18.726: the de Rham complex and topics in deformation theory

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Chapter 1

Kähler differentials and the de Rham complex

The goal of this chapter is to develop differential calculus on schemes. For what follows, the intuition comes directly from differential geometry – in the settings of manifolds, we know how to define tangent spaces, differential forms, differential of maps, etc. Perhaps surprisingly, these notions do admit very nice generalizations to the setting of morphisms between arbitrary schemes.

1.1 Kähler differentials

Let $X \to Y$ be a morphism of schemes. We want to construct a quasi-coherent sheaf of $\mathcal{O}_X$-modules $\Omega_{X/Y}$, the *sheaf of relative Kähler differentials*, together with a *derivation*

$$d : \mathcal{O}_X \to \Omega_{X/Y}.$$

In the world of differentiable manifolds, the sheaf $\Omega_{X/Y}$ is the quotient of the sheaf of differential forms of degree 1 on X by the subsheaf generated by the pullbacks of differential forms on Y.

There are at least two different ways to construct $\Omega_{X/Y}$. The first one, as in [Har83], II.8 and [BLR90], II.1, is to start with the affine case and study universal derivations in commutative algebra. One can then use glueing argument to get to the general case.

The other way is the one used by Grothendieck and Dieudonné in [Gro67], paragraph 16, where Kähler differentials are constructed in a direct way for arbitrary morphisms of schemes. We will follow this approach.

1.1.1 A special case: the Zariski tangent space and the ring of dual numbers

As a motivation for what follows and an important special case, we start by discussing the Zariski tangent space of a point in a scheme.

Definition 1.1.1. *Let X be a scheme, and let x be a point of X. Let k be the residue field of x. Let $\mathcal{O}_{X,x}$ be the local ring of X at x, and let $\mathfrak{m}_x$ be its maximal ideal.*

The Zariski cotangent space *of X at x is the k-vector space $T_x^*X = \mathfrak{m}_x/\mathfrak{m}_x^2$. The* Zariski tangent space *of X at x T_xX is its dual.*

For this definition to make sense, it is necessary to check that the $\mathcal{O}_{X,x}$-module $\mathfrak{m}_x/\mathfrak{m}_x^2$ is indeed a $k = \mathcal{O}_{X,x}/\mathfrak{m}_x$-vector space, which is obvious.

Remark 1.1.2. *This definition is perhaps not entirely satisfactory. Indeed, if X is a scheme over a field K, we will define later the cotangent sheaf $\Omega_{X/K}$ of X over K. The fiber of $\Omega_{X/K}$ at x coincides with the Zariski cotangent space of X at x only if x is a rational point of X, i.e., if $k = K$.*

The Zariski cotangent space of a scheme at a point is easy to compute in most cases.

Example 1.1.1.1. *Let k be a field.*

- *The Zariski cotangent space of $\mathbb{A}_k^n$ at 0 is of dimension n, and is canonically identified with the space of linear functions over k in n variables.*
- *Let $X = \operatorname{Spec} k[X,Y]/(XY)$. The Zariski (co)tangent space of X at any point x is one-dimensional, except if $x = 0$ in which case it has dimension 2: the jump of dimension corresponds to the fact that the scheme is singular at $x = 0$.*
- *The same phenomenon can appear for irreducible schemes, e.g. $X = \operatorname{Spec} k[X,Y]/(Y^2 - X^2(X-1))$.*

The functorial behavior of the Zariski cotangent space is as follows. Let $f : X \to Y$ be a morphism of schemes. Let x be a point of X and let y be its image under f. The morphism f induces a morphism of local rings

$$\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$$

that maps $\mathfrak{m}_y$ to $\mathfrak{m}_x$. As a consequence, it induces a morphism – the cotangent map –

$$df : T_y^*Y \to T_x^*X$$

that is linear with respect to the natural inclusion of residue fields

$$\kappa(y) \hookrightarrow \kappa(x).$$

It is to be noted that while the cotangent map is always well-defined by the construction above, the tangent map from T_xX to T_yY is only defined when x and y have the same residue field. This should be considered as an indication that – perhaps in a slightly surprising way – the cotangent space is more natural than the tangent space.

The discussion above can be rephrased using the *ring of dual numbers.*

Definition 1.1.3. *Let k be a field. The* ring of dual numbers *over k is $k[\epsilon]/(\epsilon^2)$.*

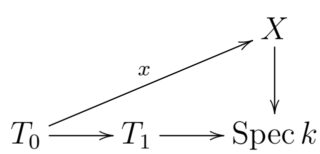
We define $T_0 = \operatorname{Spec} k$ and $T_1 = \operatorname{Spec} k[\epsilon]/(\epsilon^2)$. The scheme T_1 is a zero-dimensional scheme of length 2 over k. The morphism

$$k[\epsilon]/(\epsilon^2) \to k, \epsilon \mapsto 0$$

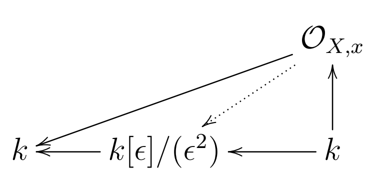
corresponds to a closed immersion $T_0 \hookrightarrow T_1$. This is a basic example of what we will call below a *first-order infinitesimal thickening.*

The topological space underlying T_1 is reduced to a point, but the scheme T_1 is not reduced. One should regard it as a point with an "infinitesimal tangent vector". We want to show that this intuition is compatible with our definition of the Zariski tangent space of a k-scheme at a rational point.

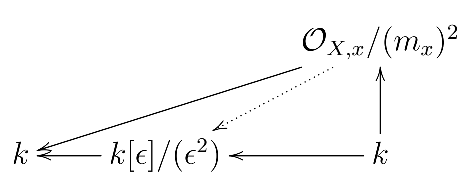
Let X be a scheme over k, and let $x \in X(k)$ be a k-point of X. In the following commutative diagram



we consider the possible lifts of the map $x : T_0 \to X$ to a map $T_1 \to X$. This is equivalent to finding the possible dotted arrows in the diagram



Furthermore, since the map from $\mathcal{O}_{X,x}$ to k maps $\mathfrak{m}_x$ to 0, the dotted map sends $\mathfrak{m}_x$ to (ϵ), so it sends $(\mathfrak{m}_x)^2$ to zero. The diagram becomes



Now we rewrite

$$\mathcal{O}_{X,x}/(m_x)^2 \simeq k \oplus T_x^*X,$$

where the multiplication rule is given by

$$(a,b)(a',b') = (aa', ab' + a'b).$$

Note that the multiplication rule only depends on the k-vector space structure of T_x^*X. Similarly,

$$k[\epsilon]/(\epsilon^2) = k \oplus k\epsilon.$$

As a consequence, completing the diagrams above with a dotted arrow is the same as giving a map

$$T_x^*X \to k,$$

i.e., is the same as giving an element of the Zariski tangent space T_xX.

Everyhing above can be rephrased in a much more general setting.

1.1.2 Aside: regular schemes

We will soon define the notion of a smooth morphism and a smooth scheme over a base. This is a relative notion. While related, regularity is an absolute notion and does not depend on a base scheme.

We start by a general theorem in commutative algebra that we do not prove completely.

Theorem 1.1.4. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. Then*

$$\dim A \leq \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Proof. Since A is finitely generated, the dimension of $\mathfrak{m}/\mathfrak{m}^2$ over k is finite. Let n be this dimension. We can find elements $f_1, \ldots, f_n$ of A whose images in $\mathfrak{m}/\mathfrak{m}^2$ form a basis of $\mathfrak{m}/\mathfrak{m}^2$. By Nakayama's lemma, we have $(f_1, \ldots, f_n) = \mathfrak{m}$. By Krull's Hauptidealsatz [Mat86], 13.6 (ii), this implies that the dimension of A is at most n. □

Definition 1.1.5. *Let A be a noetherian local ring with maximal ideal $\mathfrak{m}$ and residue field k. We say that A is a* regular local ring *if*

$$\dim A = \dim_k \mathfrak{m}/\mathfrak{m}^2.$$

Let X be a locally noetherian scheme. We say that X is a regular scheme *if all the local rings of X are regular local rings.*

The notion of regularity for schemes is very important in applications. The following is due to Serre.

Theorem 1.1.6. *Any localization of a regular local ring is regular. Any regular local ring is a unique factorization domain.*

The theorem above implies in particular that if the closed points of a scheme are dense, regularity can be checked at the closed points. A scheme that is not regular is *singular.*

1.1.3 The sheaf of relative Kähler differentials

We now give a direct – but not so intuitive at first sight – construction of the sheaf of Kähler differentials. The reader will notice that this does not have to do so much with schemes and would work in more general categories, see the presentation in the Stacks Project (the case of schemes is treated in 01UM) and of course [Ill71, Ill72].

