A theorem of the Kernel in characteristic p

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Notations $L = \mathbb{F}_p(t)^{sep}$, G is a semiabelian variety defined over L, $G^{\#} = p^{\infty}G(L)$

Question 1 *Is every point in $G^{\#}$ a torsion point?*

Update since the version from February 2011 Around the same time of the first version, Damian Rössler has given (independantly) a positive answer for Question 1 when G is an abelian variety, see [R13]. In the first version of these notes, there was a reduction from the semiabelian case to the abelian case, but the argument was quite obviously false. To my knowledge, the answer of Question 1 is still unknown in the semiabelian case. The argument given here can be seen as the baby step for the more involved argument from [R13]. On the other hand, it may also give a complete proof if the answer to Question 2 below is positive. D. Rössler also made some progress in this direction in [R13], since he proved that Question 2 has a positive answer when $\mathrm{End}_{\overline{L}}(G) = \mathbb{Z}$.

Some reductions and special cases

1. The result is true if G is defined over the field of constants $L^{p^\infty} = \mathbb{F}_p^{alg}$: We know that in this case, $G^{\#} = G(L^{p^\infty})$. As $\mathbb{F}_p^{alg}$ is a locally finite field, every element of $G(L^{p^\infty})$ is torsion.

2. If the result is true for simple abelian varieties, it is true for abelian varieties:
   It suffices to consider an isogeny from A to $A_1 \times \ldots \times A_n$, a product of simple abelian varieties, everything being defined over L. It sends $A^{\#}$ to $A_1^{\#} \times \ldots \times A_n^{\#}$, which is assumed to be torsion. As the kernel is finite, $A^{\#}$ is torsion.

So in the following we may assume that G is a simple abelian variety, of dimesion g, which does not descend to the constants.

Preparation Now we fix an embedding of G into $\mathbb{P}^d$, and R_0 a finitely generated subring of K over which G is defined. We also fix $x \in G^{\#}$ and consider the finitely generated field $K = \mathrm{Frac}(R_0[x])$. Let us consider $\mathcal{G}$ the group scheme over $R_0[x]$ whose generic fiber is G, by choosing an ideal M of $R_0[x]$ avoiding a finite number of datas, the reduction of $\mathcal{G}$ modulo M is an abelian variety. We finally define $R = (R_0[x])_M$ and $\mathcal{M}$ its maximal ideal. Then $K = \mathrm{Frac}(R)$ is a discrete valued field, with finite residue field $k = R/\mathcal{M}$, and $\mathcal{G}_R$ is an ablian scheme whith generic fiber G over K and special fiber $\overline{G}$ over k. We will denote by $(\hat{K}, \hat{R}, \hat{\mathcal{M}}, v)$ a completion of K, it is an immediate extension of K.

We will be able to prove the result under the following assumption:
(*) $x \in p^\infty G(\hat{K})$.

Proof We make use of the machinery developped in the part C.2 of [HS00] (developped a priori for number fields, but Theorem C.2.6 in particular seems to be generalizable to our context of a finite extension of $\mathbb{F}_p(t)$):
Consider the reduction map $\mathbb{P}^d(\hat{K}) \to \mathbb{P}^d(k)$, we obtain by restriction an exact sequence of abstract groups

$$0 \to G_1 \to G(\hat{K}) \to \overline{G}(k) \to 0.$$

By Theorem C.2.6 of [HS00], G_1 is isomorphic to $\mathcal{F}(\hat{\mathcal{M}})$, where $\mathcal{F}$ is the formal group over R associated to $\mathcal{G}_R$, of dimension g. Let N be the maximal ideal $(X_1, \ldots, X_g, Y_1, \ldots, Y_g)$ of $R[[X_1, \ldots, Y_g]]$; it is well known that the multiplication by p maps satisfies $[p]_i \in N^p$. In particular, if we denote, for $(x_1, \ldots, x_g)$ a tuple in $\hat{K}$, $w(x_1, \ldots, x_g) = \min\{v(x_i)\}$, we have for all $(x_1, \ldots, x_g) \in \hat{\mathcal{M}}^g$, $w([p](x_1, \ldots, x_g)) \geq pw(x_1, \ldots, x_g)$, and it follows that 0 is the only p^∞-divisible point in $\mathcal{F}(\hat{\mathcal{M}})$. Now we can conclude: as $\underline{G}(k)$ is finite, there is some n such that $[n]\overline{x} = 0$ (where $\overline{x}$ is the image of x in $\overline{G}(k)$ by the reduction map). Hence we have $[n]x \in G_1$, and $[n]x$ is p^∞-divisible (in $G(\hat{K})$). Let us choose a sequence $[n]x = y_0 \overset{[p]}{\leftarrow} y_1 \overset{[p]}{\longleftarrow} \ldots$, then for all i, $[p^i]\overline{y}_i = [n]\overline{x} = 0$ in $\overline{G}(k)$, which is finite; hence there is some i_0 such that $[p^{i_0}]\overline{y}_i = 0$ for all i, which implies that $[n]x$ is p^∞-divisible in G_1 (because $[p^{i_0}]y_i \in G_1$ and $[p^{i-i_0}]([p^{i_0}]y_i) = [n]x$ for $i \geq i_0$). By the previous argument, it implies that $[n]x = 0$. □

About the assumption (*)

Lemma 1 *Assume that $G(L)[p^\infty]$ is finite. Then $x \in p^\infty G(K)$.*

Proof Let us choose a sequence $x = x_0 \overset{[p]}{\leftarrow} x_1 \overset{[p]}{\longleftarrow} \ldots$ in $G(L)$. Recall that K is a field of definition of G, containing x, and that $L = K^{sep}$ (because of the assumption that $K \not\subseteq L^{p^\infty}$). For $\sigma \in Gal(L/K)$, $(0, x_1 - \sigma(x_1), x_2 - \sigma(x_2), \ldots) \in T_pG(L) = \{0\}$ because of the finiteness of the p^∞-torsion inside L. Hence every x_i is in K. □

If G is an elliptic curve over L, we know that either G descends to $\mathbb{F}_p^{alg}$, or $G(L)[p^\infty]$ is finite (see [BBP09]), which gives a positive answer to Question 1. A positive answer to the following question, which is open at my knowledge, would give a positive answer to Question 1 in general:

Question 2 *Let G be a simple abelian variety over a separably closed field of degree of imperfection 1. Then either G descends to L^{p^∞}, or $G(L)[p^\infty]$ is finite?*

References

[BBP09] Franck Benoist, Elisabeth Bouscaren and Anand Pillay. Semiabelian varieties over separably closed fields, maximal divisible subgroups, and exact sequences. *Preprint arXiv:0904.2083v1*, 2009.

[HS00] Marc Hindry and Joseph Silverman. *Diophantine geometry. An introduction.* Springer, 2000.

[R13] Damian Rössler. Infinitely p-divisible points on abelian varieties over function fields of characteristic $p > 0$. *Notre Dame Journal of formal logic 54*, 579-589, 2013.