TL;DR – 11/10/19

Important points

Let k be a field.

- The main players are *central simple algebras* over k: finite-dimensional k-algebras whose center is k and have no nontrivial two-sided ideals.

- *Wedderburn's theorem* characterizes central simple algebras: those are of the form $M_n(D)$, i.e. matrices algebras over a division algebra D, which has trivial center.

- Two basic operations on k-algebras are change of base field: replace A with the K-algebra $A \otimes_k K$, where K is a field extension of k, and tensor product: given k-algebras A and B, we may form the tensor product $A \otimes_k B$.

- There are two main technical properties at play for central simple algebras. First, a form of the bicommutant theorem: let A a finite-dimensional k-algebra, M a simple left A-module. Then $\mathrm{End}_A(M)$ is a division algebra D, and the natural map $A \to \mathrm{End}_D(M)$ is onto. In other words, consider the action of A as linear endomorphisms of the k-vector space M. Then D is the commutant of A, and the result states that the commutant of D is the image of A (which it obviously contains.

- Second important technical result: A is central simple if and only if the natural map $A \otimes_k A^{opp} \to A$ is an isomorphism.

- This allows us to prove the key permanence properties of central simple algebras: given a field extension K of k, A is central simple if and only if $A \otimes_k K$ is. Given central simple algebras A and B, $A \otimes_k B$ is central simple.

- In particular, central simple algebras are exactly the finite-dimensional k-algebras A such that $A \otimes_k K$ is a matrix algebra for some finite (Galois) extension K of k. In other words, central simple algebras are the *forms of matrix algebras*. This gives a link between central simple algebras and Galois theory.

- Important definition: the *Brauer group* of k is the set of central simple algebras over k modulo the equivalence $A \sim B$ if and only if $M_n(A) \simeq M_m(N)$ for some m, n. Tensor product turns the Brauer group into an abelian group. As a set, it may be identified with equivalence classes of finite-dimensional division algebras over k.

References

- Gille, Szamuely, *Central simple algebras and Galois cohomology*, Ch. 2. It also contains a thorough discussion of Galois descent, related to the expression of central simple algebras as forms of matrix algebras.
- Weil, *Basic Number Theory*, ch. IX, par. 1 and 2, if maybe a little old-fashioned.
- Stacks project, ch.11 contains all results and detailed proofs.
- Bourbaki, *Algèbre*, ch. IX in the same spirit as above.
- Jacobson, *Basic Algebra, 2*, ch.3 contains a lot of basic material for modules over (non-commutative) rings in an elementary way. In particular, 3.12 discusses Morita equivalence.