Definition 1.1.7. *A morphism of schemes $i : T_0 \to T$ is a* first-order thickening *– or equivalently, T is a first-order thickening of T_0 – if i is a closed immersion defined by a sheaf of ideals of square zero.*

The notion of n-th order thickening should be obvious.

With the notations of the preceding definition, T_0 and T have the same underlying topological space – the only difference is their structure sheaf. In particular, the sheaves on T_0 and the sheaves on T are the same. The structure sheaf $\mathcal{O}_T$ of T is a sheaf of $\mathcal{O}_{T_0}$-algebras, and we have an exact sequence

$$0 \to \mathfrak{a} \to \mathcal{O}_T \to \mathcal{O}_{T_0} \to 0. \tag{1.1.1}$$

The sheaf of ideals $\mathfrak{a}$ is a sheaf of $\mathcal{O}_T$-modules. Since $\mathfrak{a}$ is killed by $\mathfrak{a}$ by assumption, it is a sheaf of $\mathcal{O}_T/\mathfrak{a} = \mathcal{O}_{T_0}$-modules.

First-order thickenings appear in geometric situations as follows. Let $j : X \hookrightarrow Z$ be an immersion with ideal $\mathcal{I}$. This means that j factors as

$$X \to U \to Z,$$

where the first map is a closed immersion and the second one is an open immersion, and where $\mathcal{I}$ is a sheaf of ideals on U. Let Z_1 be the subsheme of Z defined by the sheaf $\mathcal{I}^2$. Then we can factor j as

$$X \xrightarrow{j_1} Z_1 \xrightarrow{h} Z$$

where j_1 is a first-order thickening with ideal $\mathcal{I}/\mathcal{I}^2$, and h is an immersion.

Remark 1.1.8. *It is important to note that the sheaf of $\mathcal{O}_X$-modules $\mathcal{I}/\mathcal{I}^2$ can be identified with $j^*\mathcal{I}$.*

Definition 1.1.9. *With the notations above, we say that the pair* (j_1, h) *– or simply* Z_1 *– is the* first infinitesimal neighborhood of X in Z.

The quasi-coherent sheaf of $\mathcal{O}_X$*-modules*

$$\mathcal{C}_{X/Z} := \mathcal{I}/\mathcal{I}^2$$

is the conormal sheaf *of* X *in* Z *– or sometimes, the conormal sheaf of* j*. The sheaf*

$$\mathcal{N}_{X/Z} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_X)$$

is the normal sheaf to X in Z.

The conormal sheaf is sometimes written $\mathcal{N}^\vee_{X/Z}$. This is a slightly misleading notation, as it is not defined as the dual of the normal sheaf. In case the conormal sheaf is locally free of finite type on X, then this is the case and the notation does not induce any confusion.

The reader should formulate and prove the universal property of the first infinitesimal neighborhood with respect to first-order thickenings.

Remark 1.1.10. *The normal sheaf – or, better, the normal bundle, i.e. the relative spectrum of the symmetric algebra built on* $\mathcal{I}/\mathcal{I}^2$ *– should be thought of as a version of the tubular neighborhood of* X *in* Z*. A version closer to geometry is that of the* normal cone $C_{X/Z}$*, i.e. the relative spectrum of the sheaf of graded algebras* $\bigoplus_n \mathcal{I}^n/\mathcal{I}^{n+1}$*. We will not discuss the geometry of the normal cone in these notes, but it has great geometric significance and does play a important role in intersection theory [Ful98].*

If the immersion $X \to Z$ *is unramified – see below – then the normal cone and the normal bundle of* X *in* Z *are the same object.*

Let $f : X \to S$ be a morphism of schemes, and let $\Delta : X \hookrightarrow X \times_S X$ be the diagonal morphism. It is an immersion – closed if f is separated. Let X_1 be the first-order neighborhood of X in $X \times_S X$.

Definition 1.1.11. *The conormal sheaf of* Δ *is the* sheaf of relative Kähler differentials of f *and is denoted by* $\Omega^1_{X/S}$.

The structure sheaf of X_1*, seen as a sheaf of rings on* X*, is the* sheaf of first order principal parts of X over S*, and is denoted by* $\mathcal{P}^1_{X/S}$.

There is of course a sheaf of n-th order principal parts of X over S. If S is the spectrum of a ring A, we will sometimes write $\Omega^1_{X/A}$ instead of $\Omega^1_{X/S}$.

The exact sequence (1.1.1) becomes in the setting above

$$0 \to \Omega^1_{X/S} \to \mathcal{P}^1_{X/S} \to \mathcal{O}_X \to 0. \tag{1.1.2}$$

Let $p_1, p_2 : X \times_S X \to X$ be the two projections. Since the composition

$$X \longrightarrow X_1 \xrightarrow{p_i} X$$

is the identity for $i = 1, 2$, they both endow the sheaf $\mathcal{P}^1_{X/S}$ with the structure of a sheaf of $\mathcal{O}_X$-algebras. We now consider $\mathcal{P}^1_{X/S}$ as an $\mathcal{O}_X$-algebra *via the algebra structure induced by* p_1.

Definition 1.1.12. *We denote by* $d^1_{X/S}$*, or simply by* d^1*, the ring morphism*

$$d^1 : \mathcal{O}_X \to \mathcal{P}^1_{X/S}$$

induced by p_2*. If* t *is any section of* $\mathcal{O}_X$*, we say that* d^1t *is* the first-order principal part of t.

We write $dt = d^1t - t$*, and we say that the morphism of sheaves of abelian groups*

$$d = d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$$

is the differential.

In the definition above, the differential d sends a priori $\mathcal{O}_X$ to $\mathcal{P}^1_{X/S}$. Since d^1 is a section of the canonical map $\Omega^1_{X/S} \to \mathcal{O}_X$, its image is actually contained in $\Omega^1_{X/S}$.

Proposition 1.1.13. *1. The sheaf* $\Omega^1_{X/S}$ *is a quasi-coherent sheaf.*

2. The morphism $d : \mathcal{O}_X \to \Omega^1_{X/S}$ *is* $f^{-1}(\mathcal{O}_Y)$*-linear and satisfies the* Leibniz rule

$$d(t_1t_2) = t_1dt_2 + t_2dt_1.$$

Proof. The first statements is a consequence of the general theory. To prove the second one, consider the exact sequence (1.1.2). It endows the sheaf $\Omega^1_{X/S}$ with the structure of a square-zero sheaf of ideals in $\mathcal{P}^1_{X/S}$. In particular, this means that $dt_1dt_2 = 0$. Expanding and using the fact that d^1 is a morphism of rings gives the result. □

To say more about $\Omega^1_{X/S}$ – in particular, to work out its functoriality properties – it is useful to prove a functorial characterization.

Definition 1.1.14. *If* $\mathcal{M}$ *is any* $\mathcal{O}_X$*-module, an* S-derivation *is a morphism*

$$D : \mathcal{O}_X \to \mathcal{M}$$

satisfying the third condition of Proposition 1.1.13: D *is* $f^{-1}(\mathcal{O}_Y)$*-linear and satisfies the Leibniz rule.*

We denote by $Der_S(\mathcal{O}_X, \mathcal{M})$ *the set of* S*-derivations.*

Note that a derivation is *not* a morphism of $\mathcal{O}_X$-modules.

In the next sections, we will prove the following theorem.

Theorem 1.1.15. *The differential*

$$d_{X/S} : \mathcal{O}_X \to \Omega^1_{X/S}$$

is a universal S-derivation.

In other words, given any S-derivation

$$D : \mathcal{O}_X \to \mathcal{M},$$

there exists a unique morphism of $\mathcal{O}_X$-modules $u : \Omega^1_{X/S} \to \mathcal{M}$ such that $D = u \circ d_{X/S}$.

As usual, the universal property above will give easily standard functoriality of the sheaf of Kähler differentials. We will spell these out later. For now, we simply explain the consequence of the theorem for the tangent sheaf.

Definition 1.1.16. *The* relative tangent sheaf of X over S, *denoted by $T_{X/S}$, is the sheaf*

$$T_{X/S} = \mathcal{H}om(\Omega^1_{X/S}, \mathcal{O}_X).$$

As an immediate consequence of Theorem 1.1.15

Corollary 1.1.17. *There is a canonical isomorphism*

$$H^0(X, T_{X/S}) \simeq Der_S(\mathcal{O}_X, \mathcal{O}_X).$$

This corresponds to the well-known fact that vector fields correspond to derivations.

