-Witt complex of step n of Z, $W_n\Omega^{\cdot}_Z$ [Il79b]. Observe that

$$i^*j_*\mathcal{O}^*_X|Z = i^*\mathcal{O}^*_{\mathcal{X}} \cdot K^*|Z \simeq (i^*\mathcal{O}^*_{\mathcal{X}}|Z) \times \mathbb{Z},$$

since any x in the left hand side can be uniquely written $u\pi^n$, $n \in \mathbb{Z}$, $u \in i^*\mathcal{O}^*_{\mathcal{X}}$, where π is a prime in A. Denote by

$$W_n\omega^{\cdot}_Y \tag{4.3.1}$$

the sub-$W_n\mathcal{O}_Y$-algebra of $v_*W_n\Omega^{\cdot}_Z$ generated by $dW_n\mathcal{O}_Y$ and the image of the map

$$d\log : i^*j_*\mathcal{O}^*_X \longrightarrow v_*W_n\Omega^1_Z \tag{4.3.2}$$

corresponding, by adjunction, to the map

$$d\log : i^*j_*\mathcal{O}^*_X|Z \longrightarrow W_n\Omega^1_Z$$

such that $d\log x$ is the *"usual"* $d\log \overline{x}$ for $x \in i^*\mathcal{O}^*_{\mathcal{X}}$ ($\overline{x}$ = image of x in $\mathcal{O}^*_Y$) and $d\log x = 0$ for $x \in K^*$. It is not difficult to show that $W_n\omega^{\cdot}_Y$ is stable under the differential d of $v_*W_n\Omega^{\cdot}_Z$ and that the operators F, V on $W.\Omega^{\cdot}_Z$ extend to operators F, V on $W.\omega^{\cdot}_Y$ satisfying the same basic relations $FV = VF = p$, $FdV = d$. The Frobenius endomorphism $\mathcal{F}$ of $W.\omega_Y^{\cdot}$ (induced by the absolute Frobenius of Y) is then given by $p^i F$ on $W.\omega_Y^i$. The whole theory of [Il79b], [IR83] can be carried over – mutatis mutandis – to this context. In particular, if

$$H^*(Y, W\omega_Y^{\cdot}) := \varprojlim H^*(Y, W_n\omega_Y^{\cdot}), \tag{4.3.3}$$

the (σ-linear) Frobenius endomorphism $\mathcal{F}$ of $H^*(Y, W\omega_Y^{\cdot})$ is an isogeny, i.e. $\mathcal{F} \otimes \mathbb{Q}$ is bijective (this is due to the existence of V), and, when $\mathcal{X}/\mathcal{S}$ is proper, $H^*(Y, W\omega_Y^{\cdot})$ is of finite type over W and satisfies a Poincaré duality formalism. It should be noted that, despite the notation, $W.\omega_Y^{\cdot}$ depends on $\mathcal{X}$, not only on Y (more precisely, one can show that $W.\omega_Y^{\cdot}$ depends only on the *"logarithmic structure"* on Y induced by $\mathcal{X}$, in the sense of Kato [Ka88a]).

Consider now the graded differential algebra

$$W_n\widetilde{\Omega}_Z^{\cdot} := W_n\Omega_Z^{\cdot} \oplus W_n\Omega_Z^{\cdot -1} \cdot \theta,$$

with θ of degree 1 satisfying $\theta^2 = 0$, $d\theta = 0$. Denote by

$$W_n\widetilde{\omega}_Y^{\cdot} \tag{4.3.4}$$

the sub-$W_n\mathcal{O}_Y$-algebra of $v_*W_n\widetilde{\Omega}_Z^{\cdot}$ generated by $dW_n\mathcal{O}_Y$ and the image of the map

$$d\log : i^*j_*\mathcal{O}_X^* \longrightarrow v_*W_n\widetilde{\Omega}_Z^1 \tag{4.3.5}$$

corresponding, by adjunction, to the map

$$d\log : i^*j_*\mathcal{O}_X^*|Z \longrightarrow W_n\Omega_Z^1 \oplus W_n\mathcal{O}_Z \cdot \theta$$

