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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 1 Matrix calculus

- Definition of a matrix

We suppose given two intergers n and m greater than 1. A matrix A with n lines and m columns is a table composed by nm numbers a_{ij}. The integer i is the index of the line $(1 \leq i \leq n)$ and j is the index of column $(1 \leq j \leq m)$. We note $A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix}$ or more simply $A = \big(a_{ij}\big)_{1\leq i\leq n,\, 1\leq j\leq m}$. The number a_{ij} is the "element of matrix (i, j) of the matrix A".

For $n = m = 1$, a matrix is a simple number.

For $n = 2$ and $m = 1$, the matrix $A = \begin{pmatrix} a \\ b \end{pmatrix}$ is called a column matrix; we speak also of a "column vector" when $m = 1$.

If $n = 1$ and $m = 2$, we have by example $A = \begin{pmatrix} \alpha & \beta \end{pmatrix}$ and the matrix A is in that case a "line vector".

If $n = m = 2$, the matrix A is a square matrix of order two: $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

We denote by $\mathscr{M}_{nm}$ the set of matrices with n lines and m columns if there is no ambiguïty for the numbers that compose the elements of matrices. If it is necessary to specify in which set belong the elements, we have the notation $\mathscr{M}_{nm}(\mathbb{R})$ in the case of real matrices and $\mathscr{M}_{nm}(\mathbb{C})$ for complex matrices.

For square matrices, we simplify the la notation and $\mathscr{M}_n \equiv \mathscr{M}_{nn}$.

In $\mathscr{M}_{nm}$, the null matrix, denoted simply by 0, is uniquely composed with zéros : $0_{ij} = 0$ for each line i and each column j.

- Equality of two matrices

The matrices A and B are equal when the three following properties are satisfied:

(i) the number n of lines of the matrix A is equal to the number of lines of the matrix B

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(ii) the number m of columns of the matrix A is equal to the number of columns of the matrix B
(iii) for each i and j such that $1 \leq i \leq n$ and $1 \leq j \leq m$, the elements of matrices a_{ij} and b_{ij} are equal: $a_{ij} = b_{ij}$ for each line i and each column j.
We observe that we can not compare two matrices that do not have the same dimensions.

- Sum of two matrices

We can add two matrices that all have the same number of lines and the same number of columns. If $A \in \mathscr{M}_{nm}$ and $B \in \mathscr{M}_{nm}$, then $A+B \in \mathscr{M}_{nm}$ and the element of matrix (i, j) of the matrix $A+B$ is equal to $a_{ij} + b_{ij}$.
The addition of matrices is commutative: we have $A+B = B+A$ as long as the sum $A+B$ can be defined.

- Product of a number by a matrix

If λ is a number and $A \in \mathscr{M}_{nm}$ a matrix with n lines and m columns, then λA is a matrix with n lines and m columns. The corresponding matrix element (i, j) is equal to $\lambda\, a_{ij}$ for all the values of indices i and j such that $1 \leq i \leq n$ and $1 \leq j \leq m$.
We have always $(\lambda + \mu)A = (\lambda A) + (\mu A)$, $(\lambda\, \mu)A = \lambda(\mu A)$ and $\lambda\,(A+B) = (\lambda A) + (\lambda B)$.
In practice, we use this multiplication by a scalar in order to factorize a number. For example for a square matrix of order two, we have $\begin{pmatrix} 2a & 2b \\ 2c & 2d \end{pmatrix} = 2 \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

- Transposition

If $A \in \mathscr{M}_{nm}$, its transpose $A^{\rm t}$ belongst to the space $\mathscr{M}_{mn}$: we just have to exchange the lines and the columns of the matrix A. If the element of matrix (i, j) of A is equal to a_{ij}, then the element (j, i) of $A^{\rm t}$ is equal to a_{ij}.
We remark that the transpose of the transpose of a given matrix is identical to the initial matrix: $(A^{\rm t})^{\rm t} = A$.

- Product of two matrices

We suppose given two matrices $A \in \mathscr{M}_{nm}$ and $B \in \mathscr{M}_{mp}$: the number of columns of the matrix A is equal to the number of lines of the matrix B. In this case, and uniquely in this case, we can compute the product AB of the matrix A by the matrix B. The matrix element (i, k) of the matrix AB is equal to $(AB)_{ik} = \sum_{j=1}^{m} a_{ij}\, b_{jk} = a_{i1} b_{1k} + a_{i2} b_{2k} + ... + a_{im} b_{mk}$. In general, even if the product existes, the product BA does not exists.

