Schemes with D-structure

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1 D-structures

Everywhere rings are commutative with a 1.

Definition 1.1 *Let $\phi : A \to B$ be ring homomorphism. A Hasse derivation on $\phi : A \to B$ is a sequence $D = (D_i)_{i \in \mathbb{N}}$ of maps $A \to B$ such that $D_0 = \phi$ and for every $i \in \mathbb{N}$, D_i is additive and satisfies the Leibniz rule:*

$$D_i(xy) = \sum_{k+l=i} D_k(x) D_l(y).$$

A Hasse derivation on a ring A is a Hasse derivation on $id : A \to A$.

Definition 1.2 *A D-ring is a ring equipped with a Hasse derivation on itself. For A, B two D-rings, a D-homomorphism $\phi : A \to B$ is a homomorphism $\phi : A \to B$ such that for all $i \in \mathbb{N}$ and $a \in A$, $D_i(\phi(a)) = \phi(D_i(a))$. We say also that (B, D) is a D-algebra over (A, D).*
A D-ideal of a D-ring is an ideal which is stable under each D_i.

We give two alternative ways of looking at a Hasse derivation. The proofs are easy and omitted.

Proposition and definition 1.3 *Let D be a Hasse derivation on $\phi : A \to B$. Define $T_D : A \to B[[X]]$ by*

$$T_D(a) = \sum_{i \in \mathbb{N}} D_i(a) X^i.$$

Then T_D is a ring homomorphism. Conversely, each ring homomorphism $T : A \to B[[X]]$ gives rise to a unique Hasse derivation D over $\phi = T \bmod X$ such that $T = T_D$.

Proposition and definition 1.4 *Let D be a Hasse derivation on A. We extend $T_D : A \to A[[X]]$ to $\Sigma_D : A[[X]] \to A[[X]]$ by*

$$\Sigma_D(\sum_i a_i X^i) = \sum_{i,j} D_j(a_i) X^{i+j}.$$

Then Σ_D is a ring automorphism of $A[[X]]$ satisfying $\Sigma_D(a) = a \bmod X$ for all a in X. Conversely, each ring automorphism $\Sigma : A[[X]] \to A[[X]]$ satisfying $\Sigma(a) = a \bmod X$ gives rise to a unique Hasse derivation D on A such that $\Sigma = \Sigma_D$.

We can make the usual algebraic constructions in the category of D-rings (with D-homomorphims), see for example [Matsumura].

Fact 1.5

1. *The kernel of a D-homomorphism is a D-ideal.*

2. *Let I be a D-ideal of an D-ring A. There is a unique way to define a Hasse derivation on the quotient A/I such that the projection map is a D-homomorphism.*

3. *Let S be a multiplicative set of a D-ring A. There is a unique way to define a Hasse derivation on $S^{-1}A$ such that the natural map $A \to S^{-1}A$ is a D-homomorphism.*

4. *Let $(A_i)_{i\in I}$ be a directed system of D-rings, with D-homomorphisms. There is a unique way to define a Hasse derivation on the direct limit $\overset{lim}{\to} A_i$ such that the maps $A_i \to \overset{lim}{\to} A_i$ are D-homomorphisms.*

5. *Consider two D-algebras (B,D) and (C,D) over a D-ring (A,D). There is a unique way to define a Hasse derivation on the tensor product $B\otimes_A C$ such that the maps $B \to B\otimes_A C$ and $C \to B\otimes_A C$ are D-homomorphisms; it is given by*
$$D_j(x\otimes y) = \sum_{k+l=j} D_k(x)\otimes D_l(y).$$
This defines a tensor product in the category of D-algebras over (A,D).