sending x to $d\log\overline{x}$ as above for $x \in i^*\mathcal{O}_{\mathcal{X}}^*$ and to $v_\pi(x)\theta$ for $x \in K^*$ (where v_π is the valuation normalized by $v_\pi(\pi) = 1$). It is easily checked that $W_n\widetilde{\omega}_Y^{\cdot}$ is stable under d and that the operators F, V on $W.\widetilde{\Omega}_Z^{\cdot}$ (wiht $F\theta = \theta$, $V\theta = p\theta$) extend naturally to $W_n\widetilde{\omega}_Z^{\cdot}$ and satisfy the usual relations. Moreover, the section θ of $W_n\widetilde{\Omega}_Z^{\cdot}$ extends to a global section θ of $W_n\widetilde{\omega}_Z^{\cdot}$, and the following sequence is exact :

$$\begin{array}{c} 0 \longrightarrow W_n\omega_Y^{\cdot -1} \longrightarrow W_n\widetilde{\omega}_Y^{\cdot} \longrightarrow W_n\omega_Y^{\cdot} \longrightarrow 0 \\ x \longmapsto x \cdot \theta \qquad \theta \longmapsto 0; \end{array} \tag{4.3.6}$$

for variable n, these sequences form an inverse system. From (4.3.6) we get a map (in the derived category)

$$N : W_n\omega_Y^{\cdot} \longrightarrow W_n\omega_Y^{\cdot}, \tag{4.3.7}$$

which satisfies

$$NF = pFN \tag{4.3.8}$$

(where $\mathcal{F}$ is, as above, the Frobenius endomorphism of $W_n\omega^{\cdot}_Y$). Therefore, the same relation holds between the corresponding operators on $H^*(Y, W\omega^{\cdot}_Y)$ (4.3.3). Observe the analogy with (4.2.2). For Y proper, since $\mathcal{F}\otimes\mathbb{Q}$ is bijective and $H^*(Y, W\omega^{\cdot}_Y)$ of finite type, (4.3.8) forces $N\otimes\mathbb{Q}$ on $H^*(Y, W\omega^{\cdot}_Y)\otimes\mathbb{Q}$ to be nilpotent. It can be shown again that $W.\tilde{\omega}^{\cdot}_Y$ and the exact sequence (4.3.6) depend only on the logarithmic structure on Y induced by $\mathcal{X}$.

The definitions of the modified de Rham-Witt complexes above are due to Kato [HK89]. There is an alternate construction due to Hyodo [Hy88b]. The cohomology $H^*(Y, W_n\omega^{\cdot}_Y)$ can also be interpreted as the cohomology of a suitable crystalline site $(Y/W_n)_{\log}$ (taking into account the log structure) with coefficients in the structural site, see [HK89]; this generalizes the canonical isomorphism $H^*(Y, W_n\Omega^{\cdot}_Y) \xrightarrow{\sim} H^*(Y/W_n)$ for Y smooth [Il79b].

The fundamental property of this new cohomology theory is the following generalization of the Berthelot-Ogus isomorphism (3.2.2) :

4.3.9. THEOREM [HK89]. — *Fix a prime $\pi\in A$. Let $\mathcal{X}/A$ be proper and flat, with semi-stable reduction, and special (resp. general) fiber Y (resp. X). Then there is a canonical isomorphism*

$$\rho_\pi : H^*(Y, W\omega^{\cdot}_Y)\otimes_W K \xrightarrow{\sim} H^*_{DR}(X/K),$$

functorial in $\mathcal{X}/A$, which coincides with (3.1.2.2) *for $\mathcal{X}/A$ smooth. For $u\in A^*$, one has*

$$\rho_\pi = \rho_{\pi u}\cdot \exp(\log(u)N) \tag{4.3.9.1}$$

(where log is the usual logarithm).

Note the analogy between (4.3.9.1) and (4.1.6).