For example with $A = \begin{pmatrix} a \\ b \\ c \end{pmatrix}$ and $B = \begin{pmatrix} \alpha & \beta \end{pmatrix}$, the product AB exists and we have

$AB = \begin{pmatrix} a\,\alpha & a\,\beta \\ b\,\alpha & b\,\beta \\ c\,\alpha & c\,\beta \end{pmatrix} \in \mathscr{M}_{32}$ whereas BA does not exist because the number of columns of the matrix B [2] is not equal to the number of lines of the matrix A [3]. Consider also the very common example with $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $X = \begin{pmatrix} x \\ y \end{pmatrix}$. We have $AX = \begin{pmatrix} ax+by \\ cx+dy \end{pmatrix}$. We remark that the product XA is not defined because the number [1] of columns of X is not equal to the number [2] of lines of A.

Even if the two products AB and BA can be computed, they define in general two matrices of different orders. Consider for example $A = \begin{pmatrix} \alpha & \beta \end{pmatrix}$ (matrix with one line and two columns) and $B = \begin{pmatrix} a \\ b \end{pmatrix}$ (matrix with two lines and one column). Then $AB = (\alpha\, a + \beta\, b)$ is a matrix with a single line and a single column, *id est* a number: $AB = (\alpha\, a + \beta\, b)$. On the other side, $BA = \begin{pmatrix} a\,\alpha & a\,\beta \\ b\,\alpha & b\,\beta \end{pmatrix}$ is a matrix with two lines and two columns.

Once the operations described in the following lines have a sense, we have, for the matrices A, B and C and the numbers λ and μ, the following identities: $A\,(B+C) = AB + AC$, $(A+B)\,C = AC + BC$, $A\,(\lambda\, B) = (\lambda\, A)\,B = \lambda\,(AB)$ and $(\lambda\,\mu)\,A = \lambda\,(\mu\, A)$.

- Associativity of the product of matrices

We suppose given three matrices $A \in \mathscr{M}_{nm}$, $B \in \mathscr{M}_{mp}$ and $C \in \mathscr{M}_{pq}$. When we consider the product AB, we find a matrix with n lines and p columns. We can in consequence multiply this matrix AB to the right by the matrix C and the product of matrices $(AB)\,C$ is well defined in $\mathscr{M}_{nq}$. In a analogous way, we can multiply B by C and define the product BC of these two matrices: it is a matrix with m lines and q columns. Then we can multiply this matrix to the left by the matrix A: the resulting matrix $A\,(BC)$ is correctly defined ans it belongs again to the set $\mathscr{M}_{nq}$. The associativity of the product of matrices express that $(AB)\,C = A\,(BC)$: we can put the parentheses as we wish if we have to consider the product of three matrices.

- Produit of square matrices and non-commutativity

Recall that a square matrix has the same number of lines and columns, also called the order of the matrix. If A and B are two square matrices with the same order, the product AB is always defined and it is a square matrix of order n. It is also the case for the product BA. In consequence, if A and B are two square matrices of order n, we can evaluate the two products AB and BA. They differ in general. The order of the product affects in general the result; the product of square matrices is not commutative.

For example, we have with $n = 2$:

$$\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} = \begin{pmatrix} 2 & -1 \\ 0 & 3 \end{pmatrix} \neq \begin{pmatrix} 3 & 0 \\ -2 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 2 & 0 \end{pmatrix} \begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix} \text{ and}$$

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.$$

- Identity matrix

The identity matrix I has all its entries equal to zero, except the diagonal elements $(j = i)$; in this case, $\mathrm{I}_{jj} = 1$. If we introduce the Kronecker symbol δ_{ij} such that $\delta_{ii} = 1$ and $\delta_{ij} = 0$ if $i \neq j$, we have $\mathrm{I}_{ij} = \delta_{ij}$. As the number 1 for the usual multiplication of numbers, the identity matrix is a neutral element for the matrix multiplication: we have $A\,\mathrm{I} = \mathrm{I}A = A$ for any arbibrary square matrix A.

- Inverse of a square matrix

We suppose given a square matrix of order n. If we can find a square matrix X such that $AX = XA = \mathrm{I}$, the matrix A is invertible. We set usually $X = A^{-1}$. Moreover, the matrix X is

not null because in this case, we would have $0 = \mathrm{I}$.
If the square matrix A is invertible, the inverse matrix A^{-1} is unique.
We have for example $\begin{pmatrix} -1 & 1 \\ 3 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & \frac{1}{3} \\ 1 & \frac{1}{3} \end{pmatrix}$, $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}^{-1} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$
and $\begin{pmatrix} 1 & 0 \\ a & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$.

- Determinant of a square matrix

The determinant of a square matrix A of order n is a number denoted by $\det A$ that satisfies the six following conditions:

(i) If $n = 1$, then $\det A = A$.

(ii) Invariance by transposition: $\det A^{\mathrm{t}} = \det A$.