Definition 1.6 *Let $\mu : R_0[[X]] \to R_0[[X]]\hat{\otimes}R_0[[X]]$ be an homomorphism defining a one dimensional formal group over some base ring R_0. Let A be an R_0-algebra equipped with a Hasse derivation D on itself. We say that D is μ-iterative if the following diagram commutes*

$$\begin{array}{ccc} A[[X]] & \xrightarrow{\mu_A} & A[[X]]\hat{\otimes}A[[X]] \\ \uparrow{\scriptstyle T_D} & & \uparrow{\scriptstyle id\otimes T_D} \\ A & \xrightarrow[T_D]{} & A[[X]] \end{array}$$

where $(id\otimes T_D)(\sum_i a_i X^i) = \sum_{i,j} D_j(a_i)X^i\otimes X^j$ and μ_A is obtained from μ by (continuous) base change.

Example 1.7 *If $\mu : X \mapsto X\otimes 1 + 1\otimes X$ is the additive formal group over $\mathbb{Z}$, a μ-iterative Hasse derivation is exactly an iterative Hasse derivation as defined in [HasseSchmidt].*

Lemma 1.8 *Let $f : (A,D) \to (B,D')$ be a D-homomorphism which is an homomorphism of R_0-algebras, and μ be a formal group over R_0. If D is μ-iterative, then $\mu_B\circ T_{D'}$ and $(id\otimes T_{D'})\circ T_{D'}$ coincide on Im(f).*

Proof : Since f is a D-homomorphism, $T_{D'}\circ f = f_{[[X]]}\circ T_D$, where $f_{[[X]]} : A[[X]] \to B[[X]]$ is deduced from f in the obvious way, and since f is an homomorphism of R_0-algebras, $\mu_B \circ f_{[[X]]} = f_{[[X]]\otimes[[X]]} \circ \mu_A$. Hence we can compute

$$(id \otimes T_{D'}) \circ T_{D'} \circ f = (id \otimes T_{D'}) \circ f_{[[X]]} \circ T_D = f_{[[X]]\otimes[[X]]} \circ (id \otimes T_D) \circ T_D =$$

$$f_{[[X]]\otimes[[X]]} \circ \mu_A \circ T_D = \mu_B \circ f_{[[X]]} \circ T_D = \mu_B \circ T_{D'} \circ f.$$

□

Corollary 1.9 *The constructions in Fact 1.5 give rise to μ-iterative derivations if we start from R_0-algebras equipped with a μ-iterative Hasse derivation.*

Proof: The cases of quotient, direct limit and tensor product are straightforward consequences of the previous lemma. Let us give some details for the case of localization. Let us denote the natural map by $i : A \to B = S^{-1}A$. The previous lemma gives us that $\mu_B \circ T_{D'} \circ i = (id \otimes T_{D'}) \circ T_{D'} \circ i =: h$. In order to prove that $\mu_B \circ T_{D'} = (id \otimes T_{D'}) \circ T_{D'}$, it suffices to show, because of the characterization of the localized ring, that $h(S)$ is contained in the set of invertible elements of $B[[X]]\hat{\otimes}B[[X]]$. But from one of the expressions of h, we get that for $s \in S$, $h(s) = \sum_{k,j} D'_k \circ D'_j(i(s))X^j \otimes X^k$, which is invertible in $B[[X]]\hat{\otimes}B[[X]]$ since its constant coefficient is $i(s)$ which is invertible in B. □

In the following, we will work:

- either in the category of D-rings with D-homomorphisms
- or, for a fixed formal group μ over a given base ring R_0, in the category of R_0-algebras equipped with a μ-iterative derivation, with D-homomorphisms which are homomorphisms of R_0-algebras as well

We usually will not insist on which category we are working in, and we will still call the later one the category of D-rings with D-homomorphisms. It is harmless in genearal, since from the previous corollary, the usual constructions that we want to do take place in the given category.

Definition 1.10 *Let X be a topological space and $\mathcal{F}$ a sheaf of rings on X. A Hasse derivation on $\mathcal{F}$ is a sequence $D = (D_i)_{i\in\mathbb{N}}$ of maps $\mathcal{F} \to \mathcal{F}$ such that for every open subset $U \subseteq X$, $D(U)$ is a Hasse derivation on $\mathcal{F}(U)$ such that for each open subsets $U \subseteq V \subseteq X$, the restriction map $\rho_{V,U} : \mathcal{F}(V) \to \mathcal{F}(U)$ is a D-homomorphism.*

We are now able to give the definition of the main notion of this paper.