4.3.10. Under the assumption of 4.3.9, we thus obtain on $H^*_{DR}(X/K)$ a structure of filtered (φ, N)-module (2.4.1), namely :

(a) the Hodge filtration $Fil^i H^*_{DR}(X/K)$ (as in 3.4 (a));

(b) the K_0-structure $H^*_{st}(X) := H^*(Y, W\omega^{\cdot}_Y)\otimes_W K_0$ (given by ρ_π);

(c) the σ-linear automorphism $\varphi = \mathcal{F}\otimes\mathbb{Q}$ of $H^*_{st}(X)$;

(d) the nilpotent endomorphism N of $H^*_{st}(X)$ satisfying $N\varphi = p\varphi N$. If $\mathcal{X}/A$ is smooth, then $N = 0$ and (a), (b), (c) is just the structure considered in 3.4.

For $\dim X \leq 1$, Raynaud [Ra89] has given another construction of a structure of type (a)-(d) above on $H^1_{DR}(X/K)$, using the Néron model of $\mathrm{Pic}^0(X)$ and rigid analytic techniques. His construction applies more generally to 1-motives over K having semi-stable reduction on A. Its compatibility with that of Hyodo-Kato has not yet been checked.

4.4. The conjecture C_{st}.

The following conjecture [Fo89b] was inspired by Jannsen (private communication, see also [Ja88]).

4.4.1 CONJECTURE (C_{st}). — *Let π and $\mathcal{X}/A$ be as in 4.3.9. Then there is a functorial B_{st}-linear isomorphism*

$$(4.4.1.1) \qquad B_{st} \otimes_{K_0} H^*_{st}(X) \xrightarrow{\sim} B_{st} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p)$$

(with $H^*_{st}(X)$ as in 4.3.10 (b) and B_{st} as in 1.4, with the embedding $\iota_\pi : B_{st} \hookrightarrow B_{DR}$ associated to the same prime π, see 1.5.3), having the following properties :

(i) it is compatible with the action of G, where $g \in G$ acts on the left (resp. *right) hand side by $g \otimes 1$* (resp. $g \otimes g$)

(ii) it is compatible with Frobenius, acting by $\varphi \otimes \varphi$ (resp. $\varphi \otimes 1$) *on the left* (resp. *right) hand side*

(iii) it is compatible with $N \otimes 1 + 1 \otimes N$ on the left hand side and $N \otimes 1$ on the right hand side

(iv) the isomorphism

$$B_{DR} \otimes_K H^*_{DR}(X/K) \xrightarrow{\sim} B_{DR} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p)$$

deduced from (4.4.1.1) *by extension of scalars (thanks to* 4.3.10 (b)*) is compatible with the filtrations $Fil^{\cdot}$ on both sides, as in* 3.2.2 *(iii).*

(v) (4.4.1.1) *is compatible with cup-products and Poincaré duality.*

4.4.2. Remark : With the terminology of 2.4.3, properties (i) to (iv) mean that $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ is a semi-stable representation, that the (φ, N)-filtered module $H^*_{st}(X)$ is admissible and that we have a canonical isomorphism $H^*_{st}(X) = D_{st}(H^*(X_{\overline{K}}, \mathbb{Q}_p))$ (or equivalently $H^*(X_{\overline{K}}, \mathbb{Q}_p) = V_{st}(H^*_{st}(X))$).

Let's recall that it means that we have canonical isomorphisms, compatible with the natural structures)

$$H^*_{st}(X) \simeq (B_{st} \otimes H^*(X_{\overline{K}}, \mathbb{Q}_p))^G, \quad H^*_{DR}(X/K) \simeq (B_{DR} \otimes H^*(X_{\overline{K}}, \mathbb{Q}_p))^G,$$

$$H^*(X_{\overline{K}}, \mathbb{Q}_p) \simeq \{x \in B_{st} \otimes H^*_{st}(X) | Nx = 0,\ \varphi x = x,\ 1 \otimes x \in Fil^0(B_{DR} \otimes H^*_{DR}(X/K)\}.$$

Note that these formulas imply that the K_0-structure $H^*_{st}(X)$ on $H^*_{DR}(X)$ together with the action of φ and N on it should not depend on the semi-stable model $\mathcal{X}$ of X. It would be interesting to have a direct proof of this, as Gillet-Messing did in the good reduction case (cf. 3.2.1).