(iii) Reduction of the order. If α is a number, B a square matrix of order $(n-1)$ and

$$A = \left(\begin{array}{c|ccc} \alpha & 0 & \cdots & 0 \\ \hline \star & & & \\ \vdots & & B & \\ \star & & & \end{array} \right) \quad \text{or} \quad A = \left(\begin{array}{c|ccc} \alpha & \star & \cdots & \star \\ \hline 0 & & & \\ \vdots & & B & \\ 0 & & & \end{array} \right), \text{ then } \det A = \alpha \, \det B.$$

(iv) If we exchange two lines [respectively two columns] of the matrix, we change the sign of the determinant.

(v) If we multiply all the elements of a line [respectively a column] by a given scalar λ, we multiply the determinant by λ.

(vi) We can add or substract to a given line [respectively a given column] any linear combination of the other lines [respectively the other columns] without changing the value of the determinant.

We deduce that whatever the dimension, the determinant of the identity matrix I is equal to 1: $\det \mathrm{I} = 1$.

- Notation for the determinant of a square matrix

$$\text{If } A = \begin{pmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nm} \end{pmatrix}, \text{ then } \det A \text{ is denoted by } \begin{vmatrix} a_{11} & \cdots & a_{1j} & \cdots & a_{1m} \\ \vdots & & \vdots & & \vdots \\ a_{i1} & \cdots & a_{ij} & \cdots & a_{im} \\ \vdots & & \vdots & & \vdots \\ a_{n1} & \cdots & a_{nj} & \cdots & a_{nm} \end{vmatrix}.$$

- Determinant of the product of two square matrices

If A and B are two square matrices of order n, the determinant of the produit is equal to the produc of the determinants: $\det\,(A\,B) = (\det A)\,(\det B)$.

Moreover, if the matrix P is invertible, we have $\det\,(P^{-1}) = \dfrac{1}{\det P}$.

- A fundamental theorem

Let A be a square matrix of order $n \geq 1$. The matrix A is invertible if and only if its determinant is not equal to zero

Exercices

- Determinant of a matrix of order two

In this exercice, we propose to prove the property that $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$.

a) Prove that this property is satisfied when $a = 0$.

b) If we suppose that $a \neq 0$, prove that the property is still satisfied.

c) Conclude for the general case.

d) Observe that the proposed relation is compatible with the fact that the determinant is invariant by transposition.

- Inverse of a square matrix of order two

We consider a general two by two matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. In this exercice we determine completely its inverse A' such that $A.A' = A'.A = \mathrm{I}$. We introduce four unknown numbers: α, β, γ, δ and we set $A' = \begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}$.

a) Evaluate the product $A.A'$.

b) Prove that if $A.A' = A'.A = \mathrm{I}$, then the coefficients α, β, γ, δ are solution of a set of two linear systemes of two equations with two unknowns.

c) Show that if moreover $\det A \neq 0$, we can solve this pair of two linear systems and determine the matrix A'.

d) Prove that the matrix A' explicited at the previous question satisfies also the relation $A'.A = \mathrm{I}$.

e) If the matrix A is not null and if $\det A = 0$, explicit a non null two by two matrix B such that $A.B = 0$.

- A commutator

We consider the following matrix with 2 lines and 2 columns: $J = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

a) Determine the set C of all matrices X with 2 lines and 2 columns that commute with the matrix J: $X \in C$ if and only if $XJ = JX$.

b) Show that if $X \in C$, there exists two real numbers α and β such that $X = \alpha \mathrm{I} + \beta J$.

- Tridiagonal matrices

For three real numbers a, b and c and we set $T(a,b,c) = \begin{pmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{pmatrix}$. It is a square matrix of order 3: $T(a,b,c) \in \mathscr{M}_3(\mathbb{R})$.

a) Show that the product $T(a,b,c)\,T(a',b',c')$ can be written under the form $T(\alpha,\beta,\gamma)$.

b) What is the value of the numbers α, β and γ as a function of the 6 numbers a, a', b, b', c and c'?

c) Show that the matrix $T(a,b,c)$ is invertible and explicit the inverse matrix $T(a,b,c)^{-1}$.

- A fourth order determinant

Show that $\Delta \equiv \begin{vmatrix} 1 & 2 & -1 & 3 \\ -1 & 1 & 4 & 2 \\ 0 & 2 & 3 & 2 \\ 8 & 10 & -11 & 2 \end{vmatrix}$ is equal to -24.

- A third order determinant

We set $\Delta \equiv \begin{vmatrix} a & b & c \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{vmatrix}$.

a) Show that if $a = b$, then $\Delta = 0$.

b) Same question if $b = c$ or if $c = a$.

c) Compute the value of Δ as a function of a, b and c. $[a\,b\,c\,(a-b)\,(b-c)\,(c-a)]$

- An other fourth order determinant

Show that the determinant defined by $D \equiv \begin{vmatrix} 1 & a & a^2 & a^3 \\ a & a^2 & a^3 & 1 \\ a^2 & a^3 & 1 & a \\ a^3 & 1 & a & a^2 \end{vmatrix}$ is equal to $\left(a^4 - 1\right)^3$.