1.1.4 Split first-order thickenings

Definition 1.1.18. *Let X be a scheme and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The* split first-order thickening *associated to $\mathcal{M}$ is the closed immersion*

$$X \to X[\mathcal{M}],$$

where $X[\mathcal{M}]$ is the relative spectrum of the sheaf of $\mathcal{O}_X$-algebras $\mathcal{O}_X \oplus \mathcal{M}$, the multiplication rule being defined by $(a, b)(a', b') = (aa', ab' + a'b)$, and where the closed immersion has ideal sheaf $\mathcal{M}$.

Lemma 1.1.19. *Let $j : X \to X_1$ be a first-order thickening. Then j is a split first-order thickening if and only if j admits a section, i.e., there exists a morphism $h : X_1 \to X$ such that $h \circ j = \mathrm{Id}_X$.*

Proof. Certainly, any split first-order thickening admits a section via the morphism $\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{M}$.

Conversely, assume that j admits a section. Then the exact sequence

$$0 \to \mathfrak{a} \to \mathcal{O}_T \to \mathcal{O}_{T_0} \to 0,$$

where $\mathfrak{a}$ is the ideal of j, splits, and since $\mathfrak{a}$ has square zero, it is readily seen that $X \to X_1$ is isomorphic to $X \to X[\mathfrak{a}]$. □

Let $X \to S$ be a morphism of schemes. The first order neighborhood X_1 of the diagonal in $X \times_S X$ is a split first-order thickening of X – as we saw above, any of the two projections induces a splitting.

The following proposition certainly implies Theorem 1.1.15.

Proposition 1.1.20. *Let $X \to S$ be a morphism of schemes, and let $\mathcal{M}$ be an $\mathcal{O}_X$-module. The following three sets are in natural bijections*

1. *$Der_S(\mathcal{O}_X, \mathcal{M})$;*
2. *The set of sections over S of the closed immersion $X \to X[\mathcal{M}]$;*
3. *$\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{M})$.*

Furthermore, the bijection between the last and the first set is the map $u \mapsto u \circ d_{X/S}$.

An immediate corollary is the following.

Corollary 1.1.21. *Let $f : X \to S$ be a morphism of schemes, and let $X \to X_1$ be a first-order thickening with ideal $\mathcal{I}$. Then the set of sections of j over S is either empty or a torsor under $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{I})$.*

Proof of Proposition 1.1.20. The proof is essentially the same as what was done in 1.1.1. We first construct the bijection between the first two sets.

Let j be the closed immersion $X \to X[\mathcal{M}]$. Sections of j over S correspond to morphism of $f^{-1}(\mathcal{O}_S)$-algebras

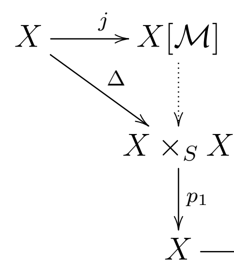
$$\mathcal{O}_X \to \mathcal{O}_X \oplus \mathcal{M}$$

that split the projection $\mathcal{O}_X \oplus \mathcal{M} \to \mathcal{O}_X$. In turn, these correspond to $f^{-1}(\mathcal{O}_S)$-linear maps $\phi : \mathcal{O}_X \to \mathcal{M}$ such that for every local sections a and a' of $\mathcal{O}_X$, we have $(a, \phi(a))(a', \phi(a')) = (aa', \phi(aa'))$, i.e.

$$a\phi(a') + a'\phi(a) = \phi(aa'),$$

which is the Leibniz rule.

We now show that the second and third sets are in bijection. By the universal property of the diagonal morphism, giving a section of j over S is the same as giving the dotted arrow in the commutative diagram



Since $\mathcal{M}$ has square zero, the dotted arrow has to factor through the first neighborhood of the diagonal $X_1 = X[\Omega^1_{X/S}]$ where the equality holds by definition of $\Omega^1_{X/S}$. As a consequence, the set of sections of j is in bijection with the set of X-morphisms

$$X[\mathcal{M}] \to X[\Omega^1_{X/S}],$$

which in turn is in bijection with $\mathrm{Hom}_{\mathcal{O}_X}(\Omega^1_{X/S}, \mathcal{M})$ by the universal property of the relative spectrum of sheaves of algebras.

It is not difficult – and not entirely uninteresting – to track the arrows in the construction above to prove the last statement. □

1.1.5 Basic properties of Kähler differentials

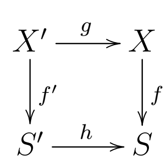
Let $f : X \to S$ be a morphism of schemes. Using Theorem 1.1.15, it is not difficult to collect basic properties of $\Omega^1_{X/S}$.

Proposition 1.1.22. *The $\mathcal{O}_X$-module $\Omega^1_{X/S}$ is generated by the image of $d_{X/S}$.*

Proof. It is straightforward to see that the $\mathcal{O}_X$-module generated by the image of $d_{X/S}$ satisfies the same universal property as $\Omega^1_{X/S}$. As a consequence, they are equal. □

The following results are fundamental. They all are easily proved using the definition of the sheaf of relative differentials and its universal property. We strongly encourage the reader to understand their proof.

Theorem 1.1.23. *Any commutative diagram of schemes*



defines a natural morphism of $\mathcal{O}_{X'}$-modules

$$g^*\Omega^1_{X/S} \to \Omega^1_{X'/S'}$$

that commutes with the differentials.

If the diagram is cartesian – i.e., if $X' = X \times_S S'$, then the morphism above is an isomorphism, and the canonical map

$$g^*\Omega^1_{X/S} \oplus h^*\Omega^1_{S'/S} \to \Omega^1_{X'/S}$$

is an isomorphism.

Corollary 1.1.24. *Assume that S is noetherian and $f : X \to S$ is a morphism of finite type. Then $\Omega^1_{X/S}$ is coherent.*

Proof. Using Theorem 1.1.23 allows us to reduce the question to the affine case, where the result is easy using definition 1.1.11.

This is also a direct consequence of Proposition 1.1.22 and the fact that the differential is $f^{-1}(\mathcal{O}_S)$-linear. □

Theorem 1.1.25. *Let*

$$X \xrightarrow{f} Y \xrightarrow{g} S$$

be morphism of schemes. Then the canonical sequence of morphisms

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact.

In the theorem above, the first map is defined by Theorem 1.1.23 and the second one can be defined using the universal property.

Theorem 1.1.26. *Let*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \downarrow & \swarrow & \\ S & & \end{array}$$

be a commutative diagram where i is an immersion. Then there is an exact sequence, called the conormal exact sequence

$$\mathcal{C}_{X/Y} \to i^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0.$$

Note that a consequence of the two preceding theorems is that the sheaf of relative differentials associated to an immersion is trivial.

Proof of Theorem 1.1.26. The second morphism is the one that appears in Theorem 1.1.25. We describe the first map. After replacing Y by an open subscheme, we can assume that i is a closed immersion with sheaf of ideals $\mathcal{I}$. By the Leibniz rule, the differential $d_{Y/S} : \mathcal{O}_Y \to \Omega^1_{Y/S}$ sends $\mathcal{I}^2$ to $\mathcal{I}\Omega^1_{Y/S}$. As a consequence, it induces a morphism

$$\mathcal{C}_{X/Y} = \mathcal{I}/\mathcal{I}^2 \to \Omega^1_{Y/S}/\mathcal{I}\Omega^1_{Y/S} = i^*\Omega^1_{Y/S}.$$

Concretely – and in a strictly equivalent way – this means the following : consider a local section f of $\mathcal{I}$ – f is a function on Y that vanishes on X. Then df is an element of $\Omega^1_{Y/S}$. Furthermore, the restriction of df to $i^*\Omega^1_{Y/S}$ only depends on the class of f modulo $\mathcal{I}^2$ by the Leibniz rule.

Using the description above and reducing to the affine case, it would be possible to prove the conormal exact sequence by an explicit computation. However, we follow the global argument of [Gro67], 16.4.20 and 16.4.21, as it emphasizes the role of the sheaf of principal parts.