4.4.3. THEOREM [Ka88b]. — *The conjecture* (C_{st}) *is true if* $\dim X < (p-1)/2$.

4.4.4. For slope reasons, one has $N^{m+1} = 0$ on $H^m_{st}(X)$. There is a better bound, which is the analog in this context of the bound 4.1.7 for the complex analytic case : one has

$$N^{a+1}|H^m_{st}(X) = 0,$$

if a is the biggest integer ≥ 0 such that it exists a rational number α such that the part of slope $\alpha - i$ in $H^m_{st}(X)$ is $\neq 0$ for $i = 0, 1, \ldots, a$. This is an immediate consequence of the relation $N\varphi = p\varphi N$.

4.4.5. Assume that $k = \mathbb{F}_q$, with $q = p^r$. Consider the monodromy filtration M_i on $H^n_{st}(X)$ relative to the nilpotent endomorphism N (note that, because of (4.3.8), N sends M_i to $M_{i-2}(-1)$, where $(-)(m)$ means as usual tensoring with the Tate F-isocrystal $(K_0, p^{-m}\sigma)$). Is it true that

$$(*) \qquad \det(1 - \varphi^r t, gr^M_i H^n_{st}(X)) = \det(1 - Ft, gr^M_i H^n(X_{\overline{K}}, \mathbb{Q}_\ell)),$$

where the right hand side is as in 4.2.3 ? When X has good reduction and is projective, then $N = 0$, the monodromy filtration is trivial, and (*) has been proven by Katz-Messing [KM74] as a consequence of Deligne's results in Weil II [De80] : the left (resp. right) hand side is $H^n(Y/W) \otimes \mathbb{Q}$ (resp. $H^n(Y_{\overline{k}}, \mathbb{Q}_\ell)$).

§ 5. The general case : C_{DR} and C_{pst}

5.1 B_{DR}-periods

The main result is the following theorem of Faltings, proving the (C_{DR}) conjecture [Fo82a] :

5.1.1 THEOREM [Fa88b]. — *Let X be a proper and smooth K-scheme. There exists a functorial B_{DR}-linear isomorphism*

$$(5.1.1.1) \qquad B_{DR} \otimes_K H^*_{DR}(X/K) \xrightarrow{\sim} B_{DR} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p)$$

compatible with :

(i) the action of G on both sides, where $g \in G$ acts by $g \otimes 1$ (resp. $g \otimes g$) *on the left* (resp. *right*) *hand side,*

(ii) the filtrations Fil$^{\cdot}$ on both sides (as in (3.2.2) (iii));

(iii) cup-products, Poincaré duality and cycle class maps.

In particular, $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ is a de Rham representation and $H^*_{DR}(X/K)$ (with its Hodge filtration) can be recovered from $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ as $D_{DR}(H^*(X_{\overline{K}}, \mathbb{Q}_p))$, i.e. $(B_{DR} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p))^G$.

5.1.2. In the case of good reduction (resp. semi-stable reduction and $\dim X < (p-1)/2$), 5.1.1 is a consequence of 3.2.2 (resp. 4.4.2). Note also that, as a corollary of 5.1.1, one gets another proof of the **Hodge-Tate decomposition**, conjectured by Tate in [Se67], i.e. the fact that $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ is Hodge-Tate and that there is a functorial B_{HT}-linear isomorphism (cf. 1.2.5)

$$B_{HT} \otimes_K H^*_{Hdg}(X/K) \xrightarrow{\sim} B_{HT} \otimes_{\mathbb{Q}_p} H^*(X_{\overline{K}}, \mathbb{Q}_p)$$

(where $H^*_{Hdg}(X/K) = \oplus H^{*-i}(X, \Omega^i_{X/K})$), compatible with the graduations and actions of G on both sides.

5.1.3. It is not possible, of course, to recover, through (5.1.1.1), the p-adic representation $H^*(X_{\overline{K}}, \mathbb{Q}_p)$ from the filtered module $H^*_{DR}(X/K)$. More structure on $H^*_{DR}(X/K)$ is needed. We will discuss this now.

