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Geometric Galois Representations

JEAN-MARC FONTAINE
UNIVERSITÉ DE PARIS-SUD
MATHÉMATIQUE-BÂTIMENT 425, 91405-ORSAY CEDEX FRANCE

BARRY MAZUR
HARVARD UNIVERSITY
DEPARTMENT OF MATHEMATICS
1 OXFORD STREET, CAMBRIDGE, MA 02138

The main point of this paper [1] is to present in detail some of our conjectures concerning p-adic representations of the Galois groups of number fields (cf. [**10**] for a brief synopsis). The motivating idea behind our conjectures is simply the hope that, given an irreducible p-adic representation ρ of G_K, the Galois group of a number field K (assumed unramified except at a finite number of places), one can find necessary and sufficient local conditions (on the restriction of the representation to the decomposition groups G_{K_V} at the finite set of places v of K of residual characteristic p) for the representation ρ to "come from algebraic geometry". The local condition we have in mind is that ρ restricted to G_{K_V} be *potentially semi-stable*. This will be made precise in Conjecture 1 of § 1 below.

There is a well known conjecture which, vaguely put, asserts that a pure motive of rank 2 and of Hodge type $(0,r),(r,0)$ "comes from" a modular newform of weight $k = r+1$ (for related conjectures, see [**28**] and the large literature concerning the connection between elliptic curves and modular curves; for spectacular recent work in the way of proving such conjectures, see [**31**] and [**29**]). Combining our Conjecture 1 with this "well known conjecture" leads to a conjectural necessary and sufficient condition, stated only in terms of a local condition at p, for an irreducible representation $\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$ to be the representation associated to a cuspidal newform.

Conjecture: *Let*

$$\rho : G_{\mathbb{Q}} \to GL_2(\overline{\mathbb{Q}}_p)$$

be an irreducible representation which is unramified except at a finite number of primes and which is not the Tate twist of an even representation which factors through a finite quotient group of $G_{\mathbb{Q}}$.

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Then ρ is associated [2] *to a cuspidal newform f if and only if ρ is potentially semi-stable at p.*

For a refinement of the above conjecture, see Conjecture 3c below.

We also include certain "finiteness conjectures" for irreducible representations which "come from algebraic geometry", which have a fixed level of ramification and a fixed Hodge-Tate type. These "finiteness conjectures" are akin to the assertion (in view of the Conjecture displayed above) that there are only a finite number of newforms of given level and weight. This discussion is collected in §3. In §7 and §8 we examine Conjecture 1 in the case where the representation is potentially abelian, and potentially everywhere unramified.

Part II of our paper is devoted to a deformation - theoretic study of the condition of *potential semi-stability* for degree two, p-adic representations of $G_{\mathbb{Q}_p}$ which are residually absolutely irreducible. The main results here are due to Ramakrishna [**23**], but we revisit Ramakrishna's theory to give a slightly more detailed picture, particularly in the case when the representations are assumed to be potentially Barsotti-Tate.

Here is our motivation for doing such a study. Fixing an absolutely irreducible representation

$$\overline{\rho}_p : G_{\mathbb{Q}_p} \to GL_2(\mathbf{F}_p),$$

Ramakrishna has shown that the universal deformation ring $R(\overline{\rho}_p)$ is isomorphic to a power series ring in five variables $\mathbb{Z}_p[[T_1, T_2, T_3, T_4, T_5]]$ (and therefore the p-adic variety $X(\overline{\rho}_p) := \mathrm{Hom}(R(\overline{\rho}_p), \mathbb{Z}_p)$ which classifies lifts of $\overline{\rho}_p$ to $GL_2(\mathbb{Z}_p)$ is smooth on five parameters). Fix a finite set of primes S including p, let $G_{\mathbb{Q},S}$ denote the Galois group of the maximal algebraic extension of $\mathbb{Q}$, unramified outside S, and fix an imbedding of $\overline{\mathbb{Q}}$ in $\overline{\mathbb{Q}}_p$ and hence a homomorphism $i : G_{\mathbb{Q}_p} \to G_{\mathbb{Q},S}$. Let $\overline{\rho}_{\mathbb{Q},S} : G_{\mathbb{Q},S} \to GL_2(\mathbf{F}_p)$ be any representation (if such exists) which extends $\overline{\rho}_p$. That is, $\overline{\rho}_p = \overline{\rho}_{\mathbb{Q},S} \circ i$. One knows that the universal deformation ring $R(\overline{\rho}_{\mathbb{Q},S})$ is of Krull dimension ≥ 4, and if smooth, is a power series ring over $\mathbb{Z}_p$ on three parameters. It is conjectured always to have Krull dimension 4.

Now, since $5 - 3 = 2$, for our "finiteness conjectures" to be feasible it would be nice to prove that all *potential semi-stable* liftings

$$\rho_p : G_{\mathbb{Q}_p} \to GL_2(\mathbb{Z}_p),$$

of $\overline{\rho}$ with fixed Hodge-Tate weights (r, s) lie in a finite union of (at most!) *two*-parameter subspaces of the universal deformation space $X(\overline{\rho}_p)$. Indeed, the best would be if there were a finite quotient ring, call it $R(\overline{\rho}_p)_{r,s}$, of $R(\overline{\rho}_p)$ through which all homomorphisms $R(\overline{\rho}_p) \to \mathbb{Z}_p$ classifying all potentially semi-stable liftings of $\overline{\rho}_p$ (with Hodge-Tate weights r, s) factors, and such that $R(\overline{\rho}_p)_{r,s}$ is of Krull dimension ≤ 3, and whose generic fiber over $\mathbb{Q}_p$ is formally smooth of dimension two.

[2] recall that "associated" is a technical term which means that one can find an isomorphism between $\overline{\mathbb{Q}}_p$ and $\mathbb{C}$ such that for all but a finite number of prime numbers l, the above isomorphism brings the trace of $\rho(\mathrm{Frob}_l)$ to the eigenvalue of the Hecke operator T_l on the newform f. Here, Frob_l denotes the Frobenius element at l, and of course, one need only consider prime numbers l which do not divide the level of f, and at which the representation ρ is unramified.

This would indeed be nice, for then we could entertain the hope that our "finiteness conjecture" is a phenomenon related to a possible transversality question; namely, do the generic fibers of Spec$R(\overline{\rho}_p)_{r,s}$ and of $R(\overline{\rho}_{\mathbb{Q},S})$ have images in the generic fiber of Spec$R(\overline{\rho}_p)$ which intersect transversally?

We do not know whether rings $R(\overline{\rho}_p)_{r,s}$ as described above exist for all Hodge-Tate weights r, s. But Ramakrishna established that the locus of crystalline liftings possessing Hodge- Tate weights contained in the interval $[0, p-1]$ indeed forms a subscheme of Spec$(R(\overline{\rho}_p))$, smooth on two formal parameters over Spec$(\mathbb{Z}_p)$. A Hodge-Tate lifting with Hodge-Tate weights r, s contained in $[0, p-1]$ will be said to be of **moderate Hodge-Tate type**. We complement Ramakrishna's theory by classifying (at least for $p \geq 5$) all weakly admissible two-dimensional irreducible potentially semi-stable filtered $(\varphi, N, G_{\mathbb{Q}_p})$ -modules (Thm. A of § 11) and then by constructing explicitly the universal such module which classifies crystalline liftings of $\overline{\rho}_p$ to $GL_2(\mathbb{Z}_p)$ of moderate Hodge-Tate type (Thm. B2 of § 12).

We also discuss the interesting case of **potential Barsotti-Tate liftings** of $\overline{\rho}_p$ to $GL_2(\mathbb{Z}_p)$. The aim here is, firstly, to preview a theory (classifying Barsotti-Tate groups "strictly of slope 1/2") which will be expounded more fully in further publications by one of us, and, secondly, to formulate a consequence of this theory which gives a classification of potentially Barsotti-Tate liftings of $\overline{\rho}_p$ to $GL_2(\mathbb{Z}_p)$. This classification is put forward in Thm. C3 of § 13 below; we give a series of hints towards its proof in Appendix C. Specifically, the locus of all liftings of $\overline{\rho}_p$ to $GL_2(\mathbb{Z}_p)$ which are potentially Barsotti-Tate (PBT, for short) is either empty, or else it is a union of two disjoint smooth two-parameter spaces in $X(\overline{\rho}_p)$. More exactly, there is a quotient-ring R_{PBT} of $R(\overline{\rho}_p)$ though which any homomorphism $R(\overline{\rho}_p) \to \mathbb{Z}_p$ which classifies a PBT lifting factors, and

$$R_{\mathrm{PBT}} \cong \mathbb{Z}_p[[Y_1, Y_2]] \times_{\mathbf{F}_p} \mathbb{Z}_p[[Y'_1, Y'_2]],$$

i.e., the fiber product of two power series rings over $\mathbb{Z}_p$ each in two variables, the fiber product being taken over their common residue field $\mathbf{F}_p$.

The PBT liftings whose classifying homomorphisms factor through $\mathbb{Z}_p[[Y_1, Y_2]]$ are quite different from those which factor through $\mathbb{Z}_p[[Y'_1, Y'_2]]$. For certain residual representations $\overline{\rho}_p$ (specifically, for those $\overline{\rho}_p$ whose invariants [3] j_1, j_2 have the property that $j_2 - j_1 = 1$) this difference can readily be seen by considering their associated admissible pst modules:

Those PBT liftings whose classifying homomorphisms factor through $\mathbb{Z}_p[[Y_1, Y_2]]$ have associated pst modules occuring in our "type I" series (cf. Thm. A of §11) and the two p-adic parameters Y_1 and Y_2 correspond to a two-parameter variation of (the two) coefficients of the characteristic polynomial $X^2 - aX + d$ of the "Frobenius"operator φ acting on the associated pst modules of these liftings. The datum of the filtration on these pst modules contributes, however, no further variation.

Those PBT liftings whose classifying homomorphisms factor through $\mathbb{Z}_p[[Y'_1, Y'_2]]$ have associated pst modules occurring in our "type IV" series (cf. Thm. A of §11) and here the two p-adic parameters Y'_1 and Y'_2 correspond to somewhat different features of the associated pst modules. The characteristic

[3] cf. Thm. C3 of §13

polynomial of the "Frobenius" operator φ acting on the associated pst modules of these liftings have the form X^2+d, and the p-adic parameter Y_1' corresponds to the possibility of varying the coefficient d. The datum of the filtration on these pst modules is important, and contributes another one-parameter of p-adic variation corresponding to the second variable Y_2'.

Full proofs of Thm. C1-3 will be given in later publications. Beyond this PBT case, however there still remains a good deal of work to do to get a completely satisfactory, and completely general, local picture of pst liftings, even in the special context of two-dimensional representations of $G_{\mathbb{Q}_p}$.

Notation:
K := number field; $\overline{K}$:= an algebraic closure of K, $G_K = \mathrm{Gal}(\overline{K}/K)$. S : a finite set of finite places of K; $G_{K,S}$:= the Galois group of the maximal algebraic extension of K in $\overline{K}$ unramified outside S; p := a fixed prime number; E := a field containing $\mathbb{Q}_p$; N := a fixed positive integer.

For each finite place v of K, we denote by K_v the completion of K at v. We choose an algebraic closure $\overline{K}_v$ of K_v and an embedding of $\overline{K}$ into $\overline{K}_v$. This gives us an identification of $G_v = \mathrm{Gal}(\overline{K}_v/K_v)$ with a decomposition subgroup of G_K. We denote by I_v the inertia subgroup of G_v.

If G is a profinite group, a **p-adic representation of** G is a finite dimensional $\mathbb{Q}_p$-vector space equipped with a continuous and linear action of G. Similarly, an **E-representation of** G is a $\mathbb{Q}_p$-vector space V of finite dimension (called the **degree of the representation**) equipped with a linear action of G which has the property that, if we choose a basis of V, the image of the corresponding map

$$\rho : G \to GL_N(E)$$

is contained in $GL_N(E_0)$ where E_0 is a suitable finite $\mathbb{Q}_p$-algebra contained in E and the map $G \to GL_N(E_0)$ is continuous (this doesn't depend on the choice of the basis).

Part I. The conjectures

§1. Geometric representations and representations coming from algebraic geometry.

A p-adic representation of G_K is called **geometric** if
(a) it is unramified outside a finite set of places of K,
(b) its restriction to every decomposition group G_v (for v ranging through all non-archimedean places of K) is *potentially semi-stable* in the sense of **[13]** (see also **[12]** for v's dividing p).

Equivalently, we may ask that its restriction to the decomposition groups for $v|p$ are potentially semi-stable, since, by a theorem of Grothendieck, the restriction to the other decomposition groups are automatically potentially semi-stable.

Remark: We do not know whether the condition (a) is satisfied for all semi-simple representations.

A continuous *irreducible* $\mathbb{Q}_p$-representation of G_K is said to **come from algebraic geometry** if it is isomorphic to a subquotient of an etale cohomology group with coefficients in $\mathbb{Q}_p(r)$ for some Tate twist $r \in \mathbb{Z}$, of an algebraic variety over K (equivalently [4] : of a projective smooth algebraic variety over K).

Conjecture 1: *An irreducible p-adic representation is geometric if and only if it comes from algebraic geometry.*

Evidence: The part of conjecture 1 saying that "irreducible representations coming from algebraic geometry are "geometric" is in no way original to this article, and goes in a direction where much more precise results are expected to be true about varieties defined over local fields. See, for example, Conjecture $\underline{C}_{\mathrm{pst}}$ of **[12]** (6.2.1) and the further conjectures in (6.2.4) there. This part of Conjecture 1 has been known for a long time to hold for abelian varieties, and, more recently, **[6]** for varieties which have good reduction at all places dividing p (in which case the representations of G_v are crystalline) and also in slightly more general instances (see **[15]** and **[19]** for a survey, see also the forthcoming work of Tsuji).

The part of Conjecture 1 saying that irreducible "geometric" representations "come from algebraic geometry" is presently known for irreducible potentially abelian representations (see §6 below). The recent work of Wiles **[31]** and Taylor-Wiles **[29]** establishes this also for a very significant class of irreducible p-adic representations V of $G_{\mathbb{Q}}$, of dimension two over $\mathrm{End}_{\mathbb{Q}_p[G_{\mathbb{Q}}]}(V)$.

§2. Hodge-Tate representations.

If the extension $E/\mathbb{Q}_p$ is finite, an E-representation of degree N of G_K is called **geometric** if it is geometric as a p-adic representation of degree $N \cdot \dim_{\mathbb{Q}_p} E$. Generally, an E-representation V of degree N of G_K is called **geometric** if there exist a finite $\mathbb{Q}_p$-algebra E_0 contained in E, a geometric E_0-representation V_0 of G_K and an isomorphism of E-representations $E \otimes_{E_0} V_0 \simeq V$.

Let V be an E-representation of G_K of degree N which is geometric. If v is a place of K dividing p and if the representation is potentially semi-stable at v then V, as a representation of G_v, is Hodge-Tate. Let us recall what this means. Put $\mathbb{C}_v :=$ the completion of $\overline{\mathbb{Q}}_v$ (on which G_v acts by continuity), and $B_{HT,v} :=$ the ring $\oplus_{r \in \mathbb{Z}} \mathbb{C}_v(r)$, with $\mathbb{C}_v(r)$ the usual Tate twist. The ring $B_{\mathrm{HT},v}$ is the ring of Laurent polynomials in the indeterminate t, a generator of $\mathbb{Z}_p(1)$, with coefficients in $\mathbb{C}_v$ and $(B_{\mathrm{HT},v})^{G_v} = K_v$. The G_v-representation V is **Hodge-Tate** in the sense that, if $\underline{D}^*_{\mathrm{HT},v}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_v]}(V, B_{\mathrm{HT},v})$, the natural map

$$B_{\mathrm{HT},v} \otimes_{K_v} \underline{D}^*_{\mathrm{HT},v}(V) \to \mathrm{Hom}_{\mathbb{Q}_p}(V, B_{\mathrm{HT},v})$$

is an isomorphism. This implies that $\underline{D}^*_{\mathrm{HT},v}(V)$ is a free $E \otimes_{\mathbb{Q}_p} K_v$-module of rank N, equipped with a $\mathbb{Z}$-gradation by sub-$E \otimes_{\mathbb{Q}_p} K_v$-modules

$$\mathrm{gr}^r \underline{D}^*_{\mathrm{HT},v}(V) = \mathrm{Hom}_{\mathbb{Q}_p[G_v]}(V, \mathbb{C}_v(r)).$$

By the (E, v)-**Hodge-Tate type** $h(v)$ of V, we mean the isomorphism class of the graded $E \otimes_{\mathbb{Q}_p} K_v$-module $\underline{D}^*_{\mathrm{HT},v}(V)$, and we call N the **degree** of $h(v)$. For instance, if $K_v = \mathbb{Q}_p$, $\underline{D}^*_{\mathrm{HT},v}(V)$ is just a graded E-vector space and to know $h(v)$ amounts to the same as to know the **Hodge-Tate numbers**, that is, the

[4] using resolutions of singularities.

non-negative integers $h_r(v) = \dim_E(\mathbb{C}_v(r) \otimes_{\mathbb{Q}_p} V)^{G_v} = \dim_E \mathrm{gr}^{-r}\underline{D}^*_{\mathrm{HT},v}(V)$ (these are almost all 0 and $\sum h_r(v) = N$).

Remark: If v divides p, the **trivial** (E, v)**-Hodge-Tate type of degree** N is the isomorphism class of $D = (E \otimes_{\mathbb{Q}_p} K_v)^N$ with $\mathrm{gr}^0 D = D$. An E-representation V of G_{K_v} is Hodge-Tate of trivial type if and only if the image of I_v in $\mathrm{Aut}_E(V)$ is finite ([**24**], th. 1 and [**25**], th. 11, cor.). In particular, if V is a geometric representation of $G_{K,S}$ and if v is not in S but divides p, the (E, v)-Hodge-Tate type of V is trivial.

By an E-**Hodge-Tate type of** K, we mean a function h assigning to each place v dividing p an (E, v)-Hodge-Tate type $h(v)$, all those $h(v)$ having the same degree N. Let $\mathbf{Geom}(K, S, h; E)$ denote the set of isomorphism classes of geometric irreducible E-representations of $G_{K,S}$ with the indicated Hodge-Tate type $h(v)$ for each v dividing p (such a representation is necessarily of dimension N).

§3.The finiteness conjectures.

If V is a geometric E-representation of G_K, for any finite place v of K, there is an unique invariant open subgroup $\mathfrak{L}_v(V)$ of I_v which is such that, if L is a finite extension of K_v contained in $\overline{K}_v$, then V, when viewed as a representation of $\mathrm{Gal}(\overline{K}_v/L)$, is semi-stable if and only if $\mathrm{Gal}(\overline{K}_v/L) \cap I_v \subset \mathfrak{L}_v(V)$. If $v \notin S$, we have $\mathfrak{L}_v(V) = I_v$.

By an **inertial level for** S, we mean a rule $\mathfrak{L}$ which assigns to each $v \in S$ an open invariant subgroup $\mathfrak{L}_v$ of I_v.

Let $\mathbf{Geom}(K, S, \mathfrak{L}, h; E)$ denote the set of isomorphism classes of geometric irreducible E-representations V of $G_{K,S}$ in the set $\mathbf{Geom}(K, S, h; E)$ such that, for each $v \in S, \mathfrak{L}_v \subset \mathfrak{L}_v(V)$.

Conjecture 2a: *For any finite set of places S of K, inertial level $\mathfrak{L}$ for S, $\overline{\mathbb{Q}}_p$-Hodge-Tate type h of K, the set $\mathbf{Geom}(K, S, \mathfrak{L}, h; \overline{\mathbb{Q}}_p)$ is finite.*

Example [**14**], prop. 1 (see [**1**] for other examples of this kind): If $K = \mathbb{Q}, S = \{7\}, \mathfrak{L}_7 = I_7$ and h, given by the Hodge numbers $(h_r)_{r\in\mathbb{Z}}$, is such that $h_r h_s \neq 0 \Rightarrow s - r \leq 3$, then $\mathbf{Geom}(K, S, Z, h; \overline{\mathbb{Q}}_7)$ is empty, unless there is $i \in \mathbb{Z}$ such that $h_{-i} = 1$ and $h_r = 0$ if $r \neq i$, in which case $\mathbf{Geom}(K, S, Z, h; \overline{\mathbb{Q}}_7)$ has one element which is the class of $\overline{\mathbb{Q}}_7(i)$.