We start by computing $i^*\mathcal{P}^1_{Y/S}$. Let ψ_1 and ψ_2 be the two morphisms of algebras $\psi_i : \mathcal{O}_Y \to \mathcal{P}^1_{Y/S}$ induced by the two projections $p_i : Y \times_S Y \to S$. With the notations above, ψ_1 is the structure morphism of $\mathcal{P}^1_{Y/S}$ seen as an $\mathcal{O}_Y$-algebra, and $\psi_2 = d^1_{Y/S}$. As a consequence, the $\mathcal{O}_X$-algebra $i^*\mathcal{P}^1_{Y/S}$ can be identified to $\mathcal{P}^1_{Y/S}/\psi_1(\mathcal{I})\mathcal{P}^1_{Y/S}$, and its quotient by the image of $\mathcal{I}/\mathcal{I}^2 = i^*\mathcal{I}$ under d^1 – or equivalently here, d – with the sheaf

$$\mathcal{P}^1_{Y/S}/(\psi_1(\mathcal{I}) + \psi_2(\mathcal{I}))\mathcal{P}^1_{Y/S}. \tag{1.1.3}$$

Since i is an immersion, the diagonal morphism $X \to X \times_Y X$ is an isomorphism. We have a cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{\ i\ } & Y \\ \downarrow{\scriptstyle\Delta} & & \downarrow{\scriptstyle\Delta} \\ X \times_S X & \xrightarrow{i \times_S i} & Y \times_S Y \end{array}$$

which induces a cartesian diagram

$$\begin{array}{ccc} X_1 & \longrightarrow & Y_1 \\ \downarrow & & \downarrow \\ X \times_S X & \xrightarrow{i \times_S i} & Y \times_S Y \end{array}$$

between the first-order neighborhoods of the diagonal. This means precisely that

$$\mathcal{P}^1_{X/S} = \mathcal{P}^1_{Y/S} \otimes_{\mathcal{O}_{Y\times_S Y}} \mathcal{O}_{X\times_S X}. \tag{1.1.4}$$

To conclude the computation, it is readily checked that we have a canonical identification

$$\mathcal{O}_{X\times_S X} = O_{Y\times_S Y}/(p_1^*(\mathcal{I}) \oplus p_2^*(\mathcal{I}))\mathcal{O}_{Y\times_S Y} = O_{Y\times_S Y}/(\psi_1^*(\mathcal{I}) \oplus \psi_2^*(\mathcal{I}))\mathcal{O}_{Y\times_S Y}$$

by definition of ψ_1 and ψ_2. Comparing (1.1.3) and (1.1.4) now gives the result. □

Remark 1.1.27. *Theorem 1.1.25 and Theorem 1.1.26 both are special cases of a general theorem that uses the* cotangent complex $\mathbb{L}_{X/S}$ *of a morphism. The cotangent complex is a (quasi-isomorphism class of) complex(es) of quasi-coherent sheaves concentrated in negative degrees, with degree* 0 *cohomology being the usual sheaf of relative differentials.*

Given diagrams of maps as in the preceding two Theorems, the cotangent complexes fit in a long exact sequence that extends the sequence of Theorem 1.1.25 on the left. The conormal exact sequence can be rephrased – and slightly extended – by saying that the cotangent complex of an immersion $X \to Y$ – more generally, of an unramified morphism, see below – is quasi-isomorphic to $\mathcal{C}_{X/Y}[1]$.

The results above are of fundamental importance, as they allow for the computation of the modules of relative Kähler differentials. We give the basic example.

Proposition 1.1.28. *Let S be a scheme, n a nonnegative integer, and let $X = \mathbb{A}^n_S = \operatorname{Spec} \mathcal{O}_S[X_1, \ldots, X_n]$ be the affine space of dimension n over S. Then the $\mathcal{O}_S$-module $\Omega^1_{X/S}$ is free of rank n with basis $(dX_1, \ldots, dX_n)$.*

The differential

$$d : \mathcal{O}_X \to \Omega^1_{X/S}$$

sends a local section s, considered as a polynomial in the X_i with coefficients in $\mathcal{O}_S$, to

$$ds = \sum_i \frac{\partial s}{\partial X_i} dX_i.$$

Proof. By proposition 1.1.22, we know that the $\Omega^1_{X/S}$ is generated as an $\mathcal{O}_X$-module by the ds, where s is a section of $\mathcal{O}_X$. Any section of $\mathcal{O}_X$ can be written as a polynomial in the X_i with coefficients in $\mathcal{O}_S$. Since the differential is $\mathcal{O}_S$-linear and satisfies the Leibniz rule, it is easily seen that the formula

$$ds = \sum_i \frac{\partial s}{\partial X_i} dX_i$$

holds – start for instance by proving it for monomials by induction of the total degree. This implies that $\Omega^1_{X/S}$ is indeed generated by the dX_i as an $\mathcal{O}_X$-module.

To show that the dX_i form a basis of $\Omega^!_{X/S}$, consider the free $\mathcal{O}_X$-module Ω' with basis $dX_1, \ldots, dX_n$ and the derivation

$$d' : \mathcal{O}_X \to \Omega', s \mapsto \sum_i \frac{\partial s}{\partial X_i} dX_i.$$

By the universal property of $\Omega^1_{X/S}$, there exists a morphism of $\mathcal{O}_X$-modules $u : \Omega^1_{X/S} \to \Omega'$ such that $d' = u \circ d$. The morphism u has to send dX_i to dX_i. This shows that u is an isomorphism. □

With this basic computation at hand and the conormal exact sequence, we can compute the sheaf of relative Kähler differentials of the subschemes of the affine space. The following statement follows directly from Theorem 1.1.26 and the above computation. It marks the appearance of the *jacobian matrix* $(\frac{\partial f_i}{\partial X_j})_{i,j}$.

Proposition 1.1.29. *Let A be a ring and n be a nonnegative integer. Let $S = \operatorname{Spec} A$ and $X = \mathbb{A}^n_S$. Let $f_1, \ldots, f_r$ be elements of $A[X_1, \ldots, X_n]$, and let Z be the subscheme of X defined by the ideal $(f_1, \ldots, f_n)$. Then, as a $B = A[X_1, \ldots, X_n]/(f_1, \ldots, f_r)$-module, we have*

$$\Omega^1_{Z/S} = (BdX_1 \oplus \ldots \oplus BdX_n)/(df_1, \ldots, df_r),$$

and $df_i = \sum_j \frac{\partial f_i}{\partial X_j} dX_j$.

As a consequence, it is possibly to compute explicitly the sheaf of relative differentials for affine morphisms of finite presentation. The global geometry of $\Omega^1_{X/S}$ is more complicated in general.

1.1.6 Functorial description of Ω^1 and relation with the Zariski tangent space

In the preceding section, we used the universal property of the sheaf of relative differentials to prove its basic properties and explain how to compute it in the affine case. This is particularly useful when working with a scheme that is defined by explicit equations. On the other hand, when working with schemes that are defined via their functor of points, we need a functorial description of the sheaf of relative differentials. This is precisely what we did in 1.1.4, and we restate – and rephrase the corresponding results here.

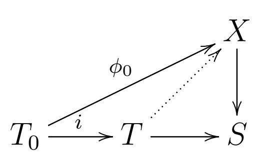
The following is essentially Proposition 1.1.20. We use the notations of 1.1.4.

Theorem 1.1.30. *Let $X \to S$ be a a morphism of schemes. The quasi-coherent sheaf $\Omega^1_{X/S}$ represents the functor*

$$QCoh(X) \to \text{Sets},\ \mathcal{M} \mapsto \{\text{sections of } X \to X[\mathcal{M}] \text{ over } S\}.$$

We give a version of this universal property in terms of points of S.

Proposition 1.1.31. *Let $X \to S$ be a morphism of schemes, and let $i : T_0 \to T$ be a first-order thickening of schemes over S with ideal $\mathcal{I}$. Let $\phi_0 : T_0 \to X$ be a T_0-point of X. Assume that the set of liftings of ϕ_0 to T, that is, to the set of dotted arrows in the diagram*



is not empty. Then this set admits a natural simply transitive action of the group $\mathrm{Hom}(\phi_0^*\Omega^1_{X/S}, \mathcal{I})$.

Proof. One possible proof is to work along the line of 1.1.20. Another possibility is to rely entirely on the universal property of $\Omega^1_{X/S}$ as expressed in Theorem 1.1.30.

To make the second strategy work, one shows that given any map ϕ_0 as above, we can construct using i a first-order thickening of X as follows. Consider the diagram of sheaves of algebras on X

$$\begin{array}{ccc} & & \phi_{0*}\mathcal{O}_T \\ & & \downarrow \\ \mathcal{O}_X & \longrightarrow & \phi_{0*}\mathcal{O}_{T_0} \end{array}$$

induced by the morphisms ϕ_0 and i. We can take the pushout of this diagram, which gives a quasi-coherent sheaf of algebras $\mathcal{A}$ on X mapping onto $\mathcal{O}_X$. Since the kernel of $\phi_{0*}\mathcal{O}_T \to \phi_{0*}\mathcal{O}_{T_0}$ is $\phi_{0*}\mathcal{I}$, the kernel of $\mathcal{A} \to \mathcal{O}_X$is $\phi_{0*}\mathcal{I}$ as well. As a consequence, we get a first-order thickening of $X \to X_1$ – which in this case is obtained by gluing X and T along T_0 – with ideal $\phi_{0*}\mathcal{I}$. By construction, this thickening splits if and only ϕ_0 lifts to T. Since

$$\mathrm{Hom}(\phi_0^*\Omega^1_{X/S}, \mathcal{I}) = \mathrm{Hom}(\Omega^1_{X/S}, \phi_{0*}\mathcal{I}),$$

this proves the result. □

In the preceding proposition, if the first-order thickening $T_0 \to T$ already splits over S, the set of liftings of ϕ_0 is never empty. This has the following consequence, where for simplicity we work over a field – but the reader will state and prove the corresponding proposition over a general base.