5.2. The p-adic monodromy conjecture

5.2.1. Conjecture (p-adic monodromy). — *For any proper and smooth variety X over K, and for any $m \in \mathbb{N}$, $H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ is potentially semi-stable.*

Granted Falting's result, this amounts to saying that the injective map

$$D_{st,\overline{K}}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) \longrightarrow D_{DR}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) = H^m_{DR}(X)$$

(see 2.5.1 for the definition of $D_{st,\overline{K}}$) is an isomorphism.

5.2.2. Remarks : i) Actually, we would like to have a more precise result. It would be nice to be able to compute directly the $D_{pst}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$'s as a suitable cohomology (a suitable generalisation of crystalline cohomology). One may expect that this cohomology should depend only on the special fiber of a suitably good model of X over the integers, equipped with a "multiplicative" or "logarithmic" structure.

ii) Let's say that X has potentially good reduction (resp. is potentially semi-stable) if there is a finite extension L of K such that $X_L = X \otimes L$ has good reduction (resp. is semi-stable). Granted the results of sections 3.2 and 4.4, the conjecture is a theorem in the potentially good reduction case and follows from the conjecture C_{st} in the potentially semi-stable case (hence is also a theorem for $m < (p-1)/2$).

In this situation, we get also the direct construction of $D_{pst}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p))$ we asked for : if $L \subset \overline{K}$ is a finite Galois extension such that there exists a semi-stable model $\mathcal{X}$ of X_L over $\mathcal{O}_L$, and if L_0 is the maximal unramified extension of K_0 contained in L, then

$$D_{pst}(H^m_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)) = K_0^{ur} \otimes_{L_0} H^m_{st}(\mathcal{X}).$$

A posteriori, the comparison conjecture (or theorem, in cases where it's known) implies that, as a sub-K_0^{ur}-vector space of $\overline{K} \otimes_K H^m_{DR}(X)$, $K_0^{ur} \otimes_{L_0} H^m_{st}(\mathcal{X})$ is independent of the choices of L and $\mathcal{X}$ and is stable under G.

iii) It is natural to ask whether or not any proper and smooth X over K is potentially semi-stable. If it was true, this conjecture (and also the stronger form asked for in remark (i)) would follow from C_{st}.

iv) If the conjecture is true, then there is a monodromy operator N acting on $H^m_{DR}(X)$. It would be interesting to give a direct definition of this operator (using rigid analytic methods ?).

5.2.3. Generalizations

It seems likely that all p-adic representations "coming from algebraic geometry in a reasonable way" are potentially semi-stable : étale p-adic cohomology (with proper support) of open and/or (not too) singular algebraic varieties, cohomology with non constant (but not too bad) coefficients.

The langage of motives (pure or mixed, à la Grothendieck or à la Deligne) is convenient to express more or less obvious comparison's conjectures between the different realizations. Partial results have already been obtained by Faltings [Fa88c] and Raynaud [Ra89].

In the case of motives of proper and smooth varieties, it's a matter of being able to check some compatibility between the way the different cohomologies are cut into pieces. For the motive associated to an elliptic modular newform of level prime to p, it has been done independently by Faltings (private communication) and Scholl ([Sc89]).

§ 6. The arithmetic case

6.1. Representations of the Galois group of a local field

6.1.1. Assume now that K is a finite extension of $\mathbb{Q}_p$. The Weil-Deligne group[7] is quite useful to discuss relations between ℓ-adic representations of G for various prime numbers ℓ.