Definition 1.11 *Let X be a scheme. A D-structure on X is a Hasse derivation on the structure sheaf $\mathcal{O}_X$.*
A morphism of schemes $f = ({}^tf, {}^sf) : X \to Y$ between two schemes (X, D^X) and (Y, D^Y) with a D-structure is a morphism of schemes with a D-structure if ${}^sf : {}^tf^*\mathcal{O}_Y \to \mathcal{O}_X$ is a D-homomorphism of sheaves. We also say that (X, D^X) is a scheme with a D-structure over (Y, D^Y), or that such a morphism is a point of (Y, D^Y) with value in (X, D^X), we denote it by $f \in (Y, D^Y)((X, D^X))$.
In particular, if (A, D) is a D-ring, $(Spec(A), D)$ is a scheme with a D-structure and we will speak about schemes with a D-structures over (A, D) or points with values in (A, D).

Proposition 1.12 *Fibered products exist in the category of schemes with a D-structure. More precisely, if (X, D^X) and (Y, D^Y) are schemes with a D-structure over (S, D^S), there is a unique D-structure on $X \times_S Y$ which turns it into the fibered product of (X, D^X) and (Y, D^Y) over (S, D^S).*

Proof : The case of affine schemes is a direct translation of the existence of tensor product in the category of D-rings. The passage from the affine case to the general case follows the same lines as the construction of fibered products, see [Hartshorne] for example. □

Definition 1.13 *Let S be a scheme with a D-structure. A group scheme G with a D-structure over S is a group scheme over S such that the underlying scheme is a scheme with a D-structure over S and such that the identity $e : S \to G$, the inverse $i : G \to G$ and the multiplication $m : G \times_S G \to G$ are morphisms of schemes with a D-structure. The morphisms in the category of group schemes with a D-structure over S are defined in the obvious way.*

Proposition 1.14 *Let (S, D^S) be a scheme with a D-structure, (X, D^X) a scheme with a D-structure over (S, D^S) and (G, D^G) a group scheme with a D-structure over (S, D^S). Let $\sigma : X \times_S G \to X$ be a morphism of schemes with a D-structure over (S, D^S) which is a group action (on the right). Assume that the quotient $p : X \to Y = X/G$ exists as an S-scheme. Then there is a unique D-structure D^Y on Y such that p is a morphism of schemes with a D-structure over (S, D^S).*

Proof : By hypothesis, Y is the quotient X/G at the level of underlying sets, with the quotient topology, and $\mathcal{O}_Y$ is isomorphic to the cokernel of the maps of sheaves on Y (cf [EGA, V]):

$$ {}^t p_* \mathcal{O}_X \underset{{}^s pr_X}{\overset{{}^s\sigma}{\rightrightarrows}} {}^t p_* \, {}^t \sigma_* \mathcal{O}_{X \times G} = {}^t p_* \, {}^t pr_{X*} \mathcal{O}_{X \times G}. $$

Since ${}^s\sigma$ and ${}^s pr_X$ are D-homomorphisms of sheaves, this cokernel is a sheaf of D-subrings of ${}^t p_* \mathcal{O}_X$. And p is a morphism of schemes with a D-structure if and only if $\mathcal{O}_Y$ is endowed with the Hasse derivation coming from the isomorphism with this cokernel. Note also that if the Hasse derivation on $\mathcal{O}_X$ is μ-iterative, the same is true for any sheaf of D-subrings, hence for $\mathcal{O}_Y$ as well. □

2 Prolongations

We use the formalism developped in [MoosaScanlon]. In this section, we work in the category of D-rings without iterativity condition.
For any D-ring R, we consider two R-algebra structures for the same ring. The standard one $s_n^R : R \to R_n^s := R[T]/(T^{n+1})$ is the usual inclusion. The exponential one $e_n^R : R \to R_n^e := R[T]/(T^{n+1})$ is given by the truncated Taylor map $T_D(r) = \sum_i D_i(r) T^i$.