Conjecture 2b: *For any finite set of places S, inertial level $\mathfrak{L}$ for S, finite extension field E of $\mathbb{Q}_p$, E-Hodge-Tate type h of K, the set $\mathbf{Geom}(K, S, \mathfrak{L}, h; \overline{\mathbb{Q}}_p)$ is finite.*

Conjecture 2c: *For any finite set of places S, finite extension field E of $\mathbb{Q}_p$, E-Hodge-Tate type h of K, the set $\mathbf{Geom}(K, S, h; E)$ is finite.*

§4. Remarks:

(a) Concerning the finiteness conjectures.

Conjecture 2c is in the spirit of the finiteness conjectures of Shafarevich (proved by Faltings [**5**]).

Obviously, *conjecture* 2a *implies conjecture* 2b. *Conjecture* 2b *and* 2c are equivalent: the implication 2c $\Rightarrow$ 2*b* is obvious. Conversely, attached to a potentially semi-stable representation of degree N (at v) one has (for this theory, see

[**12**]) a representation D_v of the Weil-Deligne group of K_v of degree N with coefficients in a field E' which is E if v does not divide p and a finite unramified extension of E if v divides p. In particular, if we choose a basis of D_v, we have an homomorphism $\rho_v : I_v \to GL_N(E')$ whose kernel is the invariant open subgroup $\mathfrak{L}_v(V)$.

If v does not divide $p, E' = E$, and, because the order of the finite subgroups of $GL_N(E)$ is bounded, there is an integer $M_v(E,N)$ such that $[I_v : \mathfrak{L}_v(V)]$ divides $M_v(E,N)$. If v divides p, one can check (cf. [**12**]) that, for any $g \in I_v$, the characteristic polynomial of g acting on D_v has coefficients in E. This implies [**7**] that there exists a linear representation of degree N of $I_v/\mathfrak{L}_v(V)$ with coefficients in the field $E_1 = E(\sqrt[p-1]{1})$ if $p \neq 2$ (resp. $E(\sqrt[4]{1})$ if $p = 2$) which has the same character as ρ_v. Therefore, because the order of the finite subgroups of $GL_N(E_1)$ is bounded, in this case as well, there is an integer $M_v(E,N)$ such that $[I_v : \mathfrak{L}_v(V)]$ divides M_v.

For any $v \in S$, it is easy to see that, given the integer $M_v(E,N)$, there is an open invariant subgroup $\mathfrak{L}_v(E,N)$ of I_v which is contained in all open invariant subgroups whose order divides $M_v(E,N)$. Therefore $\mathbf{Geom}(K,S,h;E) \subset \mathbf{Geom}(K,S,\mathfrak{L},h;E)$ where $\mathfrak{L}$ is given by $v \mapsto \mathfrak{L}_v(E,N)$.

(b) Changing K. Let L be a finite extension of K contained in $\overline{K}$. It is easy to check that an E-representation V of G_K is geometric if and only if its restriction $\mathrm{Res}_K^L V$ to G_L is geometric; similarly an E-representation W of G_L is geometric if and only if the induced representation $\mathrm{Ind}_L^K W$ of G_K is. It is also easy to compute the Hodge-Tate types and the inertial levels of $\mathrm{Res}_K^L V$ (resp. $\mathrm{Ind}_L^K W$) from those of V (resp. W). From that, using adjunction, we see easily that Conjecture 1 (resp. 2a, 2b) is true for K if and only if it is true for L.

(c) Semi-stable representations. A geometric E-representation V of G_K is said to be *semi-stable* if, for each finite place v of K, V is semi-stable as a representation of G_v, i.e. if $\mathfrak{L}_v(V) = I_v$. Let $\mathbf{Geom}_{\mathrm{st}}(K,S,h;\overline{\mathbb{Q}}_p)$ denote the set of isomorphism classes of geometric irreducible $\overline{\mathbb{Q}}_p$-representations V of $G_{K,S}$ in the set $\mathbf{Geom}(K,S,h;\overline{\mathbb{Q}}_p)$ which are semi-stable. The following is a special case of conjecture 2a (corresponding to choosing $\mathfrak{L}$ such that $\mathfrak{L}_v = I_v$ for all $v \in S$):

Conjecture 2a$'$: *For any finite set of places S of K, $\overline{\mathbb{Q}}_p$-Hodge-Tate type h of K, the set* $\mathbf{Geom}_{st}(K,S,h;\overline{\mathbb{Q}}_p)$ *is finite.*

Conversely, it is easy to check that conjecture $2a'$ for all number fields implies conjecture 2a.

(d) Concerning geometric p-adic representations, and "compatible families" of representations.

The curious implication of Conjecture 1, taken together with standard conjectures concerning the étale cohomology of algebraic varieties of number fields, is that any irreducible p-adic geometric representation V of $G_{K,S}$ has these ("motivic") properties:
(1) For all nonarchimedean places $v \notin S$, let $\Phi_v \in G_v \subset G_K$ be a lifting of the geometric Frobenius $f_v \in G_v/I_v$ and let $P_v(V;T) \in \overline{\mathbb{Q}}_p[T]$ denote the

characteristic polynomial of Φ_v acting on V. Then, there exists $r \in \mathbb{Z}$ and a finite extension E of $\mathbb{Q}$ contained in $\mathbb{Q}_p$ such that, for all $v \notin S, P_v(V,T) \in E[T]$, the complex roots of $P_v(V,T)$ (for a chosen embedding of E into $\mathbb{C}$) have their complex absolute values equal to $q_v^{r/2}$ where q_v is the cardinality of the residue field of v.
(2) There exists a finite extension E' of the above E and, for any finite place λ of E', a geometric E'_λ-representation V_λ of G_{K,S_λ} where $S_\lambda = S \cup \{v|v$ has the same residual characteristic as $\lambda\}$ such that, if $v \notin S_\lambda, P_v(V_\lambda,T)(:=$ the characteristic polynomial of Φ_v acting on the E'_λ-vector space $V_\lambda) = P_v(V,T)$.

(e) L-functions and weights.

Assume $K = \mathbb{Q}$ and let V be an irreducible geometric $\overline{\mathbb{Q}}_p$-representation of $G_\mathbb{Q}$ of degree N. For each prime l, one can associate to V an N-dimensional $\overline{\mathbb{Q}}_p$-linear representation of the Weil-Deligne group W'_l of $\mathbb{Q}_l$ (cf. [**13**] or [**17**]). Hence if we choose an imbedding of $\overline{\mathbb{Q}}_p$ into $\mathbb{C}$, one can define in the usual way a local factor $L_l(V,s)$ for each prime l and one can define the global L-function $L(V,s)$ as being the formal product of all those local factors. One conjectures that this Dirichlet series converges for $\Re(s) >> 0$ and admits a meromorphic continuation in the whole complex plane; moreover, one can give a conjectural interpretation of the order of the zero or pole at $s = 0$ in terms of Galois cohomology (see [**17**], n°3.4).

One can define the **weight** $w(V)$ of V : if $\dim_{\overline{\mathbb{Q}}_p} V = 1$, there is an unique integer i such that the action of $G_\mathbb{Q}$ on $V(i)$ is finite and $w(V) = 2i$; if $\dim_{\overline{\mathbb{Q}}_p} V = N, w(V) = w(\wedge^N V)/N$; hence this is a rational number. For $r,s \in \mathbb{Z}$, define the **Hodge numbers** $h_{r,s}(V)$ of V as being 0 unless $r+s = w(V)$ in which case $h_{r,s}(V)$ is the Hodge-Tate number $h_r(V)$.

If we choose an imbedding of $\overline{\mathbb{Q}}$ into $\mathbb{C}$, this defines the complex conjugation $c \in G_\mathbb{Q}$, hence we can define

$$h^+(V) = \dim_{\overline{\mathbb{Q}}_p}\{v \in V|cv = v\} \text{ and } h^-(V) = \dim_{\overline{\mathbb{Q}}_p}\{v \in V|cv = -v\}.$$

Define also $h^+_{r,s}(V) = h^-_{r,s}(V) = h_{r,s}(V)/2$ if $r \neq s$ and, if $j = w(V)/2 \in \mathbb{Z}$,

$$h^+_{j,j}(V) = h^+(V) - \sum_{r\neq s} h^+_{r,s}(V), \quad h^-_{j,j}(V) = h_{j,j}(V) - h^+_{j,j}(V).$$

Conjecture 3a: *Let V be an irreducible geometric $\overline{\mathbb{Q}}_p$-representation of $G_\mathbb{Q}$. Then the weight $w(V)$ of V is an integer. Moreover, for $r,s \in \mathbb{Z}$ such that $r+s = w(V)$, the numbers $h^+_{r,s}(V)$ and $h^-_{r,s}(V)$ are non negative integers and $h_{r,s}(V) = h_{s,r}(V)$.*

Assuming this conjecture, these numbers define in the usual way an isomorphism class of a linear representation of the Weil group $W_\mathbb{R}$, hence a Γ-factor $L_\infty(V,s)$ and we can consider the complete L-function $\wedge(V,s) = L_\infty(V,s)\cdot L(V,s)$.

Because we have a representation of W'_l for each prime number l and a representation of $W_\mathbb{R}$, we can define in the usual way the **conductor** N_v and the **ϵ-factor** $\epsilon(V,s)$ (cf. e.g. [**3**]). We thus have the conjectural functional equation:

Conjecture 3b: *Let V be as in* 3a. *Then $\wedge(V,s)$ converges for $\Re(s) >> 0$ and admits a meromorphic continuation in the whole complex plane, satisfying*

$$\wedge(V,s) = \epsilon(V,s) \cdot L(V^*, 1-s).$$

(f) Modular forms.

Combining conjecture 1 with classical conjectures (e.g. [**26**]), we obtain

Conjecture 3c: *Let V be an irreducible geometric $\overline{\mathbb{Q}}_p$-representation of $G_{\mathbb{Q}}$ of degree two which is not a Tate twist of a finite representation. Then there is an integer $i \in \mathbb{Z}$ such that $V(-i)$ is isomorphic to the representation associated to a "new" modular form.*

Using the previous discussion, one can be more precise: the integer i must be the smallest integer such that $h_j(V) \neq 0$. Twisting by $\mathbb{Z}_p(-i)$ if necessary, we can assume $i = 0$. Then, if w is the biggest integer such that $h_w(V) \neq 0$, the weight of V is w and we see that the weight of the corresponding modular form f must be $w+1$. The conductor of f must be the conductor N_V of V. The nebentypus can be also computed in the usual way using the representations of the Weil-Deligne groups. To prove this conjecture one "need only" to prove that the L-function of V is the Mellin transform of a modular form.

Observe also

i) that, because of the finiteness of the dimension of the space of modular forms of fixed weight and level, conjecture 2a and conjecture 2b in the two dimensional case follows from conjecture 3c;

ii) that, because of the fact that, if X is an elliptic curve over $\mathbb{Q}$, if $T_p(X)$ is its Tate module and if $V = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(X), V$ is geometric [**12**], conjecture 3c implies the Shimura-Taniyama-Weil conjecture.

(g) Tannakian categories. (for a discussion of tannakian categories, see, for instance, [**12**]). Let K be a number field, and let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_K)$ denote the tannakian category of p-adic representations of G_K. A tannakian subcategory is a full subcategory containing an object of positive dimension, and which is stable under passage to sub-object, quotient, direct sum, tensor product and dual.

Let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p,g}(G_K)$ (resp. $\underline{\mathrm{Rep}}^{ss}_{\mathbb{Q}_p,g}(G_K)$) denote the tannakian subcategory of $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_K)$ whose objects are geometric (resp. semi-simple geometric) representations of G_K.

Conjecture 4a. *The category $\underline{Rep}^{ss}_{\mathbb{Q}_p,g}(G_K)$ is the smallest tannakian subcategory of $\underline{Rep}_{\mathbb{Q}_p}(G_K)$ containing all representations of the form $H^m_{\acute{e}t}(X_{/\overline{K}}, \mathbb{Q}_p)$ where X ranges through all proper smooth varieties over K.*

Note that Conjecture 4a is equivalent to the union of Conjecture 1 and the classical conjecture, due to Tate, that if X is smooth and proper over K, the G_K-representation $H^m_{\acute{e}t}(X_{/\overline{K}}, \mathbb{Q}_p)$ is semi-simple.

Although we don't have a precise definition of "mixed motives" in mind, it is tempting to us to ask the somewhat vague question:

Question 4b: *Is the category $\underline{Rep}_{\mathbb{Q}_p,g}(G_K)$ the smallest tannakian subcategory of $\underline{Rep}_{\mathbb{Q}_p}(G_K)$ containing the p-adic realization of all mixed motives over K.*

One concrete interpretation of the above question is the following:

Question 4c: *Is the category $\underline{Rep}_{\mathbb{Q}_p,g}(G_K)$ the smallest tannakian subcategory of $\underline{Rep}_{\mathbb{Q}_p}(G_K)$ containing all representations of the form $H^m_{ét}(X_{\cdot/\overline{K}}, \mathbb{Q}_p)$ where X ranges through all simplicial schemes of finite type over K ?*

§5. "Universal" geometric deformation rings.

A preliminary remark on deformations: Let G be a profinite group and Λ a local complete noetherian ring, of residue field k. Let GSW be the category of local complete noetherian Λ-algebras, with residue field k. For short, by Λ-*algebra* we mean an object of $\mathfrak{S}_\Lambda$.

If A is any Λ-algebra, an A-**representation** of G is an A-module of finite type equipped with a linear and continuous action of G. We say that such a representation is *flat* if the underlying A-module is flat.

We give ourselves a finite dimensional k-representation $\overline{V}$ of G, which is such that the natural map $k \to \mathrm{End}_{k[G]}(\overline{V})$ is an isomorphism[5].

For each Λ-algebra A, an A-**deformation of** $\overline{V}$ is a flat A-representation V of G such that $k \otimes_A V$ is isomorphic to $\overline{V}$. Denote by $\Psi_{\overline{V}}(A)$ the set of isomorphism classes of A-deformations of $\overline{V}$. We get in this way a functor

$$\Psi_{\overline{V}} : \mathfrak{S}_\Lambda \to \underline{\text{Sets}}.$$

Then [22][6], if $\dim_k H^1(G, \mathrm{End}_k(\overline{V})) < +\infty$, this functor is representable. Therefore, one can define the universal Λ-deformation ring $R_\Lambda(\overline{V})_G$ together with a $R_\Lambda(\overline{V})_G$-deformation of $\overline{V}$, well defined up to isomorphism, with the obvious universal property.

Remark: If we choose a basis of $\overline{V}$ over k, the action of G on $\overline{V}$ gives us an homomorphism $\overline{\rho} : G \to GL_d(k)$. Usually, one considers continuous homomorphisms from $G \to GL_d(A)$ lifting $\overline{\rho}$ rather than deformations of $\overline{V}$. Recall that if $\rho, \rho' : G \to GL_d(A)$ are two such homomorphisms, one says that ρ and ρ' are isomorphic (resp. strictly isomorphic) if there exists $\alpha \in GL_d(A)$ (resp. $\alpha \in GL_d(A)$ lifting the identity in $GL_d(k)$) such that $\rho'(g) = \alpha\rho(g)\alpha^{-1}$, for all $g \in G$. From the surjectivness of A^* onto k^* and the fact that $\mathrm{End}_{k[G]}(\overline{V}) = k$, we see that the two equivalence relations are actually the same. If we denote by $\psi_{\overline{\rho}}(A)$ the set of equivalence classes, we get also in this way a functor $\psi_{\overline{\rho}} : \mathfrak{S}_\Lambda \to \underline{\text{Sets}}$. It is an easy exercise to build a natural isomorphism between the functors ψ_ρ and $\psi_{\overline{V}}$ and we will use this natural isomorphism to identify $\Psi_{\overline{\rho}}$ and $\psi_{\overline{V}}$.

Now let $\mathfrak{D}$ be a strictly full subcategory of the category $\underline{\text{Rep}}^f_\Lambda(G)$ of Λ-modules of finite length equipped with a linear and continuous action of G which is stable under subobjects, quotients and direct sums. Assume that $\overline{V}$ is

[5] actually, we will be mostly interested in the absolutely irreducible case.

[6] at least in the absolutely irreducible case, but the same proof works in general.

an object of $\mathfrak{D}$. Then we can define the subfunctor $\psi_{\overline{V},\mathfrak{D}}$ of $\psi_{\overline{V}}$ by the condition that, for any Λ-algebra A, $\psi_{\overline{V},\mathfrak{D}}(A)$ consists of elements which are represented by representations V such that, for any artinian quotient A' of $A, A' \otimes_A V$ is an object of $\mathfrak{D}$. One sees **[23]** that, if $\dim_k H^1(G, \mathrm{End}_k(\overline{V})) < +\infty$, then $\psi_{\overline{V},\mathfrak{D}}$ is representable by a quotient $R_\Lambda(\overline{V})_\mathfrak{D}$ of $R_\Lambda(\overline{V})_G$. We call $R_\Lambda(\overline{V})_\mathfrak{D}$ the **universal ring of deformations of $\overline{V}$ lying in $\mathfrak{D}$.**

We can apply that to $G = G_{K,S}$ and $\Lambda = \mathcal{O}_E$ the ring of the integers of a finite extension of $\mathbb{Q}_p$. Hence, given $\overline{V}$, we can consider the noetherian and complete $\mathcal{O}_E$-algebra $R_{\mathcal{O}_E}(\overline{V})_S := R_{\mathcal{O}_E}(\overline{V})_{G_{K,S}}$. Now, if moreover, we fix an inertia level $\mathfrak{L}$ for S and two integers $a, b \in \mathbb{Z}$ with $a \leq b$, we can consider the full subcategory $\mathfrak{D} = \underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})_{\mathfrak{L},[a,b],\mathrm{st}}$ (resp. $\underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})_{\mathfrak{L},[a,b],\mathrm{cris}}$) of $\underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})$ whose objects are the T's such that, for each $v \in S$,
i) if v doesn't divides $p, \mathfrak{L}_v$ acts trivially on T,
ii) if v divides p, one can find a semi-stable (resp. crystalline) p-adic representation V of $\mathfrak{L}_v$ satisfying $(\mathbb{C}_v(r) \otimes_{\mathbb{Q}_p} V)^{\mathfrak{L}_v} = 0$ such that T is isomorphic to a sub-quotient of V as a $\mathbb{Z}_p[\mathfrak{L}_v]$-module.

We denote by $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[ab],\mathrm{st}}$ (resp. $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[ab],\mathrm{cris}}$) the corresponding quotient of $R_{\mathcal{O}_E}(\overline{V})_S$. Observe that the second is a quotient of the first.

Conjecture 5: *For any finite set of places S of K, finite extension E of $\mathbb{Q}_p$ of residue field k, inertial level $\mathfrak{L}$, integers $a \leq b$, finite dimensional k-representation $\overline{V}$ of $G_{K,S}$ belonging to $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[ab],st}$, which is such that the natural map $k \to \mathrm{End}_{k[G]}(\overline{V})$ is an isomorphism, the ring $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[ab],st}$ is a finite $\mathcal{O}_E$-algebra.*

This conjecture implies conjecture 2b: if $S, \mathfrak{L}$ and h are given as in conjecture 2a, we can choose $a \leq b$ such that, for all v's dividing p, if D is a graded $(\overline{\mathbb{Q}}_p \otimes_{\overline{\mathbb{Q}_p}} K_v)$-module representing the class $h(v)$, we have $gr^{-r}D = 0$ if $r \notin [a, b]$. Then, if V is an E-representation of $G_{K,S}$, say of degree N, representing an element of $\mathbf{Geom}(K, S, \mathfrak{L}, h; E)$, the fact that V is irreducible implies that one can find an $\mathcal{O}_E$-lattice T of V stable under $G_{K,S}$ and such that, if $\overline{V} = k \otimes_{\mathcal{O}_E} T$, the map $k \to \mathrm{End}_{k[G]}(\overline{V})$ is an isomorphism. Obviously, each finite quotient of T lies in $\underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})_{\mathfrak{L},[a,b],\mathrm{st}}$, hence the natural homomorphism from $R_{\mathcal{O}_E}(\overline{V})_S$ to $\mathcal{O}_E$ factors through $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[a,b],\mathrm{st}}$. Therefore the conjecture implies that, given $\overline{V}$, there are only finitely many elements of $\mathbf{Geom}(K, S, \mathfrak{L}, h; E)$ within a representative of which we can find a $G_{K,S}$-stable $\mathcal{O}_E$-lattice T such that $k \otimes_{\mathcal{O}_E} T \cong \overline{V}$. The result follows from the fact, easy to check, that there are only finitely many isomorphism classes of k-representations $\overline{V}$ of degree N of $G_{K,S}$ such that the map $k \to \mathrm{End}_{k[G]}(\overline{V})$ is an isomorphism.