Proposition 1.1.32. *Let $X \to \mathrm{Spec}\, k$ be a scheme over a field k, and let x be a point of X. Let K be the residue field of x. Then we have a canonical isomorphism of K-vector spaces*

$$\Omega^1_{X/k} \otimes_{\mathcal{O}_{X,x}} K \simeq T_x^* X_K.$$

In the formula above, x is seen both as a point of X and as a point of $X_K = X \times_k K$.

Proof. By Theorem 1.1.23, the formation of the sheaf of relative differentials is compatible with base change. As a consequence, we can assume that x is a rational point of X.

Let R be the ring of dual numbers over k. Then $\mathrm{Spec}\, R = (\mathrm{Spec}\, k)[k]$. The point x corresponds to a morphism $\phi_0 : \mathrm{Spec}\, k \to X$.

By the discussion at the end of 1.1.1, we know that there is a natural bijection between the set of liftings of ϕ_0 to $\mathrm{Spec}\, R$ and the space $\mathrm{Hom}_k(T_x^* X, k)$. Furthermore, since the thickening $\mathrm{Spec}\, k \to \mathrm{Spec}\, R$ has a canonical splitting over k corresponding to the k-algebra structure of R, we get a canonical lift of ϕ_0 to $\operatorname{Spec} R$. By Proposition 1.1.31, this shows that there is a natural bijection between $\operatorname{Hom}(\Omega^1_{X/k} \otimes_{\mathcal{O}_{X,x}} k, k) = \operatorname{Hom}(\phi_0^* \Omega^1_{X/k}, k)$ and the set of liftings of ϕ_0 to $\operatorname{Spec} R$. This proves the result. □

Remark 1.1.33. *One needs to be careful about the statement above. Indeed, while T_x^*X is a vector space over K, it is not true in general that $T_x^*X_K$ and T_x^*X are isomorphic even when x is a closed point – this is certainly not true when x is not closed. Using Theorem 1.1.26, this is related to the vanishing of $\Omega^1_{K/k}$. Since K is an algebraic extension of k, this vanishing is equivalent to the fact that this extension is separable – see Problem set – which might not hold in positive characteristic.*

Corollary 1.1.34. *Assume that k is algebraically closed and that X is a regular scheme of finite type over k. Assume that X is purely of dimension d – meaning that any component of X has dimension d. Then $\Omega^1_{X/k}$ is locally free of rank d.*

Proof. We know that $\Omega^1_{X/k}$ is a coherent sheaf. By Nakayama's lemma, showing that $\Omega^1_{X/k}$ is locally free of dimension d is the same as showing that its fiber at any closed point of X has dimension d.

Now if x is any closed point of X, the regularity assumption on X ensures that T_x^*X is a k-vector space of dimension d, which by Proposition 1.1.32 shows the desired result. □

1.2 The de Rham complex

Let $X \to S$ be a morphism of schemes. As in differential geometry, the derivation $d : \mathcal{O}_X \to \Omega^1_{X/S}$ can be extended to higher-order differentials. This gives rise to the de Rham complex $\Omega^\bullet_{X/S}$.

1.2.1 Definition of the de Rham complex

Definition 1.2.1. *Let $X \to S$ be a morphism of schemes. If p is a nonnegative integer, we denote by*

$$\Omega^p_{X/S} := \bigwedge^p \Omega^1_{X/S}$$

the p-th exterior power of the $\mathcal{O}_X$-module $\Omega^1_{X/S}$.

If $p = 0$, this means by convention that $\Omega^0_{X/S} = \mathcal{O}_X$. We will write $\Omega^p_{X/S} = 0$ for $p < 0$.

We can extend the definition of d as follows.

Theorem 1.2.2. *Let $f : X \to S$ be a morphism of schemes. There exists a unique family of maps $d : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$ satisfying the following conditions:*

1. *d is an S-antiderivation of the exterior algebra $\bigoplus_p \Omega^p_{X/S}$. This means that d is $f^{-1}(\mathcal{O}_S)$-linear and satisfies the Leibniz rule*

$$d(t_1 t_2) = dt_1 \wedge t_2 + (-1)^p t_1 \wedge dt_2$$

if t_1 and t_2 are local sections of $\bigoplus_p \Omega^p_{X/S}$ and t_1 has pure degree p;

2. *$d \circ d = 0$;*

3. *If t is a local section of $\Omega^0_{X/S} = \mathcal{O}_X$, then $dt = d_{X/S}t$.*

The map $d : \Omega^p_{X/S} \to \Omega^{p+1}_{X/S}$ is called the exterior differential*.*

Proof. This is proved in [Gro67], Theorem 16.6.2, see also the Stacks Project 44.10.7. We give the essential ingredients.

Let us first assume that a collection of maps as in the Theorem exists. By Proposition 1.1.22, sections of $\Omega^p_{X/S}$ are locally sums of sections of the form

$$\omega = f dg_1 \wedge \ldots \wedge dg_p.$$

The first two conditions of the theorem imply that

$$d\omega = df \wedge dg_1 \wedge \ldots \wedge dg_p.$$

This shows that d is uniquely determined by the three conditions above.

We now prove the existence of d. By the unicity statement above and the base change property 1.1.23, we can reduce to the case where both $X = \operatorname{Spec} B$ and $S = \operatorname{Spec} A$ are affine. In that case, the universal property of Theorem 1.1.15 shows that the B-module $\Omega^1_{X/S}$ is generated as a B-module by the elements of the form df, $f \in A$, with the relations

1. $da = 0$ if $a \in A$;

2. $d(f + g) = df + dg$;

3. $d(fg) = fdg + gdf$.

We define the exterior differential

$$d : \Omega^1_{X/S} \to \Omega^2_{X/S}$$

by the formula above

$$d(fdg) = df \wedge dg.$$

To show that this is well-defined, we have to show that this map is compatible with the relations above, which is an easy computation.

Using the map above, we get a map

$$(\Omega^1_{X/S})^{\otimes p} \to \Omega^{p+1}_{X/S}, \omega_1 \otimes \ldots \otimes \omega_p \to \sum_{i=1}^{p} (-1)^{i+1} \omega_1 \wedge \ldots \wedge \omega_i \wedge \ldots \wedge \omega_p.$$

It is an easy exercise to show that this induces the required exterior differential. □

Definition 1.2.3. *Let $f : X \to S$ be a morphism of schemes. The* de Rham complex *of X over S is the complex of $f^{-1}\mathcal{O}_S$-modules*

$$\mathcal{O}_X \to \Omega^1_{X/S} \to \ldots \to \otimes^p_{X/S}$$

constructed in Theorem 1.2.2, where $\mathcal{O}_X$ is placed in degree 0*. It is denoted by $\Omega^\bullet_{X/S}$.*

Remark 1.2.4. *The de Rham complex is* not *a complex of $\mathcal{O}_X$-modules, even though its individual components are. Indeed, the exterior differential is not $\mathcal{O}_X$-linear.*

Proposition 1.2.5. *Assume that S is noetherian and $f : X \to S$ is a morphism of finite type. Then $\Omega^\bullet_{X/S}$ is a bounded complex.*

Proof. By definition, the de Rham complex is bounded below. With the assumptions of the proposition, $\Omega^1_{X/S}$ is coherent by Proposition 1.1.24. As a consequence, $\Omega^p_{X/S}$ vanishes if p is large enough. □

We can apply the machinery of derived functors to the de Rham complex. If X is a scheme, we denote by $D(X)$ the derived category of complexes of $\mathcal{O}_X$-modules. If $X \to S$ is a morphism of schemes, we denote by $D(X, f^{-1}(\mathcal{O}_S))$ the derived category of complexes of $f^{-1}(\mathcal{O}_S)$-modules on X.