Let $q = p^r$ be the cardinal of k, $f = \sigma^{-r} \in \mathrm{Gal}(\overline{k}/k)$ the geometric Frobenius. We identify $\mathrm{Gal}(\overline{k}/k)$ to $\widehat{\mathbb{Z}}$ by sending the generator f to 1, call $\nu : G = \mathrm{Gal}(\overline{K}/K) \longrightarrow \mathrm{Gal}(\overline{k}/k) = \widehat{\mathbb{Z}}$ the canonical projection and I_K the inertia subgroup, i.e. the kernel of ν. We denote by $W_K = W(\overline{K}/K)$ the Weil group of K, i.e. the sub-group of G consisting of those $g \in G$ such that $\nu(g) \in \mathbb{Z}$. Recall that, if E is any field of characteristic 0, a (finite-dimensional linear) representation of the Weil-Deligne group $'W_K$ of K over E is a finite-dimensional E-vector space D equipped with

i) a homomorphism $\rho' : W_K \longrightarrow \mathrm{Aut}_E(D)$, whose kernel contains an open sub-group of the inertia subgroup I_K;

ii) a nilpotent endomorphism N of V such that

$$N \cdot \rho'(w) = q^{\nu(w)} \rho'(w) \cdot N \quad \text{for any} \quad w \in W_K.$$

6.1.2. Let ℓ be a prime number $\neq p$. We choose $F \in W_K$ lifting f and a non-zero homomorphism

$$t'_\ell : I_K \longrightarrow \mathbb{Q}_\ell$$

(recall that there is a canonical homomorphism $t_\ell : I_K \longrightarrow \mathbb{Z}_\ell(1)$ which is onto; hence the choice of t'_ℓ amounts to choose a non-zero homomorphism from $\mathbb{Z}_\ell(1)$ to $\mathbb{Q}_\ell$).

Let's consider an ℓ-adic representation of G, that is a finite-dimensional $\mathbb{Q}_\ell$-vector space V_ℓ plus a continuous homomorphism

$$\rho_\ell : G \longrightarrow \mathrm{Aut}_{\mathbb{Q}_\ell}(V_\ell).$$

The ℓ-adic monodromy theorem of Grothendieck ([SGA7I], Exp.1) tells us that ρ_ℓ is **"potentially semi-stable"**, i.e. there is an open sub-group of I_K which acts unipotently on V_ℓ. Equivalently, there is a unique representation (ρ', N) of $'W_K$ on V_ℓ such that, for any $g \in I_K$ and any $n \in \mathbb{Z}$,

$$\rho_\ell(F^n g) = \rho'(F^n g) \cdot \exp(t'_\ell(g) \cdot N).$$

[7] cf. [De73], § 8, from which all the discussion below (n° 6.1.1 and 6.1.2) originates, except for some slight changes of notations; one has $\nu(w) = -v'(w)$, with Deligne's convention, because it is the geometrical Frobenius that we chose to identify to $1 \in \widehat{\mathbb{Z}}$.

Moreover the isomorphism class of ρ_ℓ is determined by the isomorphism class of (ρ', N) and this last one is independent of the choices we made (i.e. F and t'_ℓ).

6.1.3. For $\ell = p$, the reasonable analogue of a "general" ℓ-adic representation seems to be, in this context, a "general" potentially semi-stable representation. Actually if V is any potentially semi-stable representation of G, we know (2.5) how to associate to it

a) a discrete (φ, N, G)-module $D_{pst}(V)$, finite-dimensional as a K_0^{ur}-vector space;

b) a filtered K-vector space $D_{DR}(V)$, finite-dimensional.

Via the functor V_{pst}, the knowledge of V is equivalent to the **knowledge of both $D_{pst}(V)$ and $D_{DR}(V)$ and of a comparison map**, i.e. an isomorphism

$$(\overline{K} \otimes_{K_0}^{ur} D_{pst}(V))^G = D_{DR}(V).$$

It is remarkable that $D_{DR}(V)$ gives us the same kind of information as we would get by looking at archimedian places (Hodge numbers). On the other hand, $D_{pst}(V)$ gives us exactly the information that we would obtain from an ℓ-adic representation (with $\ell \neq p$). More precisely :

6.1.4. PROPOSITION [FM89]. — *i) Let D be a discrete (φ, N, G)-module, finite-dimensional* (as a K_0^{ur}-vector space). *For $w \in W_K$, define $\rho'(w) : D \longrightarrow D$ by*

$$\rho'(w)(d) = \varphi^{r\nu(w)}(w(d)) \ for\ d \in D;$$

then (ρ', N) is a representation of $'W_K$[8] *;*

ii) If D_1 and D_2 are finite-dimensional discrete (φ, N, G)-modules, there is a canonical isomorphism

$$K_0^{ur} \otimes_{\mathbb{Q}_p} \mathrm{Hom}_{(\varphi,N,G)-\mathrm{mod}}(D_1, D_2) \simeq \mathrm{Hom}_{'W_K}(D_1, D_2);$$

moreover D_1 and D_2 are isomorphic as (φ, N, G)-modules if and only if they are isomorphic as representations of $'W_K$.