Remarks: a) Let V be an E-representation of $G_{K,S}$ which admits a lattice T stable under $G_{K,S}$ such that each finite quotient lies in $\underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})_{\mathfrak{L},[a,b],\mathrm{st}}$ (resp. $\underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})_{\mathfrak{L},[a,b],\mathrm{cris}}$). It should not be very hard to prove that V is geometric. Up to now, we have checked this property only in special cases (for instance, this is OK for $\underline{\mathrm{Rep}}^f_{\mathcal{O}_E}(G_{K,S})_{\mathfrak{L},[a,b],\mathrm{cris}}$ if $b - a \leq p - 1$ and $K/\mathbb{Q}$ is unramified at p or if $b - a \leq 1$ and, for all v dividing p, the ramification index of $K_v/\mathbb{Q}_p$ is $\leq p - 1$).

(b) Universal geometric deformation rings and the Langlands program: Let R be either $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[a,b],\mathrm{st}}$ or $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[a,b],\mathrm{cris}}$ as above and T_R the corresponding free R-module with action of $G_{K,S}$ (well defined up to isomorphism). For any place v of K not contained in S, there is an element which might be called a **"Hecke element"** τ_v in R given by taking the trace of Frob_v acting on T_R, where Frob_v is any choice of Frobenius element at v in $G_{K,S}$.

Can one "reconstuct" the rings $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[a,b],\mathrm{st}}$ and $R_{\mathcal{O}_E}(\overline{V})_{S,\mathfrak{L},[a,b],\mathrm{cris}}$ and their systems of Hecke elements $v \mapsto \tau_v$ as completions (at maximal ideals of residual characteristic p) of rings which are generated by Hecke operators and which act (faithfully) on automorphic representations spaces for specific reductive groups (notably $GL_{N/K}$) ?

§6. The special case where ρ is potentially abelian.

A p-adic representation V of G_K is **potentially abelian** if there is an open subgroup of G_K which operates on V through an abelian quotient group.

For any abelian variety A over K, let $A[p^n](\overline{K})$ denote the group of $\overline{K}$-rational points of A annihilated by multiplication by p^n, let $T_p(A)$=lim.proj. $A[p^n](\overline{K})$
and put $V_p(A) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(A)$. Recalling the terminology of Remark (g) of § 4, let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p,CM}(G_K)$ denote the smallest Tannakian subcategory of $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_K)$ containing all representations of G_K which factor through finite groups, and also containing the $V_p(A)$ for A ranging through all abelian varieties over K which are potentially of CM type.

Proposition: *Let V be a potentially abelian p-adic representation of G_K and, for each place v of K, let V_v be the underlying representation of G_v. The following are equivalent:*
1) For every place v of K dividing p, V_v is of Hodge-Tate type and the action of I_v is semi-simple;
2) for every place v of K dividing p, V_v is of Hodge-Tate type;
3) for every place v of K dividing p, V_v is of de Rham type;
4) for every place v of K dividing p, V_v is potentially semi-stable;
5) for every place v of K dividing p, V_v is potentially crystalline;
6) the representation V is geometric;
7) the representation V is an object in $\underline{Rep}_{\mathbb{Q}_p,CM}(G_K)$.

Proof. Since potentially abelian representations are unramified outside a finite set of places ([**27**], Cor. of p. III.11) we have (4) $\Leftrightarrow$ (6). One knows that every abelian variety A which is potentially of CM type has potentially good reduction. It follows that if $V = V_p(A)$, then V satisfies (5). Moreover, any finite representation satisfies (5), and (5) is stable under tannakian operations. Therefore (7) $\Rightarrow$ (5).

The implications (5) $\Rightarrow$ (4) $\Rightarrow$ (3) $\Rightarrow$ (2) and (1) $\Rightarrow$ (2) are all trivial.

From ([**27**], Thm. 3 p. III-52) one has that (1) implies that V is "locally algebraic" which, in turn, implies (7) by ([**4**], Prop. D1).

It remains to establish the implication (2) $\Rightarrow$ (1) which is an exercise, using the Theorem of Sen [**25**] that a representation, which is Hodge-Tate and whose only Hodge-Tate weight is 0 (with any multiplicity) must factor through a finite quotient group. □

§ 7. The special case where p is potentially unramified.

After a finite base change we can make p unramified, i.e., we can view our representation as being a representation of $G_{K,S}$ where S is a finite set of primes, none of which have residual characteristic p.

Conjecture 5a: *If p is distinct from all of the residual characteristics of S, then any p-adic representation of $G_{K,S}$ factors through a finite quotient group of $G_{K,S}$.*

Remark: Conjecture 5a for all p-adic representations follows from the same conjecture, but stated only for semi-simple geometric p-adic representations. This latter conjecture follows from Conjecture 1 in conjunction with the Tate Conjecture about the subspace of étale cohomology generated by algebraic cycles.

An equivalent way of stating Conjecture 5a is that if p is distinct from all of the residual characteristics of S, then any quotient group of $G_{K,S}$ which is a p-adic analytic group, is finite. Conjecture 5a for S empty bears on the structure of the Galois group of a Golod-Shafarevich p-tower: Let $\Gamma(K,p)$ denote the Galois group of the maximal everywhere unramified pro-p-extension of the number field K. Conjecture 5a implies.

Conjecture 5b: *Any quotient group of $\Gamma(K,p)$ which is a p-adic analytic group, is finite.*

For some partial corroboration of this conjecture, see the recent work of N.Boston ([**2**], Theorem 1), and F.Hajir [**18**].

Part II. Representations of dimension 2 of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$.

In this part, $G_p = G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and I_p is the inertia subgroup. For any $\mathbb{Z}_p$-module V and any $r \in \mathbb{Z}$, we denote by $V(r) = V \otimes_{\mathbb{Z}_p} \mathbb{Z}_p(r)$ the usual Tate twist.

For $\epsilon \in \overline{\mathbb{F}}_p$, the residue field of $\overline{\mathbb{Q}}_p$, we denote $[\epsilon]$ its Teichmüller representative in $\overline{\mathbb{Q}}_p$.

For simplicity, we will assume $p \neq 2$ (and sometimes $p \geq 5$).

§8. The representations of G_p of dimension 1.

Let A be a complete noetherian local ring with finite residue field. For any profinite group G, a *character with values in A* is a continuous homomorphism $\eta : G \to A^*$. If N is any closed invariant subgroup of G contained in the kernel of η, we still denote $\eta : G/N \to A^*$ the character we get by factoring, and conversely.

If V is an A-module equipped with an action of $G, V(\eta)$ denotes the same A-module with the action of G twisted by η.

If $a \in A^*$, we denote by $\eta_a : G_p/I_p \to A^*$ the unique character such that $\eta_a(\text{arith.frob.}) = a$.

Let $\xi_1 : G_p \to \mathbb{F}_p^*$ be the character giving the action of G_p on the p-th roots of 1. For $i \in \mathbb{Z}/(p-1)\mathbb{Z}$ and $\epsilon \in \mathbb{F}_p^*$, denote by $\overline{U}_{i,\epsilon}$ the one dimensional $\mathbb{F}_p$-representation of G_p which is $\mathbb{F}_p$ on which G_p acts through $\eta_\epsilon \xi_1^i$. Then any one dimensional $\mathbb{F}_p$-representation of G_p is isomorphic to one and only one of the $\overline{U}_{i,\epsilon}$.

Define $\hat{\xi}_1(g) = [\xi_1(g)]$ and $\chi : G_p \to \mathbb{Z}_p^*$ the cyclotomic character which we can write as $\chi = \hat{\xi}_1 \cdot \chi_0$ with χ_0 taking values in $1 + p\mathbb{Z}_p$.

If we denote $R(\overline{U}_{i,\epsilon}) = R_{\mathbb{Z}_p}(\overline{U}_{i,\epsilon})_{G_p}$ the universal $\mathbb{Z}_p$-deformation ring, one sees that one can write $R(\overline{U}_{i,\epsilon}) = \mathbb{Z}_p[\![T_1, T_2]\!]$, in such a way that the corresponding character $\eta : G_p \to \mathbb{Z}_p[\![T_1, T_2]\!]^*$ giving the universal deformation has the property that if, for $r \in \mathbb{Z}_p$

$$e_r : \mathbb{Z}_p[[T_1, T_2]] \longrightarrow \mathbb{Z}_p[[T_1]]$$

is the ring-homomorphism to $\ker_\rho T_1$ to T_1 and T_2 to $(1+p)^r - 1$, then

$$e_r \cdot \eta = \eta_{[\epsilon](1+T_1)} \cdot \hat{\xi}_1^i \cdot \chi_0^{T_2}$$

for all $r \in \mathbb{Z}_p$.

Now, let U be a one dimensional $\mathbb{Q}_p$ representation of G_p. To give U up to isomorphism is the same as to give the unique $(i, \varepsilon, t_1, t_2) \in ((\mathbb{Z}/(p-1)\mathbb{Z}) \times \mathbb{F}_p^* \times p\mathbb{Z}_p \times p\mathbb{Z}_p$ such that G_p acts on U via $\eta_{[\epsilon](1+t_1)} \cdot \hat{\xi}_1^i \cdot \chi_0^r$ where $(1-p)^r - 1 = t_{2r}$. If T is a $\mathbb{Z}_p$-lattice and if $\overline{U} = T/pT$, then $\overline{U} \simeq \overline{U}_{i,\varepsilon}$; moreover U is potentially semi-stable if it is Hodge-Tate, which amounts to requiring that r be in $\mathbb{Z}$. In this case r is the (unique) Hodge-Tate weight of U.

Now fix $\overline{U}_{i,\varepsilon}$ and $r \in \mathbb{Z}$. We observe that , if U is potentially semi-stable of Hodge-Tate weight r and such that $\overline{U} \simeq \overline{U}_{i,\varepsilon}$, then there is a unique $t_1 \in p\mathbb{Z}_p$ such that G_p acts on U via $\eta_{[\epsilon](1+t_1)} \cdot \hat{\xi}_1^i \cdot \chi_0^r = \hat{\xi}_1^{i-\overline{r}} \cdot \eta_{[\epsilon](1+t_1)} \cdot \chi^r$ (where $\overline{r}$ denotes the image of r in $\mathbb{Z}/(p-1)\mathbb{Z}$), hence $U(\hat{\xi}_1^{\overline{r}-i})$ is a crystalline representation with r as unique Hodge-Tate weight. Now, if we consider the full subcategory $\mathfrak{D}$ of the category of finite $\mathbb{Z}_p$-representations T of G_p which are such that $T(\hat{\xi}_1^{\overline{r}-i})$ is isomorphic to a subquotient of a crystalline representation of unique Hodge-Tate weight r, $\mathfrak{D}$ is stable under subobjects, quotients and direct sums, so we can speak of the universal deformation of $\overline{U}_{i,\varepsilon}$ "lying in $\mathfrak{D}$". The corresponding ring $R_{\mathbb{Z}_p}(\overline{U}_{i,\varepsilon})_{\mathfrak{D}}$ is isomorphic to $\mathbb{Z}_p[\![T_1]\!]$ with the corresponding character $\eta_{[\epsilon](1+T_1)} \cdot \hat{\xi}_1^i \cdot \chi_0^r$.

§9. The absolutely irreducible $\mathbb{F}_p$-representations of G_p of dimension 2 and their deformations.

By convention, for any positive integer $n, \mathbb{Q}_{p^n} \subset \overline{\mathbb{Q}}_p$ will denote the unramified extention of $\mathbb{Q}_p$ in $\overline{\mathbb{Q}}_p$ of degree n and $\mathbb{F}_{p^n}$ the corresponding residue field. We choose $\pi_1, \pi_2 \in \overline{\mathbb{Q}}_p$ such that $\pi_1^{p-1} = -p$ and $\pi_2^{p+1} = \pi_1$ and, for $j = 1, 2$, let $F_j =$ the Galois closure over $\mathbb{Q}_p$ of $\mathbb{Q}_p(\pi_j)$; let $L = \mathbb{Q}_{p^{2(p-1)}}(\pi_2)$ and $L_0 = \mathbb{Q}_{p^{2(p-1)}}$ so that we have the diagram

$$\begin{array}{ccccccc}
 & & \mathbb{Q}_p(\pi_2) & - & F_2 = \mathbb{Q}_{p^2}(\pi_2) & - & L = \mathbb{Q}_{p^{2(p-1)}}(\pi_2) \\
 & & | & & | & & | \\
F_1 & = & \mathbb{Q}_p(\sqrt[p]{1}) = \mathbb{Q}_p(\pi_1) & - & \mathbb{Q}_{p^2}(\pi_1) & - & \mathbb{Q}_{p^{2(p-1)}}(\pi_1) \\
 & & | & & | & & | \\
 & & \mathbb{Q}_p & - & \mathbb{Q}_{p^2} & - & L_0 = \mathbb{Q}_{p^{2(p-1)}}
\end{array}$$

The $\mathrm{Gal}(L/\mathbb{Q}_p(\pi_2))(\simeq \mathrm{Gal}(L_0/\mathbb{Q}_p))$ is cyclic of order $2(p-1)$ generated by the "Frobenius element" τ satisfying $\tau x = x^p$ if $x \in \mathbb{F}_{p^{2(p-1)}}$, the residue field of L. The group $\mathrm{Gal}(L/L_0)$ is canonically isomorphic to the multiplicative group of $\mathbb{F}_{p^2}$, via the "**fundamental character**" $\xi_2 : \mathrm{Gal}(L/L_0) \to \mathbb{F}_{p^2}^*$ associated to $g \in \mathrm{Gal}(L/L_0)$, the image of $g\pi_2/\pi_2$ in $\mathbb{F}_{p^2}$. The group $\mathrm{Gal}(L/\mathbb{Q}_p)$ is a semi-direct product of $\mathrm{Gal}(L/\mathbb{Q}_p(\pi_2))$ and the normal subgroup $\mathrm{Gal}(L/L_0)$, with $\tau g \tau^{-1} = g^p$ for any $g \in \mathrm{Gal}(L/L_0)$.

Consider (ι, ε) where $\iota \in \mathbb{Z}/(p^2-1)\mathbb{Z}$ and $\varepsilon \in \mathbb{F}_p^*$. Given such data, *we choose* $\zeta \in \mathbb{F}_{p^2}$ such that $\zeta^{p+1} = -\varepsilon$. Attached to the above data, *and this choice*, we have a unique homomorphism

$$\overline{\rho}_{\iota,\varepsilon} : \mathrm{Gal}(L/\mathbb{Q}_p) \to \mathrm{Aut}_{\mathbb{F}_p}(\mathbb{F}_{p^2})$$

such that the restriction to $\mathrm{Gal}(L/L_0)$ consists in $\mathbb{F}_{p^2}$-linear automorphisms given by the one-dimensional $\mathbb{F}_{p^2}$-character ξ_2^ι, and τ acts on $\mathbb{F}_{p^2}$ via the $\mathbb{F}_{p^2}$-semilinear automorphism $x \mapsto \zeta \cdot x^p$. We denote also $\overline{\rho}_{\iota,\varepsilon} : G_p \to Aut_{\mathbb{F}_p}(\mathbb{F}_{p^2})$ the natural lifting and $\overline{V}_{\iota,\varepsilon}$ the 2-dimensional $\mathbb{F}_p$-representation of G_p which is $\mathbb{F}_{p^2}$ on which G_p acts through $\overline{\rho}_{\iota,\varepsilon}$.

Proposition . *i) The isomorphism class of the $\mathbb{F}_p[G_p]$-module $\overline{V}_{i,\varepsilon}$ is independent of the choice of ζ such that $\zeta^{p+1} = -\varepsilon$. The "determinant" $\wedge^2 \overline{V}_{\iota,\varepsilon}$ is isomorphic to $\overline{U}_{2\bar{\iota},\varepsilon}$ (where $\bar{\iota}$ is the image of ι in $\mathbb{Z}/(p-1)\mathbb{Z}$). One has $\overline{V}_{\iota',\varepsilon'} \simeq \overline{V}_{\iota,\varepsilon}$ if and only if $\varepsilon' = \varepsilon$ and $\iota' \in \{l, p\iota\}$. The representation $\overline{V}_{\iota,\varepsilon}$ is absolutely irreducible if and only if $p\iota \neq \iota$ (in $\mathbb{Z}/(p^2-1)\mathbb{Z}$).*

ii) Let $\overline{V}$ be a 2-dimensional absolutely irreducible $\mathbb{F}_p$-representation of G_p. Then $\overline{V} \simeq \overline{V}_{\iota,\varepsilon}$ for a suitable $(\iota,\varepsilon) \in (\mathbb{Z}/(p^2-1)\mathbb{Z}) \times \mathbb{F}_p^*$.

Proof. Exercise. □

Let $\iota \in \mathbb{Z}/(p^2-1)\mathbb{Z}$ such that $p\iota \neq \iota$. The residual representation $\overline{V}_{\iota,\varepsilon}$ being absolutely irreducible possesses a universal deformation ring $R_{\iota,\varepsilon} = R(\overline{V}_{\iota,\varepsilon})$ as in [**22**], and we have the theorem due to R. Ramakrishna ([**23**] Th.4.1):

Theorem . *For any absolutely irreducible $\overline{V} = \overline{V}_{\iota,\varepsilon}$, the "deformation problem is smooth" and the ring $R_{\iota,\varepsilon}$ is isomorphic to a power series ring in five variables over $\mathbb{Z}_p$, i.e. to $\mathbb{Z}_p[\![T_1, T_2, \cdots, T_5]\!]$.*

§10. **Potentially semi-stable p-adic representations and pst-modules.**

It is known[**12**] that, using the ring B_{st} [**11**], one gets an equivalence of categories between potentially semi-stable p-adic representations of $G_p = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ and the category of "admissible filtered (φ, N, G_p)-modules". We want to recall a bit of this theory.

(a) **Filtered modules** : Let F be a finite Galois extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and F_0 the maximal unramified extension of $\mathbb{Q}_p$ contained in F. A **filtered** $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$**-module** is a finite dimensional F_0-vector space D endowed with

- a "Frobenius" endomorphism $\varphi : D \to D$ bijectiv and semi-linear via the action of the absolute Frobenius on F_0,

- a "monodromy" operator $N : D \to D$ linear and satisfying $N\varphi = p\varphi N$,
- an action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ semi-linear with respect to the natural action of this group (via its quotient $\mathrm{Gal}(F_0/\mathbb{Q}_p)$) on F_0 and commuting with φ and N,
- a decreasing filtration $(\mathrm{Fil}^r D_F)_{r\in\mathbb{Z}}$ of the F-vector space $D_F = F \otimes_{F_0} D$ stable under the natural action of $\mathrm{Gal}(F/\mathbb{Q}_p)$ (acting via $(g, \lambda\otimes d) \mapsto g\lambda \otimes gd$) verifying $\mathrm{Fil}^r D_F = 0$ for $f \gg 0$ and $= D_F$ for $r \ll 0$. Observe that, if we put $D_{dR} = (D_F)^{\mathrm{Gal}(F/\mathbb{Q}_p)}$, the obvious map $F \oplus_{\mathbb{Q}_p} D_{dR} \to D_F$ is an isomorphism and the condition that the $\mathrm{Fil}^r D'_F$s are stable under $\mathrm{Gal}(F/\mathbb{Q}_p)$ means exactly that the filtration comes from a filtration of the $\mathbb{Q}_p$-vector space D_{dR}.

