# Semi-abelian varieties over separably closed fields and maximal divisible subgroup

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## Motivation: the model-theoretic proof of the conjecture of Mordell-Lang

$G$ semi-abelian variety (commutative algebraic group of a special kind)
over $K$ a differentially closed field of characteristic 0 or a non perfect separably closed field of characteristic $p > 0$

**Definition 1** *$G^\sharp$ is the smallest type-definable subgroup of $G(K)$ which is Zariski-dense in $G$.*
*$G \mapsto G^\sharp$ defines a functor.*

In order to use the machinery of Zariski geometries on $G^\sharp$, a notion of dimension is required.

- In char 0: Morley rank ($DCF_0$ is $\omega$-stable)

- In char $p > 0$: U-rank ($SCF_{p,e}$ is not superstable, but $U(G^\sharp)$ is finite)

Is it possible to give a uniform treatment of these two cases by using relative Morley rank for type-definable sets?
Is it true that $G^\sharp$ has always a relative Morley rank in char $p$?

## $G^\sharp$ in char $p$

$K \models SCF_{p,1}$, $\aleph_1$-saturated

**Proposition 1** *$G^\sharp = p^\infty G(K) = \bigcap_{n\geq 0} p^n G(K)$ the biggest divisible subgroup of $G(K)$*

**Definition 2** *$\mathcal{C}$ subfield of constants of $K$: $\mathcal{C} = K^{p^\infty} = \bigcap_{n\geq 0} K^{p^n}$*

$\mathcal{C}$ is a pure algebraically closed field, with relative Morley rank 1.

A special case: if $G$ is defined over $\mathcal{C}$, $G^\sharp = G(\mathcal{C})$, hence has relative Morley rank equal to dim($G$).

**The structure of semi-abelian varieties**

A semi-abelian variety $G$ can be written inside an exact sequence

$$0 \to T \to G \to A \to 0 \quad (*)$$

$T = \mathbb{G}_m^d$ torus

$A$ abelian variety (i.e connected projective algebraic group)

**Remark:** $T^\sharp$ has relative Morley rank (it is defined over $\mathcal{C}$)

$A^\sharp$ has relative Morley rank (look at the case of simple abelian varieties and use an appropriate version of Zilber's indecomposability theorem)

**Theorem 1** *$G^\sharp$ has relative Morley rank $\Leftrightarrow$ the sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ induced by $(*)$ is exact.*

**Sketch of the proof**

$\Leftarrow$: it is a general fact about relative Morley rank.

$\Rightarrow$: the only problem may be that $T^\sharp \subsetneq T \cap G^\sharp$.

We can show that $T^\sharp$ is the connected component of $T \cap G^\sharp$, with $(T \cap G^\sharp)/T^\sharp$ torsion free because $T$ has no $p$-torsion. But if $G^\sharp$ has relative Morley rank, this quotient has to be finite.

**Question (arbitrary characteristic)**
Does the functor $G \mapsto G^\sharp$ preserve exact sequences?

In order to exhibit a counter-example, we prove:

**Theorem 2 (char 0 or $p$)** *Let $0 \to G_1 \to G_2 \to G_3 \to 0$ be an exact sequence of semi-abelian varieties over $K$. We assume moreover that they are ordinary in the positive characteristic case. If the sequence of $G_i$'s is exact, and if $G_1$ and $G_3$ descend to $\mathcal{C}$ (i.e are isomorphic to something defined over $\mathcal{C}$), then $G_2$ descends to $\mathcal{C}$.*

**Sketch of the proof**

- Char 0: uses "D-structures" and work by Buium and Bertrand-Pillay.

- Char $p$: uses p-torsion

**Lemma 1** *$G$ ordinary semi-abelian variety over $K$. For any $n \geq 0$, if the $p^n$-torsion $G[p^n] \subseteq G(K)$, $G$ descends to $K^{p^n}$.*

**Corollary 1** *$G$ as before. $G$ descends to $\mathcal{C}$ iff $T_pG(K) = T_pG$ (Tate-module of power of $p$ torsion points)*

We obtain the theorem by the fact that, if the sequence of $G_i$'s is exact, then the sequence of $T_pG(K)$'s is exact.

**Consequence (char $p$)**
There is a semi-abelian variety $G$, written as $0 \to \mathbb{G}_m \to G \to A \to 0$, with $A$ ordinary abelian variety over $\mathcal{C}$, such that $G^\sharp$ does not have relative Morley rank.

It uses the parametrization of such extensions $G$ by $\widehat{A}$, the dual abelian variety of $A$: a point in $\widehat{A}(K) \setminus \widehat{A}(\mathcal{C})$ corresponds to a semi-abelian variety over $K$ which does not descend to $\mathcal{C}$. From the previous, the sequence $0 \to T^\sharp \to G^\sharp \to A^\sharp \to 0$ is not exact and $G^\sharp$ does not have relative Morley rank.

**Some positive results**

**Proposition 2 (arbitrary char)** *Let $E$ be an elliptic curve which does not descend to $\mathcal{C}$, and $G$ a semi-abelian variety given by the exact sequence $0 \to T \to G \to E \to 0$. Then the $\sharp$-functor preserves this exact sequence.*

**Proposition 3 (char 0)** *Let $0 \to A_1 \to A_2 \to A_3 \to 0$ be an exact sequence of abelian varieties. Then the $\sharp$-functor preserves this exact sequence.*

**Remark** This last result is false in char $p$.