Schemes for rings with a Hasse derivation

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Definition 1 *A Hasse derivation on a ring A is a sequence $D = (D_i)_{i \in N}$ of maps $A \longrightarrow A$ such that:*

- $D_0 = id_A$
- D_i *is additive for all* i
- $D_i(xy) = \sum_{m+n=i} D_m(x) D_n(y)$ *(generalized Leibniz rule)*
- $D_i \circ D_j = \binom{i+j}{i} D_{i+j}$ *(iteration rule)*

D-ring (or Hasse ring): commutative unitary ring with a Hasse derivation.

D-field (or Hasse field): commutative field with a Hasse derivation.

D-homomorphism: homomorphism which commutes with the Hasse derivation.

D-ideal: ideal stable by the application of the Hasse derivation.

$\{B\}$:= D-ideal generated by a subset B of a D-ring=ideal generated by the $D_i(b)$'s, $b \in B$, $i \geq 0$.

A model-theoretic motivation

We want an analogue of algebraically closed fields as universal domain among Hasse fields.

Definition 2 *A Hasse field K is existentially closed if every finite system of equations and inequations which has a solution in a Hasse field extension of K has a solution in K.*

Theorem 1 (Robinson-Ziegler) *For any p (0 or prime number), existentially closed Hasse fields of characteristic p are axiomatizable.*
For $p = 0$, existentially closed Hasse fields are differentially closed fields.
For $p > 0$, existentially closed fields are non perfect separably closed Hasse fields which are strict, i. e. $D_1(x) = 0$ iff $\exists y, x = y^p$.

An analogue of the Weil approach of algebraic varieties

D-algebraic (affine) variety: set of zeros of D-polynomials in a (saturated) closed Hasse field K

Some results analogue to the case of algebraic varieties:

- Analogue of Hilbert's Nullstellensatz

- Analogue of a theorem of Weil about constructible groups in an algebraically closed field:
  the category of connected infinitely definable groups in K is equivalent to the category of connected D-algebraic groups over K

Affine D-schemes

A a D-ring.
$V = Spec_D(A) :=$ set of prime D-ideals of A.
Topology on V: trace of the Zariski topology of $Spec(A)$ via the inclusion $Spec_D(A) \subseteq Spec(A)$.
Closed sets: $\mathcal{V}_D(B) := \{I \in Spec_D(A) \mid B \subseteq I\}$ for every $B \subseteq A$.
Basis of open sets: $\mathcal{D}(b) := \{I \in Spec_D(A) \mid b \notin I\}$ for $b \in A$.

Fact For $B \subseteq A$, $\sqrt{\{B\}}$ =intersection of all prime D-ideals containing B.
Corollary
- $\mathcal{V}_D(B) = \mathcal{V}_D(C)$ iff $\sqrt{\{B\}} = \sqrt{\{C\}}$
- $\mathcal{D}(b) \subseteq \bigcup_i \mathcal{D}(b_i)$ iff $b \in \sqrt{\{b_i\}_i}$
- each $\mathcal{D}(b)$ is compact for the induced topology

Structure sheaf

The structure sheaf $\mathcal{O}_V^D$ on V is induced by the structure sheaf of $Spec(A)$ via the inclusion $Spec_D(A) \subseteq Spec(A)$.
For $I \in V$, A_I has a natural structure of D-ring. It gives a natural structure of sheaf of D-rings to $\mathcal{O}_V^D$, with $\mathcal{O}_{V,I}^D \simeq A_I$.

$\widehat{A} := \mathcal{O}_V^D(V)$ is the D-ring of global sections.
Each $f \in \widehat{A}$ is given by a finite tuple $(a_i, b_i)_i$ such that:
- $V = \bigcup_i \mathcal{D}(b_i)$
- for $I \in \mathcal{D}(b_i)$, $f(I) = (a_i/b_i)_I$

An affine D-scheme is a ringed space in local D-rings of the form $(V, \mathcal{O}_V^D)$ for some $V = Spec_D(A)$.
A morphism of D-schemes is a morphism of ringed speaces in local D-rings.