Whenever $F = \mathbb{Q}_p, D$ is just a $\mathbb{Q}_p$-vector space together with two linear endomorphisms φ and N (with φ bijective and $N\varphi = p\varphi N$) and a filtration and we call D a **filtered (φ, N)-module**.

The filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules are the objects of an additive $\mathbb{Q}_p$-linear category (which is not abelian) where "morphisms" are defined in the evident way. One can define in a natural way the dual D^* of an object of this category and the tensor product $D_1 \otimes D_2$ of two objects.

Recall ([**9**], [**12**]) that for such a D, one can define its Hodge polygon (associated to the Hodge-Tate type, i.e. to the number $h^r = \dim gr^r D_F$) and its Newton polygon (associated to the slope of Frobenius). One says that D is **weakly admissible** (or **w.a.** for short) if
i) for any sub-$(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module D' of$D$, with $D'_F \subset D_F$ equipped with the induced filtration, the Hodge polygon of D' lies above the Newton polygon of D';
ii) the Hodge polygon and the Newton polygon of D ends up at the same point.

The weakly admissible modules form an abelian category [**12**]. If D is an object of this category, sub-objects of D are sub-$(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules such that the Hodge polygon and Newton polygon of D end up at the same point.

Changing F : Call $\mathfrak{F}$ the set of finite Galois extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$. Given $F, F' \in \mathfrak{F}$ with $F \subset F'$, we say that a $(\varphi, N, \mathrm{Gal}(F'/\mathbb{Q}_p))$-module D' is **F-semi-stable** if the natural map $F'_0 \otimes_{F_0} (D')^{\mathrm{Gal}(F'/F)} \to D'$ is an isomorphism. To any filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module D, one can associate a filtered $(\varphi, N, \mathrm{Gal}(F'/\mathbb{Q}_p))$-module $D_{/F'}$ in an obvious way (the underlying F'_0-vector space is $F'_0 \otimes_{F_0} D$). We get in this way an equivalence of categories between filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules and the full subcategory of filtered $(\varphi, N, \mathrm{Gal}(F'/\mathbb{Q}_p))$-modules consisting of F-semi-stable ones. Thus we can form the inductive limit over $\mathfrak{F}$ of the category of filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-modules; we call an object of this $\mathbb{Q}_p$-linear category a **pst-module**.

Each time we have a pst-module Δ, we may choose an $F \in \mathfrak{F}$ such that this object is F-semi-stable and we can speak of its **F-realisation** which is a filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module D. On this D we have an F_0-linear action of the inertia subgroup of $\mathrm{Gal}(F/\mathbb{Q}_p)$ that we can view as a continuous action of I_p whose kernel contains $I_p \cap \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)$. If $F' \in \mathfrak{F}$, Δ is F'-semi-stable if and only if $I_p \cap \mathrm{Gal}(\overline{\mathbb{Q}}_p/F')$ acts trivially on D. To know Δ is the same as to know D; the choice of F doesn't matter and we will sometimes speak of the pst-module D.

If D is a filtered $(\varphi, N, \mathrm{Gal}(F/\mathbb{Q}_p))$-module and if $F \subset F', D_{F'}$ is weakly admissible if and only if D is. Then it makes sense to speak of a **weakly admissible pst-module**. These modules form an *abelian* $\mathbb{Q}_p$*-linear category.*

(b) **Representations** : If $F \in \mathfrak{F}$, we have a natural functor $\underline{D}_{\mathrm{st},F}$ from the category of p-adic representations of G_p to the category of filtered $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-modules:

$$V \mapsto D = \underline{D}_{\mathrm{st},F}(V) = (B_{\mathrm{st}} \otimes_{\mathbb{Q}_p} V)^{\ \mathrm{Gal}(\overline{\mathbb{Q}_p}/F)}.$$

We have $\dim_{F_0} \leq \dim_{\mathbb{Q}_p} V$ and V is said to be F**-semi-stable** if equality holds. The property to be F-semi-stable is stable under taking subobject, quotient, direct sum, tensor product, dual. Say a filtered $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-module is **admissible** if it is isomorphic to $\underline{D}_{\mathrm{st},F}(V)$ for an F-semi-stable V. *It is known that admissible implies weakly admissible and conjectured that the converse is true.* The restriction of $\underline{D}_{\mathrm{st},F}$ to F-semi-stable representations is fully faithful, hence induces an *equivalence of categories between p-adic F-semi-stable representations of* G_p *and* **admissible filtered** $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$**-modules**. There is a natural quasi-inverse $\underline{V}_{\mathrm{st}}$ of $\underline{D}_{\mathrm{st},F}$ ($\underline{V}_{\mathrm{st}}(D)$ is defined as a suitable sub-$\mathbb{Q}_p$-vector space of $B_{\mathrm{st}} \otimes_{F_0} D$). This is an equivalence of tannakian categories, i.e. $\underline{D}_{\mathrm{st},F}$ is compatible in a natural way with duality and tensor product.

If D is admissible and if D' is a sub-$(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-module of D, then D' is admissible if and only if it is weakly admissible.

It is sometimes convenient to use a contravariant version of this equivalence. If V is F-semi-stable, one can define $\underline{D}^*_{\mathrm{st},F}(V)$ in three different ways (there are canonical isomorphisms between them [**12**])

$$\underline{D}^*_{\mathrm{st},F}(V) = Hom_{\mathbb{Q}_p[\ \mathrm{Gal}(\overline{\mathbb{Q}}_p/F)]}(V, B_{\mathrm{st}}) = \underline{D}_{\mathrm{st},F}(V^*) = (\underline{D}_{\mathrm{st},F}(V))^*$$

Let V be F-semi-stable and $D = \underline{D}_{\mathrm{st},F}(V)$. Then V is *de Rham* and the filtered $\mathbb{Q}_p$-vector space $\underline{D}_{dR}(V) = (B_{dR} \otimes_{\mathbb{Q}_p} V)^{G_p}$ can be identified with $D_{dR} = (D_F)^{\ \mathrm{Gal}(F/\mathbb{Q}_p)}$. Hence V is also Hodge-Tate and the Hodge-Tate multiplicities $h^r(V) = \ \dim_{\mathbb{Q}_p} \mathrm{Hom}_{\mathbb{Q}_p[G_p]}(V, \mathbb{C}_p(r))$ satisfy

$$h^r(V) = \ \dim_F \ \mathrm{Fil}^{-r} D_F / \ \mathrm{Fil}^{-r+i} D_F = \ \dim_F \ \mathrm{Fil}^r D^*_F / \ \mathrm{Fil}^{r+1} D^*_F.$$

Tate twists : For any filtered $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-module D and for $i \in \mathbb{Z}$, we denote by $D\{i\}$ the filtered $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-module which is D as a F_0-vector space with the same action of N and of $\mathrm{Gal}(F/\mathbb{Q}_p)$, and

$$\varphi_{\text{ new}} = p^{-i} \varphi_{\text{ old}} \text{ and } \ \mathrm{Fil}^r D_{F, \text{ new}} = \ \mathrm{Fil}^{r+1} D_{F, \text{ old}}.$$

If V is any p-adic representation of G_p and if $i \in \mathbb{Z}$, the usual Tate twist $V(i)$ is F-semi-stable if and only if V is. In this case, $\underline{D}_{\mathrm{st},F}(V(i))$ can be identified with $\underline{D}_{\mathrm{st},F}(V)\{i\}$ and $\underline{D}^*_{\mathrm{st},F}(V(i))$ with $\underline{D}^*_{\mathrm{st},F}(V)\{-i\}$.

Changing F : A p-adic representation V of G_p is said to be **potentially semi-stable** if it is F-semi-stable for some $F \in \mathcal{F}$. If V is F-semi-stable, with D $= \underline{D}_{\mathrm{st},F}(V)$, and if $F' \in \mathcal{F}$ contains F, V is also F'-semi-stable and

$\underline{D}_{\mathrm{st},F'}(V)$ can be identified with $D_{/F'}$. Hence we have an evident notion of **admissible** pst-**modules** and we get an equivalence between potentially semi-stable representations and the full subcategory of weakly admissible pst-modules whose objects are admissible ones (and conjecturally they are all).

Remark: A p-adic representation V of G_p is **semi-stable** if it is $\mathbb{Q}_p$-semi-stable, F-**crystalline** if it is F-semi-stable and $N = 0$ on $\underline{D}_{\mathrm{st},F}(V)$, **potentially crystalline** if it is F-**crystalline** for some F, **crystalline** if it is $\mathbb{Q}_p$-crystalline.

The one dimensional case : Consider a one dimensional pst-module Δ, choose $F \in \mathcal{F}$ such that Δ is F-semi-stable and let D be its F-realisation. This is a one dimensional F_0-vector space on which I_p acts linarly via a character of finite order with values in $\mathcal{O}_{F_0}$. The fact that this action commutes with φ implies that it takes values in $\mathbb{Z}_p$, hence its order divides $p-1$. This means that Δ is F_1-semi-stable (recall $F_1 = \mathbb{Q}_p(\sqrt[p]{1})$), hence we can choose $F = F_1$ and D is a one dimensional $\mathbb{Q}_p$-vector space; it is an easy exercise to check that i) there is a unique $(r, a, i) \in \mathbb{Z} \times \mathbb{Q}_p^* \times (\mathbb{Z}/p-1)\mathbb{Z}$ such that $D \simeq D(r; a; i)$, where $D(r; a; i)$ is defined as follows. Its underlying $\mathbb{Q}_p$-vector space we take to be $\mathbb{Q}_p$, and hence $D_{F_1} = F_1$. Put

$$\varphi 1 = a, N1 = 0, g1 = \hat{\xi}_1(g)^i \text{ if } g \in \ \mathrm{Gal}(F_1/\mathbb{Q}_p),\ \mathrm{Fil}^r F_1 = F_1,\ \mathrm{Fil}^{r+1} F_1 = 0.$$

ii) the module $D(r; a; i)$ is weakly admissible if and only if $v_p(a) = r$; in this case it is also admissible and G acts on $\underline{V}_{\mathrm{st}}(D(r; a; i))$ via the character $\chi^{-r} \cdot \eta_{(a/p^r)^{-1}} \cdot \hat{\xi}_1^i$.

All the one dimensional potentially semi-stable p-adic representations of G_p are F_1-crystalline.

§11. The irreducible 2-dimensional weakly admissible pst-modules.

If $F \in \mathcal{F}$ and D is a 2-dimensional filtered $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-module, D_F is a 2-dimensional filtered F-vector space and there are well defined integers r, s satisfying $r \leq s$ such that, if $i \in \mathbb{Z}$, $\mathrm{Fil}^i D_F = D_F \Leftrightarrow i \leq r$ and $\mathrm{Fil}^i D_F = 0 \Leftrightarrow i > s$. We call (r, s) the Hodge-Tate type of D.

We fix (r, s) and we want to describe some 2-dimensional filtered $(\varphi, N,\ \mathrm{Gal}(F/\mathbb{Q}_p))$-modules, D of Hodge-Tate type (r, s). If $r = s$, there is no choice for the filtration. If $r < s$, the filtration of D_F is determined by the knowledge of the F-line $\mathrm{Fil}^{r+1} D_F = \mathrm{Fil}^s D_F$ and we will denote it by Δ. The underlying F_0-vector space of each D will be taken to be $(F_0)^2$ and we denote $\{e_1, e_2\}$ the canonical basis.

(a) Let's first choose $F = F_1$, hence $F_0 = \mathbb{Q}_p, \Gamma_1 = \ \mathrm{Gal}(F/\mathbb{Q}_p)$ and the characters of Γ_1 with values in $\mathbb{Z}_p$ are the $\hat{\xi}_1^i$ for $0 \leq i < p-1$.

Type I : Choose $(a, d, i) \in \mathbb{Q}_p \times \mathbb{Q}_p^* \times \mathbb{Z}$ with $0 \leq i < p-1$; define $D = D_I(r, s; a, d, i)$ by $\varphi e_1 = e_2, \varphi e_2 = -de_1 + ae_2, Ne_1 = e_2 = 0, gx = \hat{\xi}_1^i(g).x \quad (x \in D, g \in \Gamma_1)$, and if $r < s, \quad \Delta = F_1.1 \otimes e_1$.

Type II : Choose $(b, c, i) \in \mathbb{Q}_p^* \times \mathbb{Q}_p \times \mathbb{Z}$ with $0 \leq i < p-1$; define $D = D_{II}(r, s; b, c; i)$ by $\varphi e_1 = pbe_1, \varphi e_2 = be_2, Ne_1 = e_1, Ne_2 = 0, gx = \hat{\xi}_1^i(g).x \quad (x \in D, g \in \Gamma_1)$, and if $r < s, \Delta = F_1.(1 \otimes e_1 + c \otimes e_2)$.

Type III ; Choose $(a_1, a_2, i_1, i_2) \in (\mathbb{Q}_p^*)^2 \times \mathbb{Z}^2$, with $0 \leq i_1 < i_2 \leq p-1$; define $D = D_{III}(r, s; a_1, a_2; i_1, i_2)$ by $\varphi e_1 = ae_1, \varphi e_2 = a_2 e_2, Ne_1 = Ne_2 = 0, ge_1 = \hat{\xi}_1^{i_1}.e_1, \quad ge_2 = \hat{\xi}_1^{i_2}.e_2 \quad (g \in \Gamma_1)$, and, if $r < s, \Delta = F_1(\pi_1^{i_2} \otimes e_1 + \pi_1^{i_1} \otimes e_1)$.

(b) We now choose $F = F_2$ with, as in §9, $F_2 = \mathbb{Q}_{p^2}(\pi_2)$ where we have chosen $\pi_2 \in \overline{\mathbb{Q}}_p$ such that $\pi_2^{p^2-1} = -p$. Then $F_0 = \mathbb{Q}_{p^2}$. Let $\Gamma_2 =$ Gal$(F_2/\mathbb{Q}_p)$ and $I\Gamma_2$ the inertia subgroup. We denote by $\hat{\xi}_2 : I\Gamma_2 \to \mu_{p^2-1}(\mathbb{Q}_{p^2})$ the isomorphism defined by $\hat{\xi}_2(g) = g\pi_2/\pi_2$. The group Γ_2 is the semi-direct product of the invariant subgroup $I\Gamma_2$ by the subgroup of order 2 generated by the unique nontrivial element $\overline{\tau}$ of Γ_2 such that $\overline{\tau}\pi_2 = \pi_2$; moreover, if $g \in I\Gamma_2, \overline{\tau}g\overline{\tau} = g^p$.
Type IV ; Choose $d \in \mathbb{Q}_p^*$, integers i_1, i_2 satisfying $0 \leq i_1 < i_2 \leq p-1$. If $s > r$, choose also $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$. If $s = r$ (resp. $s > r$) define $D = D_{IV}(r,r;d;i_1,i_2)$(resp. $D_{IV}(r,s;d;i_1,i_2;\alpha)$) by $\varphi e_1 = e_2, \varphi e_2 = -de_1, Ne_1 = Ne_2 = 0, \overline{\tau}e_1 = e_1, \overline{\tau}e_2 = e_2, ge_1 = \hat{\xi}_2^{i_1+pi_2}(g)e_1$ and $ge_2 = \hat{\xi}_2^{i_2+pi_1}(g)e_2$ for $g \in I\Gamma_2$ and , if $r < s, \Delta = F_2.(\pi_2^{(p-1)i_1} \otimes e_1 + \alpha.\pi_2^{(p-1)i_2} \otimes e_2)$ if $\alpha \neq \infty$ and $\Delta = F_2.1 \otimes e_2$ if $\alpha = \infty$.

Theorem A : i) *The following pst-modules are weakly admissible and irreducible:*
1) $D_I(r,r;a,d,i)$ *if* $v_p(a) \geq r, v_p(d) = 2r$ *and* $X^2 - aX + d$ *irreducible over* $\mathbb{Q}_p$*,*
1') $D_I(r,s;a,d,i)$ *for* $r < s$*, if* $v_p(a) > r$ *and* $v_p(d) = r+s$*,*
2) $D_{II}(r,s;b,c;i)$ *for* $s - r \geq 3$ *and odd, if* $v_p(b) = (s-r-1)/2$*,*
3) $D_{III}(r,s;a_1,a_2;i_1,i_2)$ *for* $s - r \geq 2$*, if* $v_p(a_1) > r, v_p(a_2) > r, v_p(a_1a_2) = r+s$*,*
4) $D_{IV}(r,r;d;i_1,i_2)$*(resp.* $D_{IV}(r,s;d;i_1,i_2;\alpha)$ *if* $v_p(d) = 2r$*(resp.* $r+s$*).*
ii) The above objects are all absolutely irreducible exept $D_I(r,r;a,d,i)$ *for which the ring of endomorphisms is isomorphic to* $\mathbb{Q}_p[X]/(X^2 - aX + d)$*.*
iii) Any irreducible 2-dimensional w.a. pst-module, which is F_2*-semi-stable, is isomorphic to one and only one object of the lists 1)-4).*
iv) If $p \geq 5$*, any irreducible 2-dimensional w.a. pst-module is* F_2*-semi-stable.*

Proof. See App., §A. □

This theorem reduces the problem of finding the complete list of isomorphism classes of irreducible two dimensional p-adic potentially semi-stable representations of G_p to finding out which are the pst-modules in the above list which are admissible (and conjecturally they all are). Each time that we discover that one of those D's is admissible we will denote the corresponding representation $\underline{V}_{\mathrm{st}}(D)$ (resp. the dual representation $\underline{V}_{\mathrm{st}}^*(D) = \underline{V}_{\mathrm{st}}(D)^*$) using the same notation but replacing D with V (resp. V^*) (e.g. $V_{IV}(r,s;d;i_1,i_2;\alpha) = \underline{V}_{\mathrm{st}}(D_{IV}(r,s;d;i_1,i_2;\alpha))$ and $V_{IV}^*(r,s;d;i_1,i_2;\alpha)$ is the dual of this representation).

Given a two dimensional p-adic representation V of G_p and two integers $r \leq s$, we say that V **is of Hodge-Tate type** (r,s) if V is Hodge-Tate and $\mathrm{Hom}_{\mathbb{Q}_p[G_p]}(V, \mathbb{C}_p(i)) \neq 0$ if and only if $i \in \{r,s\}$. We see that if $V = \underline{V}_{\mathrm{st}}^*(D)$ for some admissible pst-module D, than the Hodge-Tate type of V is the same as the Hodge-Tate type of D. Using this fact and checking the list of D's occurring in the previous theorem, we get the following result:

Proposition 1. *Let* V *be a two dimensional irreducible p-adic representation of* G_p *which is potentially semi-stable of Hodge-Tate type* (r,s)*. Then, we* *are in one and only one of the following cases:*

1) *There is a character of finite order* ν *such that* $V(\nu)$ *is crystalline, in which case there is a unique integer* i *satisfying* $0 \leq i < p-1$ *such that one can choose* $\nu = \hat{\xi}_1^i$ *and a unique* $(a,d) \in \mathbb{Q}_p \times \mathbb{Q}_p^*$ *such that* $D_I(r,s;a,d;i)$ *is admissible and* $V \simeq V_I^*(r,s;a,d;i)$*;*
2) *The representation is not potentially crystalline, in which case there is a unique integer* i *satisfying* $0 \leq i < p-1$ *such that* $V(\hat{\xi}_1^i)$ *is semi-stable and a unique* $(b,c) \in \mathbb{Q}_p^* \times \mathbb{Q}_p$ *such that* $D_{II}(r,s;b,c;i)$ *is admissible and* $V \simeq V_{II}^*(r,s;b,c;i)$*;*
3) *The representation is* F_1*-crystalline, but there is no character* ν *such that* $V(\nu)$ *is crystalline, in which case there is a unique couple* (i_1,i_2) *of integers satisfying* $0 \leq i_1 < i_2 \leq p-1$ *and a unique* $(a_1,a_2) \in (\mathbb{Q}_p^*)^2$ *such that* $D_{III}(r,s;a_1,a_2;i_1,i_2)$ *is admissible and* $V \simeq V_{III}^*(r,s;a_1,a_2;i_1,i_2)$*;*
4) *The representation is not* F_1*-semi-stable, in which case it is* F_2*-crystalline, there is a unique couple* (i_1,i_2) *of integers satisfying* $0 \leq i_1 < i_2 \leq p-1$*, a unique* $d \in \mathbb{Q}_p^*$ *and , if* $r < s$ *a unique* $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$*, such that, if* $r = s$ *(resp.* $r < s$*),* $D_{IV}(r,r;d;i_1,i_2)$ *(resp.* $D_{IV}(r,s;d;i_1,i_2;\alpha)$*) is admissible and* $V \simeq V_{IV}^*(r,r;d;i_1,i_2)$ *(resp.* $V \simeq V_{IV}^*(r,s;d;i_1,i_2;\alpha)$*).*

From the fact that we have an equivalence of tannakian categories, one sees that, if D is admissible, then $\wedge^2 D$ is also and that $\underline{V}_{\mathrm{st}}(\wedge^2 D) \simeq \wedge^2 \underline{V}_{\mathrm{st}}(D)$. An easy computation gives us the following result:

Proposition 2. *Let* D *be one of the w.a. pst-modules listed in the previous theorem. Assume it is admissible and let* $V = \underline{V}_{st}^*(D)$*. Then* G_p *acts on* $\det V$*, via* $\eta_u \chi^{r+s} \hat{\xi}_1^{-j}$*,*

1) *with* $u = d/p^{r+s}$ *and* $j = 2i$ *if* $V = V_I^*(r,s;a,d,i)$*,*
2) *with* $u = b^2/p^{r+s-1}$ *and* $j = 2i$ *if* $V = V_{II}^*(r,s;b,c;i)$*,*
3) *with* $u = a_1 a_2/p^{r+s}$ *and* $j = i_1 + i_2$*, if* $V = V_{III}^*(r,s;a_1,a_2;i_1,i_2)$*,*
4) *with* $u = d/p^{r+s}$ *and* $j = i_1 + i_2$*, if* $V = V_{IV}^*(r,s;d;i_1,i_2;\alpha)$ *or* $V_{IV}^*(r,s;d;i_1,i_2)$ *with* $s = r$*.*

§12. Ramakrishna's theorem : crystalline representations of dimension 2 and their deformations.