The functor f_* can be derived and we get a dericed functor

$$Rf_* : D^+(X, f^{-1}(\mathcal{O}_S)) \to D^+(S).$$

Definition 1.2.6. *Let S be a noetherian scheme, and let $X \to S$ be a morphism of finite type. The i-*th relative algebraic de Rham cohomology *of X over S is the $\mathcal{O}_S$-module*

$$H^i_{\mathrm{dR}}(X/S) = R^i f_* \Omega^\bullet_{X/S}.$$

*If S is the spectrum of a field k, the i-*th Hodge cohomology group *of X is the k-vector space*

$$\bigoplus_{p+q=i} H^q(X, \Omega^p_{X/k}).$$

One should think of algebraic de Rham cohomology as an algebraic way to compute topological invariants of X. We will prove later the following theorem of Grothendieck [Gro66].

Theorem 1.2.7. *Let X be a smooth quasi-projective variety over $\mathbb{C}$. Then there is a canonical isomorphism*

$$H^i_{\mathrm{dR}}(X/\mathbb{C}) \simeq H^i_{sing}(X(\mathbb{C}), \mathbb{C})$$

where the group on the right is the singular cohomology of the – usual – topological space $X(\mathbb{C})$ with complex coefficients.

Before making sense of this theorem, we need to define the notion of a smooth algebraic variety. We will also need some background on complex algebraic varieties which will culminate with the GAGA theorem of Serre [Ser56].

We will also need a little background on homological algebra so that to be able to compute algebraic de Rham cohomology in terms of the Hodge cohomology groups, which are much more tractable. This will be taken care of by the machinery of spectral sequences.

Before getting on to these topics, we give a construction of Atiyah as an example of the use of the de Rham complex.

1.2.2 The Atiyah class of a locally free sheaf and the cycle class map in Hodge cohomology

Let k be a field, and let X be a scheme of finite type over k. Let $\Delta : X \to X \times_S X$ be the diagonal morphism, and let $p_1, p_2 : X \times_S X$ be the two projections. Let $\mathcal{F}$ be a coherent sheaf on X. We generalize the extension (1.1.2) in the following way.

On $X \times_S X$, consider the extension (1.1.2)

$$0 \to \Delta_* \Omega^1_{X/S} \to \mathcal{O}_{\Delta_1} \to \mathcal{O}_{\Delta} \to 0.$$

We see this extension as an extension of sheaves on $X \times_S X$, so for the sake of clarity we denote by $\mathcal{O}_\Delta$ the diagonal subscheme in $X \times_S X$ instead of just X, and by Δ_1 its first infinitesimal neighborhood.

Since $\mathcal{O}_\Delta$ is flat with respect to p_2, the sequence

$$0 \to p_2^*\mathcal{F} \otimes \Delta_*\Omega^1_{X/S} \to p_2^*\mathcal{F} \otimes \mathcal{O}_{\Delta_1} \to p_2^*\mathcal{F} \otimes \mathcal{O}_\Delta \to 0$$

is exact. Note that all the tensor products above are taken with respect to $\mathcal{O}_{X\times X}$.

We now consider the sheaves above as sheaves on $X \simeq \Delta$. Via p_1, these sheaves have the structure of $\mathcal{O}_X$-modules. The first and the last one are clearly $\mathcal{F} \otimes \Omega^1_{X/k}$ and $\mathcal{F}$ respectively. We get an extension of coherent sheaves of $\mathcal{O}_X$-modules

$$0 \to \mathcal{F} \otimes \Omega^1_{X/k} \to \mathcal{P}^1_{X/k}(\mathcal{F}) \to \mathcal{F} \to 0. \tag{1.2.1}$$

Definition 1.2.8. *The coherent sheaf $\mathcal{P}^1_{X/k}(\mathcal{F})$ is the* sheaf of first-order jets of *$\mathcal{F}$.*

Definition 1.2.9. *Let X, k and $\mathcal{F}$ be as above. The extension class*

$$at(\mathcal{F}) \in \mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X/k})$$

defined by the extension (1.2.1) is the Atiyah class *of $\mathcal{F}$.*

The Atiyah class was introduced in [Ati57]. It has a geometric interpretation as follows.

Definition 1.2.10. *Let $\mathcal{F}$ be a coherent sheaf on X. A* connection *on $\mathcal{F}$ is a k-linear morphism*

$$\nabla : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X/k}$$

satisfying the Leibniz rule

$$\nabla(f\sigma) = \sigma \otimes df + f\nabla\sigma.$$

Now the exact sequence (1.2.1) always admits a k-linear splitting ϕ induced by p_2^{-1} in the same way we saw earlier in the text. Assume that π is an $\mathcal{O}_X$-linear splitting of (1.2.1). Then

$$\nabla := \pi - \phi : \mathcal{F} \to \mathcal{F} \otimes \Omega^1_{X/k}$$

is a connection by the computation of 1.1.13.

Conversely, if ∇ is a connection, then the same computation shows that $\pi := \nabla + \phi$ is an $\mathcal{O}_X$-linear splitting of (1.2.1). As a consequence, the Atiyah class measures the obstruction for the existence of an algebraic connection on $\mathcal{F}$.

Assume now that $\mathcal{F}$ is locally free. We already know that we have a canonical isomorphism

$$\mathrm{Ext}^1(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X/k}) \simeq \mathrm{Hom}_{D^b_{coh}(X)}(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X/k}[1]).$$

Taking powers of the Atiyah class, we get elements

$$at(\mathcal{F})^p \in \mathrm{Hom}_{D^b_{coh}(X)}(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X/k}[p]).$$

In general, we can construct a trace map

$$Tr : \mathrm{Hom}_{D^b_{coh}(X)}(\mathcal{F}, \mathcal{F} \otimes \Omega^1_{X/k}[p]) \to \mathrm{Hom}_{D^b_{coh}(X)}(\mathcal{O}_X, \Omega^1_{X/k}[p]) = H^p(X, \Omega^p_{X/k}).$$

Since we won't use this later, we don't give the construction, see for instance [HL97], 10.1.2.

We now assume that k has characteristic zero.

Definition 1.2.11. *Let $\mathcal{F}$ be a locally free sheaf. Then the* p-th component of the Chern character of $\mathcal{F}$ in Hodge cohomology *is*

$$ch_p(\mathcal{F}) := \frac{1}{p!} Tr(at(\mathcal{F})^p) \in H^p(X, \Omega^p_{X/k}).$$

The reader should compare the definition above with Chern-Weil theory, that is, the computation of Chern classes via the trace of the curvature of a connection for a vector bundle on a differentiable manifold.

The definition above can be readily extended to coherent sheaves with a finite locally free resolution. If X is regular, this encompasses all coherent sheaves on X.

Definition 1.2.12. *Let X be a regular variety, and let Z be a subscheme of pure codimension p in X. Then the* cycle class *of Z in Hodge cohomology is*

$$cl(Z) = ch_p(\mathcal{O}_Z) \in H^p(X, \Omega^p_{X/k}).$$

Remark 1.2.13. *We will see later that the group $H^p(X, \Omega^p_{X/k})$ is related to the de Rham cohomology group $H^{2p}_{\mathrm{dR}}(X/k)$. The factor 2 is the one that would appear if $k = \mathbb{C}$, in which case Z would have real codimension $2p$. The formula that defines the cycle class above is closely related to the Riemann-Roch theorem.*

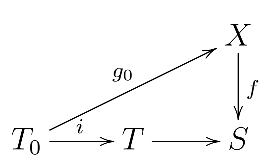
1.3 Differential properties of morphisms

In differential geometry, a map f between two smooth manifolds is said to be a submersion (resp. immersion, resp. local diffeomorphism) if its differential is onto (resp. into, resp. an isomorphism). The corresponding notions in algebraic geometry are that of smooth (resp. unramified, resp. étale) morphism and were defined by Grothendieck. We roughly follow the presentation of [Gro67], paragraph 17, as well as the summary of [Ill96]. Other presentations are given in [SGA71] and [BLR90]. We will keep this section rather short, and the reader is invited to look into the references cited above – and to the problem sets – for more material.

1.3.1 General definitions

Definition 1.3.1. *Let $f : X \to S$ be a morphism of schemes. We say that f is* formally smooth *(resp.* formally unramified*, resp.* formally étale*) if the following condition holds.*

Given any commutative diagram



where i is a first-order thickening, there exists, locally on T for the Zariski topology, at least one (resp. at most one, resp. exactly one) morphism $g : T \to X$ over S such that $gi = g_0$.

S is a scheme over k, T_0 is a k-point of S and T is a $k[\epsilon]$-point of S, this corresponds to the differential-geometric definitions we mentioned earlier. It is important to note that checking that a morphism is formally smooth, unramified or étale can be done by looking at its functor of points.

In the definition of a formally étale morphism, one can remove *locally on T for the Zariski topology.* This is because the uniqueness condition allows use to glue the liftings. This phenomenon does *not* happen for formally smooth morphism. We will work out the global obstruction later in the text. It vanishes if T is affine, see [Gro67], definition 17.1.1 and Proposition 17.1.6, (i).