$Spec_D$ is a contravariant functor from the category of D-rings into the category of affine D-schemes:
if $\phi : A \longrightarrow B$ is a D-homomorphism, we define

$$Spec_D(\phi) := ({}^t\phi, {}^s\phi) : Spec_D(B) \longrightarrow Spec_D(A) :$$

- for $I \in Spec_D(B)$, ${}^t\phi(I) = \phi^{-1}(I) \in Spec_D(A)$
- for $I \in Spec_D(B)$,

$$\begin{array}{rccc} {}^s\phi_I : & A_{{}^t\phi(I)} & \longrightarrow & B_I \\ & (a/b) & \mapsto & (\phi(a)/\phi(b)) \end{array}$$

$Spec_D(A)$ **vs** $Spec_D(\widehat{A})$

The natural D-homomorphism

$$\begin{array}{rccc} \iota_A : & A & \longrightarrow & \widehat{A} \\ & a & \mapsto & (I \mapsto a_I) \end{array}$$

is neither injective nor surjective in general.

However, does it induce an isomorphism

$$Spec_D(\iota_A) : Spec_D(\widehat{A}) \longrightarrow Spec_D(A)?$$

The "well mixed" case

Definition 3 *Let A be a D-ring.*
A proper D-ideal I of A is said to be well mixed if

$$ab \in I \Rightarrow aD_i(b) \in I, \forall i \geq 0.$$

The D-ring A is said to be well mixed if the 0 ideal is well mixed.

Remarks
- A is well mixed if every annihilators in A are D-ideal
- any radical D-ideal is well mixed
- there is a smallest well mixed D-ideal in A, denoted by 0_{wm}.

We denote by

$$\pi_A : A \longrightarrow A_{wm}$$

the projection of A onto the well mixed D-ring $A_{wm} := A/0_{wm}$.

Assume that A is well mixed. Then:

- the D-homomorphism ι_A is injective

- it is still true locally: for $b \in A$, the D-homomorphism

$$A_b \longrightarrow \mathcal{O}_V^D(\mathcal{D}(b))$$

  induced by ι_A is injective

- ι_A is almost surjective: for each prime D-ideal I of A, and $f \in \widehat{A}$, there are $a, b \in A$, with $b \notin I$, such that $\iota_A(b)f = \iota(a)$

Theorem 2 (Kovacic) *Assume A is well mixed. Then $Spec_D(\iota_A)$ is an isomorphism.*

Sketch of the proof

1. For $I \in Spec_D(A)$, we define

$$\mathcal{N}_A(I) := (eval_I)^{-1}(IA_I) \in Spec_D(\widehat{A}).$$

Then ${}^t\iota_A \circ \mathcal{N}_A = id_V$ without any assumption on A.

2. Since ι_A is almost surjective, ${}^t\iota_A$ is injective. Hence ${}^t\iota_A$ is an homeomorphism.

3. For $J \in Spec_D(\widehat{A})$, $({}^s\iota_A)_J$ is injective because of the description of the inverse $\mathcal{N}_A$ of ${}^t\iota_A$. It is surjective because ι_A is almost surjective.

A partial result in the general case

Proposition 1 *There is a commutative diagram of D-homomorphism*

$$\begin{array}{ccc} A & \xrightarrow{\iota_A} & \widehat{A} \\ {\scriptstyle \pi_A}\big\downarrow & & \big\downarrow{\scriptstyle \pi_{loc}} \\ A_{wm} & \xrightarrow[\iota_{A_{wm}}]{} & \widehat{A_{wm}} \end{array}$$

It induces a commutative diagram of homeomorphisms

$$\begin{array}{ccc} Spec_D(A) & \xleftarrow{{}^t\iota_A} & Spec_D(\widehat{A}) \\ {\scriptstyle {}^t\pi_a}\big\uparrow & & \big\uparrow{\scriptstyle {}^t\pi_{loc}} \\ Spec_D(A_{wm}) & \xleftarrow[{}^t\iota_{A_{wm}}]{} & Spec_D(\widehat{A_{wm}}) \end{array}$$

In particular, $Spec_D(A)$ and $Spec_D(\widehat{A})$ are homeomorphic as topological spaces.

Are they isomorphic as D-schemes ?

We know that the morphism of sheaves ${}^s\iota_A$ is injective without any assumption on A.

It is surjective if and only if, for every $J \in Spec_D(\widehat{A})$ and $I =^t \iota(J)$, the evaluation map $eval_I : \widehat{A}_J \longrightarrow A_I$ is injective.
A sufficient condition is that ι_A is almost surjective. We do not know any counter-examples for this property.