We say that a p-adic representation V of G_p **is absolutely irreducible mod** p if there exists a $\mathbb{Z}_p$-lattice T of V stable under G_p such that the G_p-module T/pT is absolutely irreducible; we put $\overline{V} = T/pT$ and call it the **reduction mod p** of V. This is well defined, because the other lattices stable under G_p are the $p^n T$, for $n \in \mathbb{Z}$. Of course, any p-adic representation which is absolutely irreducible mod p is absolutely irreducible, but the converse is not true.

Theorem B1. *The* D*'s of type IV which* $s - r \leq p-1$ *listed in theorem A are admissible. Moreover*

i) *If* d *is a p-adic unit, if* $X^2 - aX + d$ *is irreducible over* $\mathbb{Q}_p$ *and if* λ *is a root of this polynomial,* $V_I^*(r,r;a,d;i)$ *is isomorphic to a one dimensional* $\mathbb{Q}_p(\lambda)$*-vector space on which* G_p *acts through the character* $\eta_\lambda . \chi^r . \hat{\xi}_1^{-i}$*;*

ii) *If* $1 \le s-r \le p-1$, $V_I^*(r,s;a,d;i)$ *is absolutely irreducible mod* p *and its reduction mod* p *is isomorphic to* $\overline{V}_{\iota,\varepsilon}$ *where* ι *is the image of* $rp+s-(p+1)i$ *in* $\mathbb{Z}/(p^2-1)\mathbb{Z}$ *and* ε *the image of* d/p^{r+s} *in* $\mathbb{F}_p^*$.

Proof. See App., §B. □

Let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)$ be the category of (finite dimensional) p-adic representations of G_p and $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)$ the category of $\mathbb{Z}_p$-modules of finite length equipped with a linear and continuous action of G_p. If $a \le b$ are integers, we denote by $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{cr,[a,b]}$ the full subcategory of $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)$ consisting of those V which are crystalline and such that $h^r(V)(=\dim_{\mathbb{Q}_p} gr^r \underline{D}^*_{HT}(V)) = 0$ if $r \notin [a,b]$. Denote also by $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{cr,[a,b]}$ the full subcategory of $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)$ whose objects are T's for which one can find an object V of $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{cr,[a,b]}$ such that T is isomorphic to a subquotient of V. This is stable under subobjects, quotients, direct sums. Hence, for any (ι,ε) as in §9 such that $\overline{V}_{\iota,\varepsilon}$ is absolutely irreducible one can speak of the ring $R_{\iota,\varepsilon}(cr,[a,b])$ of the universal $\mathbb{Z}_p$-deformation of $\overline{V}_{\iota,\varepsilon}$ lying in $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{cr,[a,b]}$ as soon as $\overline{V}_{\iota,\varepsilon}$ itself is an object of this category.

Theorem B2. *Let* $\overline{V} = \overline{V}_{\iota,\varepsilon}$ *be absolutely irreducible.*

i) *The representation* $\overline{V}$ *lies in* $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{cr,[0,p-1]}$;

ii) *if* r,s *are the unique integers such that* $0 \le r < s \le p-1$ *and the image of* $r+ps$ *mod* p^2-1 *is equal to* ι *or* $p\iota$, *there is a unique isomorphism* $R_{\iota,\varepsilon}(cr,[0,p-1]) \simeq \mathbb{Z}_p[[Y_1,Y_2]]$ *such that, for any morphism* $Y_1 \mapsto y_1, Y_2 \mapsto y_2$ *of* $R_{\iota,\varepsilon}(cr,[0,p-1])$ *to* $\mathbb{Z}_p$, *if* T *is representative of the corresponding isomorphism class of* $\mathbb{Z}_p$*-deformation, then*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \simeq V_I^*(r,s;p^r y_2, p^{r+s}[\varepsilon](1+y_1),0).$$

Proof. See App., §B. This theorem is a slight refinement of a result of Ramakrishna [**23**] who proved the fact that $R_{\iota,\varepsilon}(cr,[0,p-1]) \simeq \mathbb{Z}_p[[Y_1,Y_2]]$. □

Remark If $0 < s-r \le p-1, u \in \mathbb{Z}$ and if $V_I^*(r,s;a,d;i)$ is as in theorem B1, then $V_I^*(r+u,s+u;a,d;i) \simeq V_I^*(r,s;p^u a,p^{2u}d;0)(\chi^u \hat{\xi}_1^{-i})$. If $\overline{u}$ denotes the remainder of euclidean division of u by $p-1$, the character $\chi^u \hat{\xi}_1^{-\overline{u}} = \chi_0^u$ takes values in $1+p\mathbb{Z}_p$. For each $u \in \mathbb{Z}$, twisting the representation on $\mathbb{Z}_p[[Y_1,Y_2]]$ which is the "universal deformation of $\overline{V}_{\iota,\varepsilon}$ for which all finite quotients lie in $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{cr,[0,p-1]}$" by χ_0^u, we get the "universal deformation of $\overline{V}_{\iota,\varepsilon}$ for which all finite quotients T are such that $T(\hat{\xi}_1^{\overline{u}})$ lie in $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{cr,[u,u+p-1]}$". Any $V = \underline{V}^*_{\mathrm{st}}(D)$ for D absolutely irreducible of type I and Hodge-Tate type (r',s') with $s'-r' \le p-1$ such that $\overline{V} \simeq \overline{V}_{\iota,\varepsilon}$ can be obtained by a suitable specialization of one of those universal deformations.

§13. Potentially Barsotti-Tate representations of dimension 2 and their deformations.

We will only give a sketch of the proof of the main results of this section (see Appendix, § C). Details will be published elsewhere.

Consider (ι, ε) where $\iota \in \mathbb{Z}/(p^2-1)\mathbb{Z}$ and $\varepsilon \in \mathbb{F}_p^*$. We denote by $V_{\iota,\varepsilon}$ the p-adic representation of G_p which is the "canonical lifting" of $\overline{V}_{\iota,\varepsilon}$: this is $\mathbb{Q}_{p^2}$ on which G_p acts through the homomorphism

$$\rho_{\iota,\varepsilon} : \operatorname{Gal}(L/\mathbb{Q}_p) \to \operatorname{Aut}_{\mathbb{Q}_p}(\mathbb{Q}_{p^2})$$

such that the restriction to $\operatorname{Gal}(L/L_0)$ consists in $\mathbb{Q}_{p^2}$-linear automorphisms given by the one-dimensional $\mathbb{Q}_{p^2}$-character $\hat{\xi}_2^{\iota}$, and τ acts on $\mathbb{Q}_{p^2}$ via the $\mathbb{Q}_{p^2}$-semilinear automorphism $x \mapsto [\zeta] \cdot \sigma x$ (where, as in §9, $\zeta^{p+1} = -\varepsilon$).

Let $\mathbb{Q}_{p^2}^{ab}$ be the maximal abelian extension of $\mathbb{Q}_{p^2}$ contained in $\overline{\mathbb{Q}}_p$ and $\theta : \mathbb{Q}_{p^2}^* \to \operatorname{Gal}(\mathbb{Q}_{p^2}^{ab}/\mathbb{Q}_{p^2})$ the inverse of the local reciprocity map[7]. The restriction θ_0 of θ to the group of units is an isomorphism onto the inertia subgroup $\operatorname{In}(\mathbb{Q}_{p^2}^{ab}/\mathbb{Q}_{p^2})$ of $\operatorname{Gal}(\mathbb{Q}_{p^2}^{ab}/\mathbb{Q}_{p^2})$; as the natural map $I_p \to \operatorname{In}(\mathbb{Q}_{p^2}^{ab}/\mathbb{Q}_{p^2})$ is onto, the inverse of θ_0 can be viewed as a character

$$\nu : I_p \to \mathbb{Q}_{p^2}^* .$$

Given $d \in \mathbb{Q}_p$ with $v_p(d) = 1$ and $\iota \in \mathbb{Z}/(p^2-1)\mathbb{Z}$, one sees easily that, up to conjugacy, there is one and only one continuous homomorphism

$$\rho_{LT,d,\iota} : G_p \to \operatorname{Aut}_{\mathbb{Q}_p}(\mathbb{Q}_{p^2})$$

("*LT*" for "Lubin-Tate") such that the restriction of $\rho_{LT,d,\iota}$ to I_p is the character $\nu \cdot \hat{\xi}_2^{\iota}$ and the deteminant of $\rho_{LT,d,\iota}$ is the character $\eta_{d/p} \cdot \chi \cdot \hat{\xi}_1^{\iota}$. We denote by $V_{LT,d,\iota}$ a chosen representative of the corresponding isomorphism class of p-adic representations. One sees that, if $(d', \iota',) \neq (d, \iota)$, then $V_{LT,d',\iota'} \not\simeq V_{LT,d,\iota}$ and that $V_{LT,d,\iota}$ is absolutely irreducible mod p if and only if $p+1$ doesn't divide $\iota + 1$ in which case $\overline{V}_{LT,d,\iota} \simeq \overline{V}_{\iota+1,\varepsilon}$ with ε the image of d/p in $\mathbb{F}_p^*$. also $V_{LT,d,0} \simeq V_I^*(0,1;0,d;0) \simeq \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(J_d)$ where J_d is a p-divisible group over $\mathbb{Z}_p$ which viewed as a p-divisible over the ring $\mathbb{Z}_{p^2}$ of the integers of $\mathbb{Q}_{p^2}$ is a Lubin-Tate formal group for $\mathbb{Q}_{p^2}$.

Observe that $D_{IV}(r,r;d;i_1,i_2) \simeq D_{IV}(0,0;p^{-2r}d;i_1,i_2)\{-r\}$, hence one of those modules is admissible if and only if the other is, in which case $V_{IV}^*(r,r;d;i_1,i_2) \simeq V_{IV}^*(0,0;p^{-2r}d;i_1,i_2)(r)$. When $r < s$, we have a similar statement for $D_{IV}(r,s;d;i_1,i_2;\alpha) \simeq D_{IV}(0,s-r;p^{-2r}d;i_1,i_2;p^r\alpha)\{-r\}$ with, in case of admissiblity, $V_{IV}^*(r,s;d;i_1,i_2;\alpha) \simeq V_{IV}^*(0,s-r;p^{-2r}d;i_1,i_2;p^r\alpha)(r)$. In particular, the first representation is absolutely irreducible mod p if and only if the second is.

Theorem C1. *The D's of type 4 with $s - r \leq 1$ listed in theorem A are admissible. Moreover, if $0 \leq i_1 < i_2 \leq p-1$ and if $\bar{i}$ denotes the image of $i_1 + pi_2$ in $\mathbb{Z}/(p^2-1)\mathbb{Z}$, then*

i) if $d = [\varepsilon]u^2$ with $\varepsilon \in \mathbb{F}_p^*$ and $u \in 1 + p\mathbb{Z}_p$, $V_{IV}^*(0,0;d;i_1,i_2) \simeq V_{-\bar{i},\varepsilon}(\eta_u)$ (hence this representation is absolutely irreducible mod p and its reduction mod p is isomorphic to $\overline{V}_{-\bar{i},\varepsilon}$),

[7] hence the image of $\theta(p)$ in $\operatorname{Gal}(\overline{\mathbb{F}}_{p^2}/\mathbb{F}_{p^2})$ is the arithmetic Frobenius

ii) *if* $d = p[\varepsilon]u^2$ *with* $\varepsilon \in \mathbb{F}_p^*, u \in 1 + p\mathbb{Z}_p$ *and if* $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$, *then* $V = V_{IV}^*(0,1;d;i_1,i_2;\alpha)$ *is absolutely irreducible mod p if and only if - either* $v_p(\alpha) \geq 0$, *in which case* $\overline{V} \simeq \overline{V}_{1-\bar{i},\varepsilon}$ *- or* $v_p(\alpha) \leq -2$ *and* $i_2 - i_1 > 1$, *in which case* $\overline{V} \simeq \overline{V}_{1-p\bar{i},\varepsilon}$;

iii) *if* $v_p(d) = 1, V_{IV}^*(0,1;d;i_1,i_2;0) \simeq V_{LT,d,1-\bar{i}}$ *and* $V_{IV}^*(0,1;d;i_1,i_2;\infty) \simeq V_{LT,d,1-p\bar{i}}$.

We observed that all w.a. pst modules listed in theorem A with $s - r \leq 1$ are admissible because they are either of type I or of type IV. Say that a p-adic representation V of G_p is **potentially Barsotti-Tate** if one can find a finite extension E of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and a Barsotti-Tate group J defined over the ring of integers of E such that $V \simeq V_p(J) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} T_p(J)$ as a $\mathbb{Q}_p[\,\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)]$-module, where $T_p(J)$ is the Tate module of J. If this is the case, one knows that V is potentially crystalline with Hodge-Tate weights $\in \{0,1\}$. One conjectures the converse is true and this is clearly the case whenever 0 is the only weight (this means that the image of inertia is finite, hence that J is étale) or whenever 1 is the only weight (this means that the image of inertia on $V(-1)$ is finite, hence that J is of multiplicative type, i.e. is the Cartier dual of an étale Barsotti-Tate group).

Assume V is irreducible of dimension 2. Then the Hodge-Tate type can be $(0,0),(1,1)$ or $(0,1)$. If it is $(0,0)$ this means that $V \simeq \underline{V}_{\mathrm{st}}^*(D)$ with D equal to one of the w.a. modules $D_I(0,0;a,d,i)$ or $D_{IV}(0,0;d;i_1,i_2)$ listed in theorem A. If it is $(1,1)$ this means that $V(-1)$ is of type $(0,0)$. The problem for type $(0,1)$ is solved by the next theorem, for which we need two more definitions:

Definitions: Let J_0 be a Barsotti-Tate group over $\mathbb{F}_p$. Then J_0 is equipped with two endomorphisms, the Frobenius $\varphi = F$ and the Verschiebung V satisfying $\varphi V = V\varphi = p$. We say that J_0 **is strictly of slope** $1/2$ if there is an automorphism u of J_0 (necessarily unique) such that $\varphi = Vu$.

Let $\mathcal{O}_E$ be the ring of the integers of a finite, totally ramified extension E of $\mathbb{Q}_p$. We say that a Barsotti-Tate group J over $\mathcal{O}_E$ **is strictly of slope** $1/2$ if its special fiber is strictly of slope $1/2$ and if moreover, given any invariant differential form ω on J, one can find differential forms ω_1, ω_2 with ω_1 exact and $\lambda \in \mathcal{O}_E$ with $v_p(\lambda) \geq 1/2$ such that $\omega = \omega_1 + \lambda\omega_2$.

Theorem C2. *Assume* $p \geq 5$. *Let* V *be an irreducible two dimensional p-adic representation of* G_p *which is Hodge-Tate of Hodge-Tate type* $(0,1)$. *Then the following are equivalent:*

i) V *is potentially Barsotti-Tate,*

ii) V *is potentially crystalline,*

iii) V *is potentially semi-stable,*

iv) *There is a w.a. pst module* D *which is either one of the modules* $D_I(0,1;a,d,i)$ *or one of the modules* $D_{IV}(0,1;d,i_1,i_2;\alpha)$ *listed in Theorem A such that* $V \simeq \underline{V}_{st}^*(D)$.

Moreover,

a) *if* $V \simeq V_I^*(0,1;a,d,i)$, *there is a Barsotti-Tate group* J *defined over* $\mathbb{Z}_p$, *strictly of slope 1/2, such that $V \simeq V_p(J)(\hat{\xi}_1^{-i})$ when viewed as a G_p-module;*

b) *Let $\pi = (\pi_2)^{p-1}$ (hence $\pi^{p+1} = -p$) and $\hat{\xi}_\pi$: $Gal(\mathbb{Q}_p(\pi_2)/\mathbb{Q}_p(\pi)) \to \mathbb{Z}_p^*$ the character defined by $\hat{\xi}_\pi(g) = g\pi_2/\pi_2$. If $V \simeq V_{IV}^*(0,1;d;i_1,i_2;\alpha)$ there is a Barsotti-Tate group J defined over the integers of $\mathbb{Q}_p(\pi)$, strictly of slope 1/2, such that $V \simeq V_p(J)(\hat{\xi}_\pi^{-i_1-i_2})$ when viewed as a $Gal(\overline{\mathbb{Q}}_p/\mathbb{Q}_p(\pi))$-module.*

From now on, $\mathcal{O}_E = \mathbb{Z}_p[\pi]$ and $E = \mathbb{Q}_p(\pi)$. We denote by $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{BT,E,1/2}$ the full subcategory of $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)$ consisting of those T's for which we can find a Barsotti-Tate group J over $\mathcal{O}_E$ strictly of slope 1/2 such that T, when viewed as a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/E)$-module, is isomorphic to a quotient of $T_p(J)$). This category is stable under passage to subobjects, quotients, direct sums. Hence for any (ι,ε) as in §9 such that $\overline{V}_{\iota,\varepsilon}$ is absolutely irreducible, one can speak of the ring $R_{\iota,\varepsilon}(BT,E,1/2)$ of the universal $\mathbb{Z}_p$-deformation of $\overline{V}_{\iota,\varepsilon}$ lying in $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{BT,E,1/2}$ as soon as $\overline{V}_{\iota,\varepsilon}$ itself is an object of this category.