We will require a finiteness assumption.

Definition 1.3.2. *Let $f : X \to S$ be a morphism of schemes. We say that f is* smooth *(resp.* unramified*, resp.* étale*) if f is a morphism of finite presentation and f is formally smooth (resp. formally unramified, resp. formally étale).*

Since we will work over noetherian bases, the condition above becomes that f is of finite type.

Remark 1.3.3. *It is readily checked that an immersion of noetherian schemes is unramified.*

Remark 1.3.4. *In the definitions above, one could simply assume that $i : T_0 \to T$ is a closed immersion defined by a nilpotent ideal. Indeed, let $\mathcal{I}$ be an ideal defining T_0 in T, such that $\mathcal{I}^n = 0$. If m is a nonnegative integer, let T_m be the subscheme of T defined by the ideal $\mathcal{I}^{m+1}$. Then i can be factored as a composition of first-order thickenings*

$$T_0 \to T_1 \to \ldots \to T_n.$$

The notion of a smooth morphism corresponds to that of a submersion in differential geometry. The notion of étale morphism corresponds to that of a local diffeomorphism, or covering. If K/k is a finite extension of fields, the extension is separable if and only if the morphism $\operatorname{Spec} K \to \operatorname{Spec} k$ is étale.

The following proposition summarizes easy – but essential ! – properties of smooth (resp. unramified, resp. étale) morphisms.

Proposition 1.3.5.
1. *The composition of two smooth (resp. unramified, resp. étale) morphisms is smooth (resp. unramified, resp. étale).*
2. *If $f : X \to S$ is smooth (resp. unramified, resp. étale), then any morphism $f' : X' \to S'$ obtained by a base change $S' \to S$ is smooth (resp. unramified, resp. étale).*

3. *Being smooth (resp. unramified, resp. étale) is a local property for the Zariski topology on the source of a morphism.*

4. *If $f_1 : X_1 \to S$ and $f_2 : X_2 \to S$ are smooth (resp. unramified, resp. étale), then $f_1 \times_S f_2 : X_1 \times_S X_2 \to S$ is smooth (resp. unramified, resp. étale).*

5. *If $\mathcal{E}$ is a locally free sheaf of finite type on S, then $\mathbb{P}(\mathcal{E}) \to S$ and $\mathbb{V}(\mathcal{E}) \to S$ are smooth.*

Proof. Apart from the last item, all the properties are direct consequences of the definition. By the third property, property 5 is readily reduced to showing that $\mathbb{P}^n_S \to S$ and $\mathbb{A}^n_S \to S$ are smooth. The first one follows from the second by the locality property again, and to prove the smoothness of $\mathbb{A}^n_S \to S$ we can assume that $n = 1$. Then the result is an easy exercise. □

By the third property above, we can define smoothness (resp. unramifiedness, resp. étaleness) at a point.

Definition 1.3.6. *Let $f : X \to S$ be a morphism of schemes, and let x be a point of X. We say that f is smooth (resp. unramified, resp. étale) at x if there exists an open subset U of X containing x such that $f_{|U}$ is smooth (resp. unramified, resp. étale).*

Clearly, the set of points $x \in X$ such that f is smooth (resp. unramified, resp. étale) at x is open.

1.3.2 Differential properties and characterizations

As expected, the properties above are related to properties of the sheaf of Kähler differentials. We do not give proofs of propositions 1.3.7, 1.3.8 and 1.3.11 as these will be the object of a problem set – they follow without too much difficulty from the preceding sections.

Proposition 1.3.7. *Let $f : X \to S$ be a morphism of finite presentation. Then f is unramified if and only if $\Omega^1_{X/S} = 0$.*

If f is smooth, then $\Omega^1_{X/S}$ is locally free of finite type.

Note that the first criterion holds in particular if f is étale.

As in Theorem 1.1.25 and Theorem 1.1.26, we fix a diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \downarrow & \swarrow_{g} & \\ S & & \end{array}$$

Proposition 1.3.8. *1. If f is smooth, then the sequence*

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0,$$

i.e., the sequence of Theorem 1.1.25 prolonged by zero on the left, is exact and locally split. In particular, if f is étale, then $f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$ is an isomorphism.

2. Assume that $X \to S$ is smooth and that $f = i$ is an immersion. Then the sequence

$$0 \to \mathcal{C}_{X/Y} \to i^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0,$$

that is, the conormal exact sequence prolonged by zero on the left, is exact and locally split. In particular, if $X \to S$ is étale, then $\mathcal{C}_{X/Y} \to i^*\Omega^1_{Y/S}$ is an isomorphism.

Remark 1.3.9. *In the second case of the proposition, if X and Y are smooth, then the conormal sheaf $\mathcal{C}_{X/Y}$ is locally free.*

Remark 1.3.10. *The proposition above is the manifestation of the fact that the cotangent complex of a smooth morphism is the sheaf of Kähler differentials set in degree 0.*

Proposition 1.3.8 has a converse.

Proposition 1.3.11. *1. Assume that $X \to S$ is smooth. Then if the sequence*

$$0 \to f^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to \Omega^1_{X/Y} \to 0$$

is exact and locally split, then f is smooth. If the canonical morphism

$$f^*\Omega^1_{Y/S} \to \Omega^1_{X/S}$$

is an isomorphism, then f is étale.

2. Assume that $Y \to S$ is smooth and $f = i$ is an immersion. If the sequence

$$0 \to \mathcal{C}_{X/Y} \to i^*\Omega^1_{Y/S} \to \Omega^1_{X/S} \to 0$$

is exact and locally split, then $X \to S$ is smooth. If the canonical morphism

$$\mathcal{C}_{X/Y} \to i^*\Omega^1_{Y/S}$$

is an isomorphism, then $X \to S$ is étale.

Used in conjunction with Proposition 1.1.28, 1.3.7 and 1.3.11 give concrete criteria to check whether a morphism is smooth (resp. unramified, resp. étale) at a point. We give two examples.

Proposition 1.3.12. *Let $f : X \to S$ be a morphism of schemes. Let x be a point of X, and assume that f is smooth at x. Then there exists an open neighborhood U of x in X, and an étale morphism $s : U \to \mathbb{A}^n_S$, for some nonnegative integer n, such that $f_{|U}$ is the composition*

$$U \to \mathbb{A}^n_S \to S.$$

Proof. Let k be the residue field of x. By Proposition 1.3.7, $\Omega^1_{X/S}$ is locally free. Let $s_1, \ldots, s_n$ be sections of $\mathcal{O}_X$ on a neighborhood U of x such that $ds_1, \ldots, ds_n$ form a basis of $\Omega^1_{X/S}$ over U. Up to shrinking U and by Nakayama's lemma, this is the case as soon as the images of the ds_i in $\Omega^1_{X/S} \otimes k$ form a basis of this k-vector space.

The n sections s_i define a morphism

$$s : U \to \mathbb{A}^n_S.$$

Certainly, $f_{|U}$ is the composition

$$U \to \mathbb{A}^n_S \to S.$$

We claim that s is étale. By Proposition 1.3.11, we need to check that the canonical morphism $s^*\Omega^1\mathbb{A}^n_S/S \to \Omega^1_{X/S}$ is an isomorphism. For this, with the assumptions above, we only have to show that $s^*\Omega^1\mathbb{A}^n_S/S \otimes k \to \Omega^1_{X/S} \otimes k$ is an isomorphism. Now by Proposition 1.1.28, $\Omega^1_{\mathbb{A}^n_S/S}$ is free with basis $(dX_i)_{i=1,\ldots,n}$. By definition of the map s, this means that $s^*\Omega^1\mathbb{A}^n_S/S$ has basis $(ds_i)_{i=1,\ldots,n}$. This shows the result. □

In the situation of the preceding proposition, we say that the s_i form *a system of local coordinates* on X at x.

Proposition 1.3.13. *Consider a diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ S & & \end{array}$$

where g is smooth and i is an immersion with ideal $\mathcal{I}$. Let x be a point of X. Then f is smooth at x if and only if there exist sections $s_1, \ldots, s_r$ of $\mathcal{I}$ on a neighborhood of x such that

1. *The s_i generate $\mathcal{I}_x$;*

2. *The $(ds_i)(x)$ are linearly independent in $\Omega^1_{Y/S} \otimes k(x)$.*

Proof. By Proposition 1.3.8 and Proposition 1.3.11, f is smooth at x if and only if the canonical morphism

$$\mathcal{C}_{X/Y} \to i^*\Omega^1_{Y/S}$$

is a locally split injection on a neighborhood of x.