Definition 2.1 *(R field, X quasi-projective (?)) The n-th prolongation of a scheme X over R is the scheme over R*

$$ \Delta_n X := \Pi_{R_n^s/R}(X \times_R R_n^e), $$

where Π *is the Weil restriction.*

In other words, $\Delta_n X$ represents the following functor, from the category of R-algebras to the category of sets:

$$A \mapsto \Delta_n X(A) = (X \times_R R_n^e)(A \otimes_R R_n^s).$$

Define the projection maps $\pi_{m,n} : \Delta_m X \to \Delta_n X$ for $m \geq n$.

Proposition 2.2 *Let* R *and* X *be as before.*
There is a 1-1 correspondance between:

- *the set of* D*-structures of* X *over* (R, D)
- *the set of sequences of morphisms over* R $(s_n : X \to \Delta_n X)_{n \geq 1}$ *satisfying* $s_0 = id_X$ *and for all* $m \geq n \geq 0$, $\pi_{m,n} \circ s_m = s_n$.

3 The functor of D-points

Definition 3.1 *Let* X *be a scheme with a* D*-structure. A point* $x \in X$ *is called a* D*-point if the maximal ideal of the local* D*-ring* $\mathcal{O}_{X,x}$ *is a* D*-ideal.*

Lemma 3.2

1. *Let* X *be a scheme with a* D*-structure, and* $x \in X$ *be a point in an affine open subset* $U \subseteq X$*. Then* x *is a* D*-point if and only if the corresponding prime ideal* I_x *of* $\mathcal{O}_X(U)$ *is a* D*-ideal.*
2. *Let* $f : X \to Y$ *be a morphism of schemes with a* D*-structure. Then* f *sends the* D*-points of* X *to the* D*-points of* Y*.*

Definition D-scheme.

Proposition and definition 3.3 *There is a functor* $(X, D) \mapsto (X, D)^\sharp$, $f \mapsto f^\sharp$ *from the category of schemes with a* D*-structure to the category of* D*-schemes, with a natural injective continuous map* $i : (X, D)^\sharp \to X$ *(as topological spaces). It is defined in the following way:*

- $(X, D)^\sharp$ *is the set of* D*-points of* X*, with the topology induced from the Zariski topology on* X
- $i : (X, D)^\sharp \to X$ *is the inclusion map*
- *The sheaf* $\mathcal{O}^D_{(X,D)^\sharp}$ *of* D*-rings on* $(X, D)^\sharp$ *is given by* $\mathcal{O}^D_{(X,D)^\sharp} = i^*(\mathcal{O}_X, D)$
- *If* $f = ({}^t f, {}^s f) : (X, D) \to (Y, D)$ *is a morphism of schemes with a* D*-structure,* $f^\sharp = ({}^t f^\sharp, {}^s f^\sharp)$ *is such that* ${}^t f^\sharp = {}^t f \circ i_X$ *is the restriction of* ${}^t f$ *on* $(X, D)^\sharp$ *and* ${}^s f^\sharp = i_X^* \, {}^s f : i_X^* \, {}^t f^* \mathcal{O}_Y \to i_X^* \mathcal{O}_X$*, with* $i_X^* \mathcal{O}_X = \mathcal{O}^D_{(X,D)^\sharp}$ *and* $i_X^* \, {}^t f^* \mathcal{O}_Y = {}^t f^{\sharp *} i_Y^* \mathcal{O}_Y = {}^t f^{\sharp *} \mathcal{O}^D_{(Y,D)^\sharp}$.

Proposition 3.4 *The image of* $(X, D)^\sharp$ *via the map* i *is Zariski-dense in* X*.*

Theorem 3.5 *The functor of* D*-points* $(X, D) \mapsto (X, D)^\sharp$ *induces an equivalence of categories between the category of locally entire schemes with a* D*-structure and the category of locally entire* D*-schemes.*

4 Classification of D-structures

5 Some descent results