Theorem C3. *Let j_1, j_2 be integers satisfying $0 \leq j_1 < j_2 \leq p-1, \iota$ the image of $j_1 + pj_2$ in $\mathbb{Z}/(p^2-1)\mathbb{Z}$, $\varepsilon \in \mathbb{F}_p^*$ and $\overline{V} = \overline{V}_{\iota,\varepsilon}$. Then $\overline{V}$ belongs to $\underline{Rep}^f_{\mathbb{Z}_p}(G_p)_{BT,E,1/2}$. One can build a $\mathbb{Z}_p[\![Y_1,Y_2]\!] \times_{\mathbb{F}_p} \mathbb{Z}_p[\![Y_1',Y_2']\!]$-deformation $T_{\iota,\varepsilon}(BT,E,1/2)$ of $\overline{V}$ in such a way that all finite quotients lie in $\underline{Rep}^f_{\mathbb{Z}_p}(G_p)_{BT,E,1/2}$ and that*

i) *if $y_1, y_2 \in p\mathbb{Z}_p$ and if T is the $\mathbb{Z}_p$-deformation of $\overline{V}$ obtained from $T_{\iota,\varepsilon}(BT,E,1/2)$ via the map $Y_1 \mapsto y_1, Y_2 \mapsto y_2, Y_1' \to 0, Y_2' \to 0$, then*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \simeq V_I^*(0,1;p[\varepsilon](1+y_1);j_2) \text{ if } j_2 - j_1 = 1 \quad {}^{8}$$

$$(\text{ resp. } V_{IV}^*(0,1;p[\varepsilon](1+y_1);j_1+1,j_2,p^{-1}y_2) \text{ if } j_2 - j_1 > 1);$$

ii) *if $y_1', y_2' \in p\mathbb{Z}_p$ and if T is the $\mathbb{Z}_p$-deformation of $\overline{V}$ obtained from $T_{\iota,\varepsilon}(BT,E,1/2)$ via the map $Y_1 \mapsto 0, Y_2 \mapsto 0, Y_1' \mapsto y_1', Y_2' \mapsto y_2'$, then*

$$\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T \simeq V_{IV}^*(0,1;p[\varepsilon](1+y_1');0,j_1+1,p^{-1}y_2') \text{ if } j_2 = p-1$$

$$(\text{ resp. } V_{IV}^*(0,1;p[\varepsilon](1+y_1');j_1,j_2+1,1/y_2') \text{ if } j_2 < p-1).$$

Remarks :

1) We observe that each isomorphism class of p-adic representation V as in Theorem C2 which is absolutely irreducible mod p is isomorphic to $\mathbb{Q}_p \otimes_{\mathbb{Z}_p} T$ for one and only one T obtained via the construction referred to in Theorem C3.
2) Assume ι and ε are as in Theorem C3. From this theorem, we get a natural homomorphism

$$R_{\iota,\varepsilon}(BT,E,1/2) \to \mathbb{Z}_p[\![Y_1,Y_2]\!] \times_{\mathbb{F}_p} \mathbb{Z}_p[\![Y_1',Y_2']\!]$$

It is likely that this map is onto with kernel killed by p.

[8] with the convention that $V_I^*(r,s;a,d;p-1) = V_I^*(r,s;a,d;0)$.

3) With the above conventions, we see that the determinant of the considered representation is $\eta_{[\varepsilon](1+y_1)}\cdot\chi\cdot\hat{\xi}_1^{-i}$ in case (i) and $\eta_{[\varepsilon](1+y'_1)}\cdot\chi\cdot\hat{\xi}_1^{-i}$ in case (ii). Thus, if we consider $\mathbb{Z}_p$-deformations of $\overline{V}_{\iota,\varepsilon}$ with a given determinant, we have one less parameter.
4) The families of type I and of type IV are of different nature : if y_1, y_2 or y'_1, y'_2 are as above, we can look at the characteristic polynomial of Frobenius acting on the Dieudonné module of the Barsotti-Tate group J defined over $\mathcal{O}_E$: if $j_2 - j_1 = 1$, for $V_I^*(0,1;y_2,p[\varepsilon](1+y_1);j_2)$, this is $X^2 - y_2X + p[\varepsilon](1+y_1)$, hence if we fix this polynomial, there is only one representation; if we require this polynomial to have coefficients in $\mathbb{Q}$ with roots Weil numbers, there are only finitely many representations. For all the other cases, the characteristic polynomial is $X^2 + p[\varepsilon](1+y_1)$ (or $X^2 + p[\varepsilon](1+y'_1)$) and y_2 (or y'_2) may vary.

Appendix : Proof of results on potentially semi-stable representations

§A : Proof of theorem A.

By twisting, we see that it is enough to consider the case where $r = 0$, i.e. the case of type $(0,s)$ with $s \in \mathbb{N}$.

(a) *The case of twisted $\mathbb{Q}_p$-semi-stable modules (type I and II)*: Assume D is a two dimensional filtered (φ, N, Γ_1)-module of type $(0,s)$ which has the property that the action of $\Gamma_1 = \mathrm{Gal}(F_1/\mathbb{Q}_p)$ on it is diagonal. There is a unique integer i satisfying $0 \leq i < p-1$ such that the action is given by $\hat{\xi}_1^i$. Twisting by $\hat{\xi}_1^{-i}$, we can assume the action of Γ_1 is trivial i.e. we can suppose that D is just a (φ, N)-module, i.e. there is no action of Γ_1 and the filtration is defined on D. This D is a two dimensional $\mathbb{Q}_p$-vector space and the Frobenius φ acts linearly on it. The fact that the Newton polygon of its characteristic polynomial $X^2 - aX + d$ must be above the Hodge polygon and ends up at the same point means $v_p(d) = s$ and $a \in \mathbb{Z}_p$.

Suppose $s = 0$. We must have $N = 0$. Such a D is determined by (a,d), is weakly admissible and is irreducible if and only if there is no line in D stable under φ, i.e. if $X^2 - aX + d$ is irreducible, putting us in case 1 of the statement of Theorem A, part (i).

Suppose now $s \geq 1$ and $N = 0$. Observe that $\Delta = \mathrm{Fil}^s D$ can't be stable under φ. Otherwise Δ would be stable under φ and N and one could find $c \in \mathbb{Z}_p$ with $0 \leq v_p(c) \leq s$ such that $\varphi x = cx$ if $x \in \Delta$. But,

-if $v_p(c) < s$ the Newton polygon (of φ acting on Δ) is not above its Hodge polygon and D woould not be weakly admissible;

-if $v_p(c) = s, D$ would be weakly admissible but Δ would be a proper weakly admissible sub-object and D would not be irreducible.

Hence, if we choose a basis e of Δ, $\{e, \varphi e\}$ forms a basis of D. Now D is weakly admissible. As Δ is not stable under φ, the possible proper weakly admissible sub-objects are lines Δ' stable under φ such that $\varphi x = ux$ for $x \in \Delta'$ and u a unit. This doesn't occur if and only if p divides a and we are in case 1′ of the statement of Theorem A, part (i).

The last case to consider is $s \geq 1$ and $N \neq 0$. The condition $N\varphi = p\varphi N$ implies, that φ must have two distinct eigenvalues $b, pb \in \mathbb{Z}_p$ with $v_p(b) + v_p(pb) = s$, hence s must be odd and $v_p(b) = (s-1)/2$. If e is a non-zero eigenvector corresponding to the eigenvalue pb, then e and Ne form a basis of D, and we have $\varphi e = pbe, \varphi Ne = b \cdot Ne, N^2 e = 0$. The unique proper subobject of D is the $\mathbb{Q}_p$-subspace Δ' spanned by Ne and weak admissibility amounts to requiring that $\Delta \neq \Delta'$. There is a unique $c \in \mathbb{Q}_p$ such that Δ is the line spanned by $e + c \cdot Ne$. Now D is weakly admissible; it is irreducible if and only if Δ' is not weakly admissible, i.e. if $v_p(b) > 0$ which amounts to saying that $s \geq 3$ and we are in case 2 of the statement of Theorem A, part(i).

Hence we have proved that *the D's of type I or II listed in theorem A are irreducible weakly admissible and that they exhaust the possibilities of D's which are F_1-semi-stable with a diagonal action of* Γ_1.

(b) *The case of F_1-semi-stable modules which are not twists of $\mathbb{Q}_p$-semi-stable modules (type III)* : Let D be a two dimensional (φ, N, Γ_1)-module of type $(0, s)$. The group Γ_1 acts linearly on D and because it is cyclic of degree $p-1$, this action is diagonalisable. To ask that it is not diagonal amounts to requiring that there are interes $0 \leq i_1 < i_2 < p-1$ and two lines D_1 and D_2 in D such that $gx = (\hat{\xi}_1(g))^{i_j} \cdot x$ for all $g \in \Gamma_1$ and all $x \in D_j, j \in \{1, 2\}$.

The Frobenius φ acts linearly on D. The fact that φ commutes with the action of Γ_1 means that D_1 and D_2 are stable under φ, hence there are elements $a_1, a_2 \in \mathbb{Q}_p$ such that $\varphi x = a_j \cdot x$ for all $x \in D_j$. The fact that the Newton polygon lies above the Hodge polygon and ends at the same point implies $a_1, a_2 \in \mathbb{Z}_p$ and $v_p(a_1) + v_p(a_2) = s$. As N is nilpotent, the fact that it commutes with the action of Γ_1 implies $N = 0$.

Now, if $s = 0$, one sees easily that such a D is weakly admissible, but D_1 and D_2 are proper sub-objects and D is not irreducible. Assume $s \geq 1$. For $j = 1, 2$, the line $F_1 \otimes D_j$ is stable under the action of Γ_1, but if Δ were this line, either $v_p(a_j) < s$ and D would not be weakly admissible or $v_p(a_j) = s$ and D would be weakly admissible but D_j would be a proper sub-object and D would not be irreducible. Therefore Δ must be a line stable under Γ_1 and $\neq D_1, D_2$. Then it is easy to see that one can choose a basis e_1 of D_1 and a basis e_2 of D_2 in such a way that Δ is generated by $\pi_1^{i_2} \otimes e_1 + \pi_1^{i_1} \otimes e_1$. It is now easy to check that such a D is weakly admissible. It will be irreducible if there is no proper sub-object. A proper sub-object must be stable under φ, hence must be D_1 or D_2. But, one sees that D_j is a proper subobject if and only if $v_p(a_j) = 0$.

Hence we have proved that *the D's of type III listed in theorem A are irreducible weakly admissible and that they exhaust the possibilities of D's which are F_1-semi-stable with a non-diagonal action of* Γ_1.

(c) *The case of F_2-, but not F_1-semi-stable modules (type IV)* : Let D be a two dimensional (φ, N, Γ_2)-module of type $(0, s)$. Let D_0 the sub-$\mathbb{Q}_p$-vector space of D consisting of those x such that $\overline{\tau} x = x$. One deduces easily from the fact that $\overline{\tau}$ acts semi-linearly on the $\mathbb{Q}_{p^2}$-vector space D that D_0 is of dimension 2 over $\mathbb{Q}_p$ and spans D as a $\mathbb{Q}_{p^2}$-vector space. The fact that φ commutes with the action of $\overline{\tau}$ is then equivalent to the fact that the restriction of φ to D_0 is a $\mathbb{Q}_p$-linear automorphism of D_0.

The group $I\Gamma_2$ is cyclic of order $p^2 - 1$ and acts linearly on D and, if we want D not to be semi-stable, the inertia group of F_2/F_1, which is the subgroup of index $p+1$, cannot act trivially. Therefore $I\Gamma_2$ acts on D through characters and at least one of them is of order not dividing $p-1$. This means that one can find a line D' of D and an integer i satisfying $0 \leq i < p^2 - 1$ and i not divisible by $p+1$ such that $gx = \hat{\xi}_2^i(g) \cdot x$ if $g \in I\Gamma_2$ and $x \in D'$. Then, $g(\varphi x) = \varphi(gx) = \varphi(\hat{\xi}_2^i(g) \cdot x) = \hat{\xi}_2^{pi}(g) \cdot \varphi x$; but i not divisible by $p+1$ means $pi \not\equiv i \pmod{p^2-1}$ which implies that $\varphi x \in D'$; we see that $D = D' \oplus D''$ with $gy = \hat{\xi}_2^{pi}(g) \cdot y$ if $g \in |\Gamma_2$ and $y \in D''$. Permuting i and pi mod p^2-1 if necessary, we can assume that $i = i_1 + pi_2$ and $pi \equiv i_2 + pi_1 \pmod{p^2-1}$ with integers i_1, i_2 satisfying $0 \leq i_1 < i_2 \leq p-1$.

Now if $x \in D'$, we have $g\overline{\tau}x = \overline{\tau}g^p x = \overline{\tau}(\hat{\xi}_2^{pi}(g) \cdot x) = \hat{\xi}_2^i(g) \cdot \overline{\tau}x$, hence $\overline{\tau}$ leaves D' fixed; similarly it leaves also D'' fixed. If $x \in D_0$ and if $x = x' + x''$, with $x' \in D'$ and $x'' \in D''$, we then see that $\overline{\tau}x' = x'$ and $\overline{\tau}x'' = x''$, hence we can write $D_0 = D_0' \oplus D_0''$ with $D_0' = D' \cap D_0$ and $D_0'' = D'' \cap D_0$. If $x' \in D_0'$, we must have $g\varphi x' = \varphi g x' = \varphi(\hat{\xi}_2^i(g) \cdot x') = \hat{\xi}_2^{pi}(g) \cdot \varphi x'$, hence $\varphi x' \in D_0''$. Similarly $\varphi^2 x' \in D_0'$. Summarizing, we see that we can find $d \in \mathbb{Q}_p$ and a basis e_1, e_2 of D such that

$$\overline{\tau}e_1 = e_1, \overline{\tau}e_2 = e_2, ge_1 = \hat{\xi}_2^{i_1+pi_2}(g)e_1 \text{ and } ge_2 = \hat{\xi}_2^{i_2+pi_1}(g)e_2$$

$$\text{for } g \in I\Gamma_2, \quad \varphi e_1 = e_2, \quad \varphi e_2 = -de_1.$$

As N is nilpotent, the fact that it commutes with the action of $I\Gamma_2$ implies $N = 0$. Now we have actions of φ, N and $\mathrm{Gal}(F_2/\mathbb{Q}_p)$ satisfying the required properties. If we want D to be weakly admissible, the fact that the Hodge polygon lies below the Newton polygon and that they end up at the same point amounts to requiring that $v_p(d) = s$.

If $s = 0$, we then have a weakly admissible module. There is no line stable under φ and Γ_2, hence D is irreducible and we get $D \simeq D_{IV}(0,0;d;i_1,i_2)$.

If $s \geq 1, \Delta$ can be any line of D_{F_2} stable under the action of Γ_2. One sees immediately that there is a unique $\alpha \in \mathbb{P}^1(\mathbb{Q}_p)$ such that this is the line generated by $\pi_2^{(p-1)i_1} \otimes e_1 + \alpha \cdot \pi_1^{(p-1)i_2} \otimes e_2$. Again because there is no line of D stable under φ and Γ_2, this D is weakly admissible and does not contain any proper weakly admissible subobject. Hence we have proved that *the D's of type IV listed in theorem A are irreducible weakly admissible and that they exhaust the possibilities of D's which are F_2-semi-stable but not F_1-semi-stable.*

(d) The only thing which is left to prove is the fact that, if $p \geq 5$, *any irreducible 2-dimensional weakly admissible pst-module is F_2-semi-stable.* : One can assume that this module is F-semi-stable, for F a finite Galois extension of $\mathbb{Q}_p$ containing F_2 and contained in $\overline{\mathbb{Q}_p}$. The inertia group $\mathrm{In}(F/\mathbb{Q}_p)$ of the extension $F/\mathbb{Q}_p$ acts linearly on the corresponding two dimensional F_0-vector space D. If $g \in \mathrm{In}(F/\mathbb{Q}_p)$, the fact that the action of g commutes with the action of φ implies that the characteristic polynomial of g has coefficients in $\mathbb{Q}_p$. Because $p \geq 5, [\mathbb{Q}_p(\sqrt[p]{1}) : \mathbb{Q}_p] > 2$ and any element of $\mathrm{In}(F/\mathbb{Q}_p)$ of order a power of p acts trivially; so the p-Sylow group P of $\mathrm{In}(F/\mathbb{Q}_p)$ acts trivially. Replacing F by F^P, one can assume $F/\mathbb{Q}_p$ tame, i.e. that $\mathrm{In}(F/\mathbb{Q}_p)$ is cyclic of order prime to p; if g is a generator of this group, the roots of the polynomial characteristic of g acting on D are in $\mathbb{Q}_{p^2}$, hence of order dividing p^2-1. Therefore g^{p^2-1} acts trivially and the result follows from the fact that this element generates the inertia group of F/F_1. □

§B : Proof of theorems B1 and B2.

(a)*Proof of theorem* B1: By twisting, it is enough to prove the theorem for $r = 0$ and $i = 0$. In this case we observe that $D_I(0, s; a, d; 0)$ can be viewed as a weakly admissible filtered φ-module over $\mathbb{Q}_p$ ($N = 0$ and the Galois group action is trivial) which satisfies $\mathrm{Fil}^0 D = D$ and $\mathrm{Fil}^p D = 0$ hence is admissible ([**15**],[**30**]).

To prove (i) we are reduced to checking that $V_I^*(0, 0; a, d; 0)$ is isomorphic to a one dimensional $\mathbb{Q}_p(\lambda)$-vector space on which G_p acts through the unramified character η_λ. But $\mathrm{Fil}^0 B_{\mathrm{cris}}$ and $\mathrm{Fil}^0 B_{\mathrm{st}}$ contain the completion $\hat{\mathbb{Q}}_p^{nr}$ of the maximal unramified extension of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and hence we have an injective map $\mathrm{Hom}_{\mathbb{Q}_p[\varphi]}(D_I(0, 0; a, d; 0), \hat{\mathbb{Q}}_p^{nr}) \to V_I^*(0, 0; a, d; 0)$, which is an isomorphism for reasons of dimensions. The assertion is then an easy exercise.

(ii) is a consequence of theorem B2 which will be prove below. □

(b) *The category* $\underline{MF}^f_{]-p+1,0]}$ *and the functor* $\overline{V}_{cr}$: Let $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{\mathrm{cr},[0,p-1[}$ be the full subcategory of $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{\mathrm{cr},[0,p-1]}$ consisting of those V which have no non-trivial subobject V' such that $V'(-p+1)$ is unramified. Similarly denote by $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1[}$ the full subcategory of $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1]}$ consisting of those T which are isomorphic to a subquotient of an object of $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{\mathrm{cr},[0,p-1[}$. It is easy to see that, for any V in $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{\mathrm{cr},[0,p-1]}$, one can find a short exact sequence

$$0 \to V' \to V \to V'' \to 0$$

such that $V'(-p+1)$ is unramified and V'' is in $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{\mathrm{cr},[0,p-1[}$.

Let $\overline{V}$ be a 2-dimensional $\mathbb{F}_p$-vector space on which the action of the inertia group is irreducible. We see that, for any local artinian $\mathbb{Z}_p$-algebra A of residue field $\mathbb{F}_p$, any A-deformation of $\overline{V}$ which lies in $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1]}$ actually lies in $\underline{\mathrm{Rep}}_{\mathbb{Q}_p}(G_p)_{\mathrm{cr},[0,p-1[}$.

For $a, b \in \mathbb{Z}$ satisfying $a \leq b$, let $\underline{MF}^f_{[a,b]}$ be the following category:
-the objects are $\mathbb{Z}_p$-modules M of finite length, equipped with
i) a filtration of M by sub-$\mathbb{Z}_p$-modules

$$M = \mathrm{Fil}^a M \supset \mathrm{Fil}^{a+1} M \supset \cdots \supset \mathrm{Fil}^j M \supset \cdots \supset \mathrm{Fil}^b M \supset \mathrm{Fil}^{b+1} M = 0;$$

ii) for each j, a $\mathbb{Z}_p$-linear map $\varphi^j : \mathrm{Fil}^j M \to M$ such that $\varphi^{j+1}(x) = p\varphi^j(x)$ if $x \in \mathrm{Fil}^{j+1} M$, and such that $M = \Sigma_{a \leq j \leq b} \varphi^j(\mathrm{Fil}^j M)$.
-the morphisms are $\mathbb{Z}_p$-linear maps compatible with all the structures.

Recall [**16**] that this is an abelian category and that for any object M the $\mathrm{Fil}^j M$'s are direct summands (as $\mathbb{Z}_p$-modules). If $a < b$, we denote by $\underline{MF}_{]a,b]}$ the full subcategory of $\underline{MF}_{[a,b]}$ consisting of those M which have no non-trivial subobject N such that $\mathrm{Fil}^{a+1} N = 0$.

Let $\overline{k}$ be the residue field of $\overline{\mathbb{Q}}_p$ and σ the absolute Frobenius acting on $\overline{k}$ (via $x \mapsto x^p$) and on the ring $W(\overline{k})$ of Witt vectors with coefficients in $\overline{k}$ by functoriality. Let A_{cris} be the ring constructed in [**11**]. Recall that this is a $W(\overline{k})$-algebra which is a domain equipped with i) an action of G_p compatible with the ring structure and with the obvious action on $W(\overline{k})$, ii) a Frobenius $\varphi : A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ compatible with the ring structure, commuting with the action of G_p and σ-semi-linear, iii) a decreasing filtration $(\mathrm{Fil}^j A_{\mathrm{cris}})_{j\in\mathbb{N}}$ by ideals, which are direct summands as $W(\overline{k})$-modules, stable under G_p.