By Nakayama's lemma, this holds if and only if there exist sections of the conormal sheaf $\mathcal{C}_{X/Y}$ that generate $\mathcal{C}_{X/Y}$ and have linearly independent images in $i^*\Omega^1_{Y/S} \otimes k(x)$. Considering the definition of the map in 1.1.26, this shows the result. □

In view of Proposition 1.1.29, the criterion above is called the *jacobian criterion*.

We can combine the ideas of 1.3.12 and 1.3.13 as follows. The proof is left to the reader.

Proposition 1.3.14 (the implicit function theorem in algebraic geometry)**.** *Consider a diagram*

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ {\scriptstyle f}\downarrow & \swarrow{\scriptstyle g} & \\ S & & \end{array}$$

where f, g are smooth at x. Then we can find an open neighborhood U of x in Y, a nonnegative integer n, an étale morphism $s : U \to \mathbb{A}^n_S$, and a linear subspace V of $\mathbb{A}^n_S$ such that we have a cartesian diagram

$$\begin{array}{ccc} U \cap X & \xrightarrow{i} & U \\ \downarrow & & \downarrow{\scriptstyle s} \\ V & \longrightarrow & \mathbb{A}^n_S \end{array}$$

1.3.3 Regularity, flatness, relative dimension

Schemes are always noetherian.

The preceding results give a local model for smooth morphisms in terms of étale maps. There exists a local description of étale maps, but its proof is more difficult as it relies on Zariski's main theorem [Gro67], 18.12.13, which is a major result. The proof of the following can be found in [Ray70], chapter V.

Proposition 1.3.15. *Let $f : X \to S$ be a morphism of schemes. Let x be a point of X, and let $s = f(x)$. Assume that f is étale at x. Then there exist open affine neighborhood $U = \operatorname{Spec} B$ and $V = \operatorname{Spec} A$ of x and s respectively, with $f(U) \subset V$, and an S-immersion $U \to \mathbb{A}^1_S$ that identifies U with an open subscheme of a closed subscheme $Z \subset \mathbb{A}^1_S$ defined by a monic polynomial $P \in A[X]$ such that P' does not vanish on U.*

Remark 1.3.16. *Conversely, it is easy to check that such a U is étale over S. The theorem above should be considered as a generalization of the primitive element theorem for separable extensions of fields.*

Using the proposition, we can prove the following important fiberwise characterization of smooth morphisms.

Proposition 1.3.17. *Let $f : X \to S$ be a morphism of finite type. Let x be a point of X and let $s = f(x)$. Let $\kappa(s)$ be the residue field of s. Then the following conditions are equivalent:*

1. *f is smooth at x;*
2. *f is flat at x and the fiber X_s is smooth over $\operatorname{Spec}\kappa(s)$ at x.*

In particular, smooth morphisms are flat.

Proof. To show that the first statement implies the second one, we only have to show that smooth morphisms are flat. By proposition 1.3.12, we can assume that f is étale. By proposition 1.3.15, we only have to show that if A is a local ring and P is a monic polynomial, then $B = A[T]/(P)$ is flat over A. Since B has finite rank over A, this amounts to showing that B has no torsion, which is clear.

Let us show the converse. We can assume that X is a closed subscheme of $\mathbb{A}^n_S$ for some n. Let $\mathcal{I}$ be the ideal of X in $\mathbb{A}^n_S$. The scheme X_s is defined by the ideal $\mathcal{I}_s = \mathcal{I}_{|\mathbb{A}^n_s}$.

We apply Proposition 1.3.13. Since X_s is smooth at x, we can find sections $s_1, \ldots, s_r$ of $\mathcal{I}$, defined on a neighborhood of x in $\mathbb{A}^n_S$, such that X_s is defined by the images $\overline{s}_1, \ldots, \overline{s}_r$ of the s_i in $\mathcal{I}_s$, and such that the $d\overline{s}_i(x)$ are linearly independent in $\Omega^1_{\mathbb{A}^n_s/s}$. Now let Z be the subscheme of $\mathbb{A}^n_S$ defined by the s_i. Then by Proposition 1.3.13 again, Z is smooth at x. Furthermore, Z contains X as a closed subscheme, and $Z_s = X_s$ in a neighborhood of x. We can assume that $S = \operatorname{Spec} R$ is affine, as well as $X = \operatorname{Spec} A$ and $Z = \operatorname{Spec} B$. Let J be the ideal of X in Z. We have an exact sequence

$$0 \to J \to B \to A \to 0.$$

Since A is flat over R at x, the sequence remains exact after tensoring with $\kappa(s)$ over R. Since $X_s = Z_s$ around x, this implies, by Nakayama's lemma, that $J_x = 0$. This means precisely that $X = Z$ in a neighborhood of x. □

The preceding result relates smoothness of a morphism to smoothness of the fibers over their residue field. In this case, smoothness is related to regularity. We leave the following to the reader, who should be able to prove this using the preceding sections, most notably Corollary 1.1.34.

Proposition 1.3.18. *Let k be a field, and let X be a scheme of finite type over k. Then the following holds.*

1. *f is smooth if and only if X is geometrically regular, i.e., if and only if the base change of X to an algebraic closure of k is regular.*

2. *If f is smooth, then X is regular. In that case, if x is a closed point of X, then the residue field of x $\kappa(x)$ is a separable extension of k and we have*

$$\dim \mathcal{O}_{X,x} = \dim_x X = \dim \Omega^1_{X/k} \otimes \kappa(x),$$

where $\dim_x X$ *is the dimension of the irreducible component of X that contains x.*

3. *If k is a perfect field and X is regular, then f is smooth.*

Remark 1.3.19. *A regular scheme over a field k is not geometrically regular in general. Indeed, if K/k is a non-separable finite extension, then $K \otimes_k K$ is not reduced. By the proposition above, see also , this essentially the only such phenomenon.*

Remark 1.3.20. *We just used that a regular scheme is reduced. This is true in general, and this is very easy to prove in the zero-dimensional case above.*

Definition 1.3.21. *Let $f : X \to S$ be a morphism of schemes, and let x be a point of X. Let $s = f(x)$. The* relative dimension *of f at x is the integer* $\dim \mathcal{O}_{X,x} = \dim_x X_s = \dim \Omega^1_{X/S} \otimes \kappa(x)$.

We say that f is smooth of pure relative dimension d if f is smooth and the relative dimension of f at any point of X is d.

Since in the situation above $\Omega^1_{X/S}$ is locally free, the relative dimension is a locally constant function. Note that the relative dimension at x vanishes if and only if f is étale.

The following is obvious, but very important.

Proposition 1.3.22. *Let $f : X \to S$ be a smooth morphism of pure relative dimension d. Then the de Rham complex $\Omega^\bullet_{X/S}$ vanishes in degree$> d$, and $\Omega^i_{X/S}$ is locally free of rank $\binom{r}{i}$ if $0 \leq i \leq d$.*

The case where $i = d$ is maximal is especially important.

Definition 1.3.23. *Let $f : X \to S$ be a smooth morphism of pure relative dimension d. Then the line bundle $\Omega^d_{X/S}$ is called the* relative canonical bundle *of X over S. It is often denoted by $K_{X/S}$.*

The canonical bundle of a variety over a field encodes curvature properties of this variety. For instance, the canonical bundle of a smooth projective curve of genus g is ample if $g > 1$, trivial if $g = 0$, and anti-ample if $g = 0$. A *Fano variety* is a smooth variety with anti-ample canonical bundle.

Example 1.3.3.1. *By the Euler exact sequence, the canonical bundle of $\mathbb{P}^n_S$ is $\mathcal{O}(-n-1)$.*

The conormal exact sequence gives a very useful formula for the canonical bundle of a smooth hypersurface of a smooth variety.

Theorem 1.3.24 (adjunction formula)**.** *Let X be a smooth variety over a field k, and let V be a smooth subvariety of codimension 1 in X. Then*

$$K_V \simeq (K_X)_{|V} \otimes \mathcal{O}_X(V)_{|V}.$$

In the theorem above, $\mathcal{O}(V)$ is by definition the dual of the ideal sheaf of V.

Proof. Since V is smooth, the conormal exact sequence of Theorem 1.1.26 can be written as

$$0 \to \mathcal{O}(-V)_{|V} \to (\Omega^1_{X/k})_{|V} \to \Omega^1_{V/k} \to 0.$$

Taking the determinant of all the terms, we get the result. □

Example 1.3.3.2. *By the adjunction formula above, if n is a positive integer, a degree $n+2$ hypersurface in $\mathbb{P}^n_k$ has trivial canonical bundle. If $n = 2$, these are called* elliptic curves*. If $n = 3$, we get $K3$* surfaces*. In general, these are all examples of* Calabi-Yau varieties*.*

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