6.2. Applications to motives

6.2.1. Let M a motive over K (we don't want to be precise, there are three typical examples we have in mind : (a) M "is" $H^m(X)$, with X proper and smooth over K and $m \in \mathbb{N}$, (b) M "is" the motive associated

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[8] If $D = D_{pst}(V)$, where V is a potentially semi-stable p-adic representation, the monodromy operator changes if we change the valuation which was used to define it ; the isomorphism class of the representation of $'W_K$ just constructed doesn't.

to an elliptic modular newform, (c) M is a 1-motive in the sense of Deligne [De74b], n° 10.1).

For each prime number ℓ, we have an ℓ-adic representation V_ℓ of G, the ℓ-adic realization of M (if $M = H^m(X)$, this is $H^m(X_{\overline{K}}, \mathbb{Q}_\ell)$). Assume that M satisfies the p-adic monodromy conjecture, i.e. that V_p is potentially semi-stable. Then for each prime number ℓ, we have a representation D_ℓ of $'W_K$:

— for $\ell \neq p$, $D_\ell = V_\ell$ a $\mathbb{Q}_\ell$-vector space, on which $'W_K$ acts as explained in 6.1.2;

— for $\ell = p$, $D_p = D_{pst}(V_p)$ a K_0^{ur}-vector space on which $'W_K$ acts as explained in 6.1.4.

There are three natural questions (or conjectures, it's a matter of taste) :

Q1 : are the D_ℓ's all compatible?

Q2 : are the D_ℓ's all F-semi-simple?

Q3 : are the F-semi-simplifications of the D_ℓ's all compatible, (see [De73], § 8 for a precise definition of the F-semi-simplification of a representation of $'W_K$ and of the notion of compatible representations).

Question 3 seems more accessible and there is no doubt that the answer should be yes (observe also that a positive answer to this question would imply a positive answer to question 4.4.5). This is known in the case of $M = H^m(X)$ with X proper and smooth over K with good reduction ([De74a] for $\ell \neq p$ and [KM74] for $\ell = p$) and in the case of a 1-motive [Ra89].

6.2.2. Remark : We don't know any example of a de Rham representation which is not potentially semi-stable. It seems likely that, in the situation we are considering in this paragraph ($K/\mathbb{Q}_p$ finite), any de Rham representation is potentially semi-stable, a result which would be the p-adic analog of Grothendieck ℓ-adic monodromy theorem. If it was true, Falting's theorem (n° 5.1) would imply the p-adic monodromy conjecture for proper and smooth varieties.

6.2.3. Let us finish with a few words about global Galois representations. Let ℓ be a prime number and V an irreducible ℓ-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$. For each prime p, choose an embedding of $\overline{\mathbb{Q}}$ into $\overline{\mathbb{Q}}_p$. Let's say that V is **geometric** if it is unramified outside finitely many p's and if it is potentially semi-stable at $p = \ell$. It seems reasonable to conjecture that such a representation "comes from algebraic geometry". At least, for each place p of $\mathbb{Q}$ (finite or not) there is a well defined isomorphism class of a representation of the Weil-Deligne group of $\mathbb{Q}_p$ associated to it (for $p = \infty$, one use the action of the complex conjugation and the filtration on $D_{DR}(V)$ to define it). Modulo the conjecture that the characteristic polynomials of the Frobenii should have coefficients in $\mathbb{Q}$, we can therefore associate to V an L-function, an ε-factor, a conductor. These questions are discussed with more details in [FM89].

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