Moreover, for $0 \leq j \leq p-1$, we have $\varphi(\mathrm{Fil}^j M) \subset p^j \mathrm{Fil}^j M$, therefore, one can define $\varphi^j : \mathrm{Fil}^j A_{\mathrm{cris}} \to A_{\mathrm{cris}}$ by $\varphi^j x = p^{-j}\varphi x$.

For any object M of $\underline{MF}^f_{[0,p-1]}$ and $0 \leq j \leq p-1$ we can use the natural map $\mathrm{Fil}^j A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathrm{Fil}^{-j} M \to M$, which is injective, to identify $\mathrm{Fil}^j A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathrm{Fil}^{-j} M$ with a sub-A_{cris}-module of $A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} M$ and define $\mathrm{Fil}^0(A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} M) = \Sigma_{0\leq j\leq p-1} \mathrm{Fil}^j A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} \mathrm{Fil}^{-j} M$. It is easy to see that there is a unique σ-semi-linear map

$$\varphi : \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} M) \to A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} M$$

such that, for $0 \leq j \leq p-1$, $\varphi(\lambda \otimes x) = \varphi^j \lambda \otimes \varphi^{-j} x$ if $\lambda \in \mathrm{Fil}^j A_{\mathrm{cris}}$ and $x \in \mathrm{Fil}^j M$.

Recall ([**16**], see also [**30**]) that if for M in $\underline{MF}_{[-p+1,0]}$, we define

$$\underline{V}_{\mathrm{cr}}(M) = \{v \in \mathrm{Fil}^0(A_{\mathrm{cris}} \otimes_{\mathbb{Z}_p} M) | \varphi v = v\},$$

then $\underline{V}_{\mathrm{cr}}(M)$ is a finite $\mathbb{Z}_p$-representation of G_p. If M is in $\underline{MF}_{]-p+1,0]}$, this is an object of $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1[}$, which has the same length as M as a $\mathbb{Z}_p$-module. The functor

$$\underline{V}_{\mathrm{cr}} : \underline{MF}^f_{]-p+1,0]} \to \underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1[}$$

defined in this way induces an equivalence between those two categories[9]. We denote by $\underline{D}_{\mathrm{cr}}$ a quasi-inverse.

(c) *The mod p representation*: Recall that $r, s \in \mathbb{Z}$ with $0 \leq r < s \leq p-1$ and $\varepsilon \in \mathbb{F}_p^*$. We give to the two dimensional $\mathbb{F}_p$-vector space $(\mathbb{F}_p)^2$ (with $\overline{u}_1, \overline{u}_2$ the canonical basis) the structure of an object $\overline{M} = \overline{M}(r, s; \varepsilon)$ of $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1[}$ by defining

$$\mathrm{Fil}^{-s}\overline{M} = \overline{M},\ \mathrm{Fil}^{-s+1}\overline{M} = \mathrm{Fil}^{-r}\overline{M} = \mathbb{F}_p\overline{u}_1,\ \mathrm{Fil}^{-r+1}\overline{M} = 0,$$
$$\varphi^{-r}\overline{u}_1 = \overline{u}_2, \varphi^{-s}\overline{u}_2 = -\varepsilon^{-1} \cdot \overline{u}_1.$$

It is not hard to check (compare with [**16**]): i) that one can identify $\underline{V}_{\mathrm{cr}}(\overline{M})$ with the residue field $\mathbb{F}_{p^2}$ of $\mathbb{Q}_{p^2}$ in such a way that the inertia subgroup I_p acts via the character ξ_2^{r+ps}; ii) that the determinant of the action of G_p is the character $\eta_\varepsilon \chi^{r+ps}$; all together this proves that $\underline{V}_{\mathrm{cr}}(\overline{M}) \simeq \underline{V}_{i,\varepsilon}$ and this proves the first part of the assertion (ii) of Theorem B1.

(d) Let's prove the following lemma:

Lemma . *Let A be a local artinian ring, $\mathfrak{S}$ an abelian A-linear category, F, G two covariant functors from $\mathfrak{S}$ to the category of A-modules of finite length which are A-linear, exact, faithful and such that, for any object N of $\mathfrak{S}$, the A-modules $F(N)$ and $G(N)$ have the same length. Then, if $d \in \mathbb{N}$ and M is an object of $\mathfrak{S}$, $F(M)$ is free of rank d over A if and only $G(M)$ is.*

Indeed, for $a \in A$ and N an object of $\mathfrak{S}$, call $[a]_N$ the endomorphism of N which is the action of a. Let m_A be the maximal ideal of A and $k = A/m_A$. The

[9] actually, what is defined in [**16**] is a contravariant version of this construction but the passage from one construction to the other is straightforward.

faithfulness of F implies that, for any object N of $\mathfrak{S}$, one has $[a]_N = 0$ for all $a \in m_A$ if and only if $F(N)$ is a k-vector space. Now let $\overline{M}$ be the biggest quotient of M such that $[a]_{\overline{M}} = 0$ for all $a \in m_A$. It is clear that $F(\overline{M}) = F(M)/m_AF(M)$ and $G(\overline{M}) = G(M)/m_AG(M)$. Assume that $F(M)$ is free of rank d. Then $\dim_k F(\overline{M}) = d$. Hence we have also $\dim_k G(\overline{M}) = d$ and if $e_1, e_2, \cdots, e_d$ are lifting in $G(M)$ of a basis of $G(\overline{M})$ over k, the e_j generate $G(M)$. But they are linearly independent over A, for otherwise, $\text{length}_A G(M) < d{\cdot}\text{length}_A A = \text{length}_A F(M)$.

(e) *Deformations*: Let A be a local artinian $\mathbb{Z}_p$-algebra. Any A-representation T of V can be viewed as a $\mathbb{Z}_p$-representation together with an imbedding of A into the ring of the endomorphisms of T. Therefore, if T is an object of $\underline{\text{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\text{cr},[0,p-1[}$, $\underline{D}_{\text{cr}}(T)$ is an **A-object of** $\underline{MF}^f_{]-p+1,0]}$; that is, there is a natural structure of A-module on it, the filtration is given by sub-A-modules and the maps φ^r are A-linear. Conversely, given any A-object M 0f $\underline{MF}^f_{]-p+1,0]}$, $\underline{V}_{\text{cr}}(M)$ is an A-representation of G_p. Therefore, for any A, the functor $\underline{V}_{\text{cr}}$ induces an equivalence between the category of A-objects of $\underline{MF}^f_{]-p+1,0]}$ and the category of A-objects in $\underline{\text{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\text{cr},[0,p-1[}$. Applying the lemma to the first of those two categories and to the functors which associate to M respectively the underlying A-module and the A-module underlying $\underline{V}_{\text{cr}}(M)$, we see, that for any A-object M of $\underline{MF}^f_{]-p+1,0]}$, $\underline{V}_{\text{cr}}(M)$ is flat as an A-module if and only if M is.

Now let V be an A-deformation of $\overline{V}$ and $M = \underline{D}_{\text{cr}}(V)$; if m_A is the maximal ideal of A, one can identify M/m_AM with $\overline{M}$; as M must be a free-A-module, any couple $\{u_1, u_2\}$ of elements of M lifting $\{\overline{u}_1, \overline{u}_2\}$ is a basis of M over A; one knows **[16]** that any morphism in $\underline{MF}^f_{]-p+1,0]}$ is strictly compatible with the filtrations. This implies that $\text{Fil}^{-s}M = M$ and that one can choose $u_1 \in \text{Fil}^{-r}M$; as $\varphi^{-r}\overline{u}_1 = \overline{u}_2$, one can choose $u_2 = \varphi^{-r}u_1$; then we must have $\varphi^{-s}u_2 = -[\varepsilon^{-1}](1 + x_1)u_1 + x_2u_2$ with $x_1, x_2 \in m_A$.

Now $\text{Fil}^{-r}M$ contains the free-A-module of rank one spanned by u_1. *It can't be bigger*, for otherwise one could find a nonzero $\lambda \in A$ such that $\lambda u_2 \in \text{Fil}^{-r}M$; multiplying λ by a suitable power of p, one could assume $p\lambda = 0$; but then $p^{s-r} \cdot \varphi^{-r}(\lambda u_2) = \varphi^{-s}(\lambda u_2) = \lambda(-[\varepsilon^{-1}](1 + x_1)u_1 + x_2u_2) \neq 0$ because λ is a unit in A; on the other hand, $p^{s-r}\varphi^{-r}(\lambda u_2) = p^{s-r-1}\varphi^{-r}(p\lambda \cdot u_2) = 0$ giving us a contradiction. So we see that

$$\text{Fil}^{-s}M = M,\ \text{Fil}^{-s+1}M = \text{Fil}^{-r}M = Au_1,\ \text{Fil}^{-r+1}M = 0,$$

$$\varphi^{-r}u_1 = u_2, \quad \varphi^{-s}u_2 = -[\varepsilon^{-1}](1 + x_1)u_1 + x_2u_2.$$

Now if we change the lifting of u_1, it must be to the element of the form λu_1 with λ a unit in A; then $\varphi^{-r}(\lambda u_1) = \lambda u_2$ and $\varphi^{-s}u_2 = -[\varepsilon^{-1}](1 + x_1) \cdot \lambda u_1 + x_2 \cdot \lambda u_2$ and we see that x_1 and x_2 depends only on the isomorphism class of M.

Conversely it is clear that if we now identify $\{u_1, u_2\}$ with the canonical basis of A^2 we have equipped A^2, with the structure of an A-object $M = M_A(r, s, \varepsilon; x_1, x_2)$ of $\underline{MF}^f_{]-p+1,0]}$ and that

$$V_A(r, s, \varepsilon; x_{,1}, x_2) = \underline{V}_{\text{cr}}(M(r, s, \varepsilon; x_{,1}, x_2))$$

is an A-deformation of $\overline{V}$ and that for any A-deformation V of $\overline{V}$, there is a unique $(x_1, x_2) \in m_A$ such that $V \simeq V(r, s, \varepsilon; x_{,1}, x_2)$.

Now let $\mathfrak{a}$ be the set of artinian quotients of the ring $\mathbb{Z}_p[\![X_1, X_2]\!]$. For any $A \in \mathfrak{a}$, consider $M_A = M_A(r, s, \varepsilon; x_{,1}, x_2)$ and $V_A = V_A(r, s, \varepsilon; x_1, x_2)$ with $x_i =$ the image of x_i in A. Then the M_A's form a projective system of objects of $\underline{MF}^f_{]p+1,0]}$ and therefore the V_A's form a projective system of finite representations of G_p which are in $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)_{\mathrm{cr},[0,p-1[}$. Then it is clear that $\lim{\cdot}\mathrm{proj}{\cdot}_{A\in\mathfrak{a}} V_A$ is a free $\mathbb{Z}_p[\![X_1, X_2]\!]$-module of rank two equipped with an action of G_p which gives an identification of $\mathbb{Z}_p[\![X_1, X_2]\!]$ with $R_{i,\varepsilon}(\mathrm{cr}, [0, p-1])$. Now if we put $Y_1 = (1+x)^{-1} - 1$ and $Y_2 = [\varepsilon](1+x_1)^{-1}x_2$, we have $\mathbb{Z}_p[\![X_1, X_2]\!] = \mathbb{Z}_p[\![Y_1, Y_2]\!]$.

(f) *End of the proof of* (ii): It is easy to see that if we send $\mathbb{Z}_p[\![X_1, X_2]\!]$ to $\mathbb{Z}_p$ via $X_1 \mapsto x_1, X_2 \mapsto x_2$ if $y_1 = (1+x_1)^{-1} - 1$ and $y_2 = -[\varepsilon](1+x_1)^{-1}x_2$ and if T and V are as in the statement of the theorem, then V is crystalline and that the dual D of $D^* = \underline{D}^*_{\mathrm{cris}}(V) = \underline{D}_{\mathrm{st},\mathbb{Q}_p}(V)$ is the two dimensional $\mathbb{Q}_p$-vector space $D = \mathbb{Q}_p u_1 + \mathbb{Q}_p u_2$ with

$$\mathrm{Fil}^{-s} D = D,\ \mathrm{Fil}^{-s+1} D = \ \mathrm{Fil}^{-r} D = \mathbb{Q}_p u_1,\ \mathrm{Fil}^{-r+1} D = 0,$$

$$\text{and } \varphi u_1 = p^{-r} u_2, \quad \varphi u_2 = p^{-s}(-[\varepsilon^{-1}](1+x_1)u_1 + x_2 u_2).$$

Hence D is of Hodge-Tate type $(-s, -r)$ and the characteristic polynomial of φ acting on D is $X^2 - p^{-s}x_2 \cdot X + p^{-r-s}[\varepsilon]^{-1}(1+x_1)$. Therefore, D^* is of Hodge-Tate type (r, s) and the characteristic polynomial of φ acting on D^* is $X^2 - p^r x_2[\varepsilon](1+x_1)^{-1} \cdot X + p^{r+s}[\varepsilon](1+x_1)^{-1}$; hence $D^* \simeq D_I(r, s; p^r x_2[\varepsilon](1+x_1)^{-1}, p^{r+s}[\varepsilon](1+x_1)^{-1} = D_I(r, s; p^r y_2, p^{r+s}[\varepsilon](1+y_1){\cdot}0)$.

§C : About the proof of theorems C1, C2 and C3:

a) *Proof of theorem C1 for $s = r$*: Twisting if necessary, we are reduced to case $r = s = 0$ and this is an easy exercise on representations of G_p on which the action of I_p is finite. □

b) *About theorem C2* : Let V be as in theorem C2. The inplications i) ⇒ ii) ⇒ iii) are well known. If V is potentially semi-stable, there is a D listed in theorem A which is of Hodge-Tate type $(0,1)$ and admissible such that $V \simeq \underline{V}^*_{\mathrm{st}}(D)$. Looking at the list, we see that D is either one of the modules $D_I(0, 1; a, d; i)$ or one of the modules $D_{IV}(0, 1; d, i_1, i_2, \alpha)$, hence iii) ⇒ iv).

Remark : Before giving a sketch of proof of the other statements, let us explain how one could easily deduce theorem C2 and the admissibility statement of theorem C1 from results of Laffaille : We already know (th.B1) that any $D = D_I(0, 1; a, d; i)$ is admissible; moreover $V_I^*(0, 1; a, d; i)(\hat{\xi}_1^i) \simeq V_I^*(0, 1; a, d; 0)$ is a crystalline representation of G_p of Hodge-Tate type $(0,1)$ and **[21]** gives that this representation comes from a Barsotti-Tate group defined over $\mathbb{Z}_p$; using the fact that the p-adic valuation of each root of the characteristic polynomial of φ acting on D, which is $X^2 - aX + d$ is $1/2$, it is easy to see that Γ is strictly of slope $1/2$, hence we get *the assertion (a) of theorem C2.*

Similarly, let D be any of the $D_{IV}(0,1;d,i_1,i_2,\alpha)$'s of theorem A. Denote by D_0 the sub-$\mathbb{Q}_p$-vector space fixed under $\overline{\tau}$. Observe that the filtration on D_{F_2} is actually defined on E and that D_0 equipped with the action of φ and the filtration on $E \otimes D_0$ can be viewed as a "weakly admissible filtered φ-module over E" of dimension 2 and of Hodge-Tate type $(0,1)$. Laffaille's **[20]** gives that this module is admissible and that there is a Barsotti-Tate group J defined over $\mathcal{O}_E$ such that the associated p-adic representation of $\mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$, $\mathrm{Hom}_{\mathrm{filt}\cdot\varphi-\mathrm{mod}}(D_0, B_{\mathrm{cris}})$, is isomorphic to $V_p(J)$. From that, we easily get the admissibility of D and *the assertion (b) of theorem C2*, except for the fact that one can choose J strictly of slope $1/2$; this latter fact can be shown by a direct computation. By twisting, we get the *admissibility of all* $D_{IV}(r,r+1;d,i_1,i_2,\alpha)$'s. From the above discussion we have that all the w.a. modules $D_I(0,1;a,d;I)$ and $D_{IV}(0,1;d,i_1,i_2,\alpha)$ are actually admissible and that the corresponding representations are potentially Barsotti-Tate, hence iv) $\Rightarrow$ i).

Sketch of the proof of the other statements:

1. - If S is any scheme, let us call a **p-group scheme over** S any inductive system $(J_n)_{n\in\mathbb{N}}$ of finite and flat commutative group schemes over S such that map $J_n \to J_{n+1}$ identifies J_n with the kernel of the multiplication by p^n in J_{n+1}. Thus, if the map $J_n \to J_{n+1}$ is also an isomorphism for n big enough, we have a **finite p-group scheme**; that is, a finite and flat commutative group scheme killed by a power of p. If there is an integer h such that J_n is free of rank p^{nh} for all n, we get a **Barsotti-Tate group**.

2. - Any formal group, and therefore also any p-group scheme, over $\mathbb{F}_p$ is equipped with two endomorphisms, the Frobenius $F = \varphi$ and the Verschiebung V satisfying $\varphi V = V\varphi = p$.

A **slope 1/2 structure** on a p-group scheme J over $\mathbb{F}_p$ is an automorphism u of J such that $\varphi = Vu$. If J is a Barsotti-Tate group over $\mathbb{F}_p$, there is at most one slope $1/2$ structure u on J and there is such a u if and only if J is strictly of slope $1/2$.

The p-group schemes over $\mathbb{F}_p$ with slope $1/2$ structure form, in an obvious way an additive category $\underline{pGS}_{\mathbb{F}_p,1/2}$ which turns out to be abelian.

Forgetting u we get an additive functor from $\underline{pGS}_{\mathbb{F}_p,1/2}$ to the category $\underline{pGS}_{\mathbb{F}_p}$ of p-group schemes over $\mathbb{F}_p$ which is exact and faithful; if (J_1,u) and (J_2,u) are two objects of $\underline{pGS}_{\mathbb{F}_p,1/2}$, the cokernel of the injective map

$$\mathrm{Hom}_{\underline{pGS}_{\mathbb{F}_p,1/2}}((J_1,u)(J_2,u)) \to \mathrm{Hom}_{\underline{pGS}_{\mathbb{F}_p}}(J_1,J_2)$$

is killed by p.

3. - Let CW be the formal group of covectors over $\mathbb{F}_p$ ([**8**], chap.II, §4). Let us recall that, for any finite $\mathbb{F}_p$-algebra A, $CW(A)$ consists of covectors $a = (\cdots, a_{-n}, \cdots, a_{-1}, a_0)$ with $a_{-n} \in A$ for all n and $a_{-n} \in r_A$ for almost all n, where r_A denotes the radical of A; with the obvious notation, we have $a+b=c$, with

$$c_{-n} = S_m(a_{-n-m},\cdots,a_{-n-1},a_{-n};b_{-n-m},\cdots,b_{-n-1},b_{-n}) \text{ for } m >> 0,$$

where the S_m's are the polynomials which define the addition on Witt vectors. We have also

$$\varphi a = (\cdots, (a_{-n})^p, \cdots, (a_{-1})^p, (a_0)^p) \text{ and } V_a = (\cdots, a_{-n-1}, \cdots, a_{-2}, a_{-1}).$$

To any p-group scheme J over $\mathbb{F}_p$, one can associate its "contravariant Dieudonné-module" $\underline{M}(J) = \mathrm{Hom}(J, CW)$. One can view $\underline{M}$ as a contravariant additive functor from the category of p-group schemes over $\mathbb{F}_p$ to the category of $\mathbb{Z}_p[\varphi, V]$-modules which are $\mathbb{Z}_p$-modules of finite type. This is an anti-equivalence of categories. A quasi-inverse to $\underline{M}$ is given by associating to such a $\mathbb{Z}_p[\varphi, V]$-module M the p-group scheme $\underline{J}(M)$ defined by $\underline{J}(M)(A) = \mathrm{Hom}_{\mathbb{Z}_p,[\varphi,V]}(M, CW(A))$ for all finite $\mathbb{F}_p$-algebras A ([8], chap.III).

Denote by $\mathcal{O}_{1/2}$ the ring $\mathbb{Z}_p[u, u^{-1}, \varphi]/(\varphi^2 - pu)$. Setting $V = \varphi u^{-1}$, we can view $\mathbb{Z}_p[F, V]$ as a subring of $\mathcal{O}_{1/2}$. If (J, u) is an object of $\underline{pGS}_{\mathbb{F}_p,1/2}$, then $\underline{M}(J)$ has a natural structure of $\mathcal{O}_{1/2}$-module; we get in this way an anti-equivalence between $\underline{pGS}_{\mathbb{F}_p,1/2}$ and the category of $\mathcal{O}_{1/2}$-modules which are $\mathbb{Z}_p$-modules of finite type.

4. - For any finite $\mathbb{F}_p$-algebra A, denote $BW_{1/2}(A)$ the set of the $a = (a_n)_{n\in\mathbb{Z}}$ with $a_n \in r_A$ for all n. With obvious the notation, one sees that, for all fixed $n \in \mathbb{Z}$, if $a, b \in BW_{1/2}(A)$, the sequence

$$S_m((a_{n-m})^{p^m}, \cdots, (a_{n-1})^p, a_n; (b_{n-m})^{p^m}, \cdots, (b_{n-1})^p, b_n) \text{ for } m \in \mathbb{N}$$

is stationary; if we denote by c_n its limit and if we set

$$a + b = c = (c_n)_{n\in\mathbb{Z}},$$

$BW_{1/2}(A)$ becomes an abelian group and $BW_{1/2}$ may be viewed as a commutative formal group over $\mathbb{F}_p$ which is equipped with an automorphism u defined by

$$u((a_n)_{n\in\mathbb{Z}}) = (a_{n+1})_{n\in\mathbb{Z}}.$$

We have $\varphi(a_{n\in\mathbb{Z}}) = ((a_n)^p)_{n\in\mathbb{Z}}$ and $V = \varphi u^{-1}$.

We have also a natural morphism from $BW_{1/2}$ to CW sending $(a_n)_{n\in\mathbb{Z}}$ to $(\cdots, (a_{-n})^{p^n}, \cdots, (a_{-1})^p, a_0)$. If (J, u) is an object of $\underline{pGS}_{\mathbb{F}_p,1/2}$, this map induces an isomorphism from $\mathrm{Hom}((J, u), BW_{1/2})$ to the $\mathcal{O}_{1/2}$-module $\underline{M}(J)$ and we use it to identify these two modules.

5. - We denote by $\underline{pGS}_{\mathcal{O}_E}$ the additive category of p-group schemes over $\mathcal{O}_E$. The couples (J, u) consisting of a p-group scheme J over $\mathcal{O}_E$ and a slope $1/2$ structure u on the special fiber $J_{\mathbb{F}_p}$ are in an obvious way the objects of an additive category $\underline{pGS}_{\mathcal{O}_E,1/2}$. We denote by $\underline{pGS}_{\mathcal{O}_E,1/2,\text{strict}}$ the full subcategory of $\underline{pGS}_{\mathcal{O}_E,1/2}$ consisting of those (J, u), for which one can find two Barsotti-Tate groups J' and J'' over $\mathcal{O}_E$, strictly of slope $1/2$ and morphisms $J \to J'$, $J' \to J''$ of p-group schemes such that the sequence

$$0 \to J \to J' \to J'' \to 0$$

is exact and the diagram

$$\begin{array}{ccccccccc} 0 \to & J_{\mathbb{F}_p} & \to & J'_{\mathbb{F}_p} & \to & J''_{\mathbb{F}_p} & \to 0 \\ & \downarrow u & & \downarrow u' & & \downarrow u'' & \\ 0 \to & J_{\mathbb{F}_p} & \to & J'_{\mathbb{F}_p} & \to & J''_{\mathbb{F}_p} & \to 0 \end{array}$$

(where u' (resp. u'') is the unique slope 1/2 structure which exists on $J'_{\mathbb{F}_p}$ (resp. $J''_{\mathbb{F}_p}$)) is commutative.

6. - Set $\mathcal{O}_{E,1/2} = \mathcal{O}_E[v, v^{-1}]$. The ring $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_{1/2}$ is a domain and, setting $1 \otimes \varphi = \pi^{(p+1)/2} \cdot v$ and $1 \otimes u = -v^2$, we identify $\mathcal{O}_{E,1/2}$ with the normalization of $\mathcal{O}_E \otimes_{\mathbb{Z}_p} \mathcal{O}_{1/2}$ in its field of fractions. Also $\mathcal{O}_{E,1/2}$ is a faithfully flat $\mathcal{O}_{1/2}$-algebra.

Define the **category** $\underline{MF}_{E,1/2}$ as follows:

- an object is a couple (M, Λ) where M is an $\mathcal{O}_{1/2}$-module, which is of finite type as a $\mathbb{Z}_p$-module, and Λ is a sub $\mathcal{O}_E$-module of $\mathcal{O}_{E,1/2} \otimes_{\mathcal{O}_{1/2}} M$ such that the map from $\Lambda \oplus \Lambda$ to $\mathcal{O}_{E,1/2} \otimes_{\mathcal{O}_{1/2}} M$ sending (x, y) to $x + vy$, is an isomorphism;
- a morphism $(M, \Lambda) \to (M', \Lambda')$ is an $\mathcal{O}_{1/2}$-linear map from M to M' such that the $\mathcal{O}_{E,1/2}$-linear map induced by scalar extension sends Λ into Λ'.

7. - Now, for any finite and flat $\mathcal{O}_E$-algebra $\mathfrak{a}$, there is a unique $\mathcal{O}_E$-linear map

$$\lambda_{\mathfrak{a}} : \mathcal{O}_{E,1/2} \otimes_{\mathcal{O}_{1/2}} BW_{1/2}(\mathfrak{a}/\pi\mathfrak{a}) \to \mathfrak{a}[1/p]/\pi\mathfrak{a}$$

such that, if $a = (a_n)_{n \in \mathbb{Z}} \in BW_{1/2}(\mathfrak{a}/\pi\mathfrak{a})$ and if $\hat{a}_n$ is a lifting of a_n in $\mathfrak{a}$, then $\lambda_{\mathfrak{a}}(1 \otimes a) = \Sigma_{m \in \mathbb{N}} p^{-m} (\hat{a}_{-m})^{p^{2m}} (\bmod \pi\mathfrak{a})$ and $\lambda_{\mathfrak{a}}(v \otimes a) = \pi^{-(p+1)/2} \cdot \Sigma_{m \in \mathbb{N}} p^{-m} (\hat{a}_{-m})^{p^{2m+1}} (\bmod \pi\mathfrak{a})$.

Let (M, Λ) be an object of $\underline{MF}_{E,1/2}$. Then M defines an object $(J_{\mathbb{F}_p}, u)$ of $\underline{pGS}_{\mathbb{F}_p,1/2}$: for any finite $\mathbb{F}_p$-algebra A,

$$J_{\mathbb{F}_p}(A) = \operatorname{Hom}_{\mathcal{O}_{1/2}}(M, BW_{1/2}(A)).$$

If for any finite and flat $\mathcal{O}_E$-algebra $\mathfrak{a}$, we define

$$J(\mathfrak{a}) = \{\alpha \in \operatorname{Hom}_{\mathcal{O}_{1/2}}(M, BW_{1/2}(\mathfrak{a}/\pi\mathfrak{a})) | \Lambda \subset \operatorname{Ker} \lambda_{\mathfrak{a}} \circ (id \otimes \alpha)\},$$

we get a functor J from the category of finite and flat $\mathcal{O}_E$-algebras to abelian groups; one can check that J is actually a p-group scheme over $\mathcal{O}_E$. Moreover (J, u) is an object of $\underline{pGS}_{\mathcal{O}_E,1/2,\text{strict}}$.

The correspondence $(M, \Lambda) \mapsto (J = \underline{J}(M, \Lambda), u = \underline{u}(M))$ defines a contravariant additive functor

$$\underline{Ju} : \underline{MF}_{E,1/2} \to \underline{pGS}_{\mathcal{O}_E,1/2,\text{strict}},$$

which turns out to be an anti-equivalence of categories. If the $\mathbb{Z}_p$-module underlying M is free, J is a Barsotti-Tate group.

8. - Let $\mathcal{O}'_{1/2}$ be the noncommutative ring generated over the commutative ring $\mathbb{Z}_{p^2}[u, u^{-1}]$ by an element φ with relations $\varphi^2 = pu$, $\varphi u = u\varphi$ and $\varphi a = \sigma a \cdot \varphi$, if $a \in \mathbb{Z}_{p^2}$ (where σ is the Frobenius). This ring contains $\mathbb{Z}_{p^2}$ and $\mathcal{O}_{1/2}$ as commutative subrings and can be identified as a $\mathbb{Z}_p$-module with

$\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \mathcal{O}_{1/2}$. For any finite $\mathbb{F}_{p^2}$-algebra A, there is unique structure of $\mathbb{Z}_{p^2}$-module on $BW_{1/2}(A)$ such that $[\varepsilon]\cdot(a_n)_{n\in\mathbb{Z}} = (\varepsilon a_n)_{n\in\mathbb{Z}}$ if $\varepsilon \in \mathbb{F}_{p^2}$ and $(a_n)_{n\in\mathbb{Z}} \in BW_{1/2}(A)$. Together with the structure of $\mathcal{O}_{1/2}$-module, $BW_{1/2}(A)$ becomes an $\mathcal{O}'_{1/2}$-module. Moreover, for any $\mathcal{O}_{1/2}$-module M, $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M$ has a natural structure of $\mathcal{O}'_{1/2}$-module and the obvious map

$$\mathrm{Hom}_{\mathcal{O}_{1/2}}(M, BW_{1/2}(A)) \to \mathrm{Hom}_{\mathcal{O}'_{1/2}}(\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M, BW_{1/2}(A))$$

is an isomorphism: we use it to identify these two groups.

9. - Let $F = \mathbb{Q}_{p^2}(\pi)$ be the Galois closure of E in $\overline{\mathbb{Q}}_p$; this is a subfield of $F_2 = \mathbb{Q}_{p^2}(\pi_2)$ and $\Gamma_2 = \mathrm{Gal}(F/\mathbb{Q}_p)$ acts on it. Recall (§11) that we have defined an isomorphism $\hat{\xi}$ from the inertia subgroup $I\Gamma$, of Γ_2 onto the group $\mu_{p^2-1}(\mathbb{Q}_{p^2})$ and called $\overline{\tau}$ the only nontrivial element of $\mathrm{Gal}(F_2/\mathbb{Q}_p(\pi_2))$.

Consider now a couple (M, gr) where M is an $\mathcal{O}_{1/2}$-module which is a $\mathbb{Z}_p$-module of finite type, and gr is a gradation on M indexed by $\mathbb{Z}/(p+1)\mathbb{Z}$,

$$M = \bigoplus_{s\in\mathbb{Z}/(p^2-1)\mathbb{Z}} gr^s M,$$

by sub-$\mathbb{Z}_p[u, u^{-1}]$-modules, such that $\varphi(gr^s M) \subset gr^{ps} M$ for all s. The group Γ_2 acts naturally on $\mathcal{O}'_{1/2} = \mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \mathcal{O}_{1/2}$ (via $g(a\otimes b) = a\otimes b$ if $g \in I\Gamma_2$ and $\overline{\tau}(a\otimes b) = \sigma a\otimes b$). We can define a semi-linear action of Γ_2 on the $\mathcal{O}'_{1/2}$-module $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M$ by setting, if $a \in \mathbb{Z}_{p^2}$ and $x \in gr^s M$,

$$\overline{\tau}(a\otimes x) = \sigma a \otimes x \text{ and } g(a\otimes x) = (\hat{\xi}_2(g))^s \cdot a \otimes x \text{ for all } g \in I\Gamma_2.$$

Let $\overline{\mathbb{Z}}_p$ be the integral closure of $\mathbb{Z}_p$ in the chosen algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. Define $BW_{1/2}(\overline{\mathbb{Z}}_p/\pi)$ to be the inductive limit of the $BW_{1/2}(\mathcal{O}_{F'}/\pi\mathcal{O}_{F'})$ for F' running through finite Galois extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and containing F. Set

$$J_{\mathbb{F}_p}(\overline{\mathbb{Z}}_p/\pi) = \mathrm{Hom}_{\mathcal{O}_{1/2}}(M, BW_{1/2}(\overline{\mathbb{Z}}_p/\pi)) = \mathrm{Hom}_{\mathcal{O}'_{1/2}}(\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M, BW_{1/2}(\overline{\mathbb{Z}}_p/\pi)).$$

This abelian group is equipped with an action of G_p: if

$$u : \mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M \to BW_{1/2}(\overline{\mathbb{Z}}_p/\pi)$$

is an $\mathcal{O}'_{1/2}$-linear map and if $\gamma \in G_p$, then $\gamma(u) = \gamma \circ u \circ \gamma^{-1}$.

10. - The group Γ_2 acts naturally on $\mathcal{O}_F$ and on $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \mathcal{O}_{E,1/2} = \mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \mathcal{O}_E[v, v^{-1}] = \mathcal{O}_F[v, v^{-1}]$ (with $gv = v$ if $g \in I\Gamma_2$ and $\overline{\tau}v = -v$). If (M, gr) is as above, the action of Γ_2 on $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M$ extends uniquely to a semi-linear action on $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} (\mathcal{O}_{E,1/2} \otimes_{\mathcal{O}_{1/2}} M)$. Define the category $\underline{MF}_{E/\mathbb{Q}_p,1/2}$ as follows:

-an object is a triple $(M, \wedge, gr)$ with $(M, \wedge)$ an object of $\underline{MF}_{E/\mathbb{Q}_p,1/2}$ and (M, gr) as above such that $\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} \wedge \subset \mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} (\mathcal{O}_{E,1/2} \otimes_{\mathcal{O}_{1/2}} M)$ is stable under Γ_2;

- a morphism is a morphism of the underlying objects of $\underline{MF}_{E,1/2}$ which is compatible with the gradations.

This category is abelian.

If $(M, \wedge, gr)$ is an object of $\underline{MF}_{E/\mathbb{Q}_p,1/2}$, and if J is the p-group scheme over $\mathcal{O}_E$ associated to $(M, \wedge)$, one can view $J(\overline{\mathbb{Z}}_p)$ (= the inductive limit of the $J(\mathcal{O}_{F'})$ for F' describing the finite Galois extensions of $\mathbb{Q}_p$ contained in $\overline{\mathbb{Q}}_p$ and containing F) as a subgroup of the group $J_{\mathbb{F}_p}(\overline{\mathbb{Z}}_p/\pi)$ defined above and the condition that $\wedge$ is stable under Γ implies that $J(\overline{\mathbb{Z}}_p)$ is stable under G_p.

If $\underline{\mathrm{Rep}}_{p-\mathrm{tor}}(G_p)$ denotes the category of p-torsion abelian groups V such that the kernel of multiplication by p is finite, equipped with a linear and continuous action of G_p, we can see $J(\overline{\mathbb{Z}}_p)$ as an object of this category.

The correspondence $(M, \wedge, gr) \mapsto J(\overline{\mathbb{Z}}_p)$ can be viewed as a contravariant additive functor

$$\underline{J}_{1/2} : \underline{MF}_{E/\mathbb{Q}_p,1/2} \to \underline{\mathrm{Rep}}_{p-\mathrm{tor}}(G_p).$$

11. - The function $\underline{J}_{1/2}$ is exact and faithful. Moreover, if $(M, \wedge, gr)$ and $(M', \wedge', gr')$ are two objects of $\underline{MF}_{E/\mathbb{Q}_p,1/2}$, the cokernel of the map

$$\mathrm{Hom}((M, \wedge, gr), (M', \wedge', gr')) \to \mathrm{Hom}(\underline{J}_{1/2}(M, \wedge, gr), \underline{J}_{1/2}(M', \wedge', gr'))$$

is killed by p.

If the $\mathbb{Z}_p$-module underlying M is free and if $J = \underline{J}(M, \wedge)$ is the corresponding Barsotti-Tate group over $\mathcal{O}_E$, $\underline{J}_{1/2}(M, \wedge, gr) = J(\overline{\mathbb{Z}}_p)$, when viewed as a $\mathrm{Gal}(\overline{\mathbb{Q}}_p/F_2)$-module. Therefore,

$$V = V_p((\underline{J}_{1/2})(M, \wedge, gr)) = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} \text{lim.proj} \cdot \underline{J}_{1/2}(M, \wedge, gr)_{p^n}$$

is a potentially Barsotti-Tate p-adic representation of G_p, hence a fortiori is potentially crystalline. It is F_2-semi-stable; the corresponding admissible filtered (φ, N, Γ_2)-module $D = \underline{D}^*_{\mathrm{st},F_2}(V)$ can be identified with $\underline{D}(M, \wedge, gr)$ defined as follows: the underlying $\mathbb{Q}_{p^2}$-vector space is $\mathbb{Q}_{p^2} \otimes_{\mathbb{Z}_p} M = \mathbb{Q}_p \otimes_{\mathbb{Z}_p} (\mathbb{Z}_{p^2} \otimes_{\mathbb{Z}_p} M)$ with the given action of φ and of Γ_2 and with $N = 0$; the filtration on

$$D_{F_2} = F_2 \otimes_{\mathbb{Q}_{p^2}} D = F_2 \otimes_E (E \otimes_{\mathbb{Q}_{p^2}} D) = F_2 \otimes_{\mathcal{O}_E} (\mathcal{O}_{E,1/2} \otimes_{\mathcal{O}_{1/2}} M)$$

is given by $\mathrm{Fil}^0 D_{F_2} = D_{F_2}$, $\mathrm{Fil}^1 D_{F_2} = F_2 \otimes_{\mathcal{O}_E} \wedge$, $\mathrm{Fil}^2 D_{F_2} = 0$.

12. - With these results in mind, the proof of the theorems becomes an exercise in the category $\underline{MF}_{E/\mathbb{Q}_p,1/2}$:

a) For each w.a. pst-module $D = D_I(0,1; a, d, i)$ or $D = D_{IV}(0,1; d, i_1, i_2; \alpha)$ one exhibits an object $(M, \wedge, gr)$ of $\underline{MF}_{E/\mathbb{Q}_p,1/2}$ such that $D \simeq \underline{D}(M, \wedge, gr)$; this gives us statement a) and b) of the theorem C2, hence also, as we already explained in the above remark, the admissibility statement of theorem C1 and the implication iv) $\Rightarrow$ i) of theorem C2, whose proof is completed. □

b) To this $(M, \wedge, gr)$, we can associate

$$T_p((\underline{J}_{1/2})(M, \wedge, gr)) = \text{lim.proj} \cdot \underline{J}_{1/2}(M, \wedge, gr)_{p^n}$$

which can be identified with a lattice T of $V = \underline{V}^*_{\mathrm{st}}(D)$ stable under G_p. Reducing $(M, \wedge, gr)$ mod p we can compute explicitly the two dimensional $\mathbb{F}_p$-representation $\overline{V} = T/pT = \underline{J}_{1/2}((M, \wedge, gr) \bmod p)$ of G_p and check the assertions of theorem C1 for $(r, s) = (0, 1)$. By twisting we deduce the assertions for $s - r = 1$ and the proof of theorem C1 is completed. □

c) To prove the theorem C3, we consider the category $\mathfrak{M}$ opposite to the full subcategory of $\underline{MF}_{E/\mathbb{Q}_p,1/2}$ whose objects are torsion objects. We can view an object of $\mathfrak{M}$ as a $\mathbb{Z}_p$-module N of finite length and an object $(M, \wedge, gr)$ of $\underline{MF}_{E/\mathbb{Q}_p,1/2}$ such that the $\mathbb{Z}_p$-module underlying M is $N^\wedge = \mathrm{Hom}_{\mathbb{Z}_p}(N, \mathbb{Q}_p/\mathbb{Z}_p)$ the Pontrjagin dual of N. Now, $\underline{J}_{1/2}$ can be viewed as a *covariant* additive exact functor from $\mathfrak{M}$ to $\underline{\mathrm{Rep}}^f_{\mathbb{Z}_p}(G_p)$. The proof then consists of playing the same game on $\mathfrak{M}$ that we played with the category $\underline{MF}^f_{]-p+1,0]}$ to prove the theorem B2 (see Appendix, §B). □

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