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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 3 Changing the basis

- Linear map

We consider two vector spaces E and F and a map u from E to F: for each $x \in E$, there exists a unique vector $y = u(x)$ image of x by the map u. We say that the map u is linear if and only if the two following conditions of compatibility are satisfied: compatibility with the addition $\forall x, y \in E, u(x+y) = u(x) + u(y)$, and compatibility with the external multiplication $\forall \lambda \in \mathbb{R}, u(\lambda . x) = \lambda . u(x)$.

We use the following example constructed as follows. We denote by P_1 the vector space of all affine functions. In particular the function f_0 defined by $\mathbb{R} \ni t \longmapsto f_0(t)) = 1 \in \mathbb{R}$ and the function f_1 is such that $\mathbb{R} \ni t \longmapsto f_1(t)) = t \in \mathbb{R}$. The affine functions f_0 and f_1 are vectors in the space P_1. The family (f_0, f_1) is a basis of P_1. Each $f \in P_1$ can be decomposed in the following way: $f = b\, f_0 + a\, f_1$ and the real coefficients a and b are unique. The application w from P_1 to P_1 is defined by the relation $w(b\, f_0 + a\, f_1) = (2\,a + 3\,b)\, f_1$. It is a linear map defined on P_1 and taking its values in P_1.

- Kernel

We consider a linear map $u \in \mathscr{L}(E, F)$ between the vector spaces E and F. The kernel Ker u is a subset of E defined by the following condition: $x \in \mathrm{Ker}\, u$ if and only if $u(x) = 0$. The kernel Ker u is a vector subspace of the space E. In particular, $\mathrm{Ker}\, u \subset E$.

With the previous example $w \in \mathscr{L}(P_1)$ and we have
$\mathrm{Ker}\, w = \{ f \in P_1, \exists a \in \mathbb{R}, \forall t \in \mathbb{R}, f(t) = a \left(t - \frac{2}{3}\right) \} = < \varphi >$ with $\varphi(t) = t - \frac{2}{3}$.

- Image

We consider a linear map $u \in \mathscr{L}(E, F)$ between the vector spaces E and F. The Image $\mathrm{Im}\, u$ is a subset of F defined by the condition that $y \in \mathrm{Im}\, u$ if and only if there exists $x \in E$ such that $y = u(x)$. The image $\mathrm{Im}\, u$ is a vector subspace of the space F and $\mathrm{Im}\, u \subset F$.

For the previous example with $w \in \mathscr{L}(P_1)$, we have
$\mathrm{Im}\, w = \{ f \in P_1, \exists \alpha \in \mathbb{R}, f = \alpha\, f_1 \} = < f_1 >$.

- Conservation of the dimension

We consider a vector space E with a finite dimension: $\dim E = n$, where n is a nonnegative integer, and we introduce also $u \in \mathscr{L}(E)$. Then the spaces $\mathrm{Ker}\, u$ and $\mathrm{Im}\, u$ are of finite dimensions and we have the relation $\dim \mathrm{Ker\ u} + \dim \mathrm{Im\ u} = \dim E$.

François Dubois, 25 october 2022.

For the previous example with $w \in \mathscr{L}(P_1)$, we have $\dim \mathrm{Ker\ w} = 1$ and $\dim \mathrm{Im\ w} = 1$ whereas $\dim P_1 = 2$ as we observed in the previous chapter.

- Matrix of a linear map relatively to a set of bases

We consider a vector space E of finite dimension n and we introduce a basis $(e_1, e_2, \dots, e_n)$ of this space. We suppose given also a vector space F of dimension p and we introduce a basis $(f_1, f_2, \dots, f_p)$ of the vector space F. For $j = 1, \dots, n$, the vector $u(e_j) \in F$ can be secomposed in a unique way in the basis $(f_1, f_2, \dots, f_p)$: there exists unique coefficients $a_{1j}, a_{2j}, \dots, a_{pj}$ in such a way that $u(e_j) = \sum_{i=1}^{p} a_{ij} \cdot f_i$. We regroup these $n\,p$ coefficients into a matrix $M_u \equiv \big(a_{ij}\big)_{1 \leq i \leq p,\, 1 \leq j \leq n}$ with p lines and n columns. This matrix is the matrix of the linear map u relatively to the bases $(e_1, e_2, \dots, e_n)$ of E and $(f_1, f_2, \dots, f_p)$ of F. We can write it in the following way:
$$M_u = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix}.$$

With the linear map $w \in \mathscr{L}(P_1)$ introduced previously, the associated matrix M_w is given by the relation $M_w = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix}$ relatively to the basis (f_0, f_1).

- Output of a given vector

With the previous notations, we regroup the components $x_1, x_2, \dots, x_n$ of the vector $x = \sum_{j=1}^{n} x_j \cdot e_j$ in the basis $(e_1, e_2, \dots, e_n)$ of E into a single vector X with one column and n lines:
$$X = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}.$$

Analogously, the coordinates $y_1, y_2, \dots, y_p$ of the vector $y = u(x) = \sum_{i=1}^{p} y_i \cdot f_i$ in the basis $(f_1, f_2, \dots, f_p)$ of F are presented with a vector Y with one column and p liges :
$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_p \end{pmatrix}.$$

Then the coordinates $y_i = \sum_{j=1}^{n} a_{ij} x_j$ can be expressed with the help of the product of the matrix M_u with the vector X:
$$Y = \begin{pmatrix} y_1 \\ y_2 \\ \vdots \\ y_i \\ \vdots \\ y_p \end{pmatrix} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots & & \vdots \\ a_{p1} & a_{p2} & \cdots & a_{pj} & \cdots & a_{pn} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_j \\ \vdots \\ x_n \end{pmatrix} = M_u \cdot X.$$

The coordinates Y of the image vector $u(x)$ are obtained by the mutiplication of the matrice M_u of operator u by the coordinates X of the vector $x \in E$: $Y = M_u X$.

With the previous linear map $w \in \mathscr{L}(P_1)$ and the vector $x = 4 f_0 - f_1$, we have $X = \begin{pmatrix} 4 \\ -1 \end{pmatrix}$.

We can perform the product and $Y = M_w X = \begin{pmatrix} 0 & 0 \\ 3 & 2 \end{pmatrix} \cdot \begin{pmatrix} 4 \\ -1 \end{pmatrix} = \begin{pmatrix} 0 \\ 10 \end{pmatrix}$. Thus $w(x) = 10 f_1$.

- Bijectivity

Racall that a map u from E to F is bijective if and only if for each $y \in F$, the equation $u(x) = y$ has unique solution x that belongs to the domain E.

Theorem. Let E be a vector space of finite dimension: $\dim E = n$ with $n \in \mathbb{N}$, and let u be a linear map from E to E ($u \in \mathscr{L}(E)$). Then u is bijective if and only if one of the following conditions is satisfied: (i) u is injective, (ii) $\operatorname{Ker} u = \{0\}$, (iii) u is surjective, (iv) $\operatorname{Im} u = E$, (v) u transforms a given basis of E into a new basis of E, (vi) the matrix M_u of the operator u relatively to a given basis is invertible in $\mathscr{M}_n$.

The linear map $w \in \mathscr{L}(P_1)$ introduced previously is not bijectie. We have for example $\operatorname{Ker} u$ of dimension 1. We remark also that the matrix M_w is clearly not invertible.

The linear map $\theta \in \mathscr{L}(P_1)$ defined by $P_1 \ni f = b f_0 + a f_1 \longmapsto \theta(f) = a f_0 + b f_1 \in P_1$ is bijective. Its matrix M_θ is equal to $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ and is invertible.

- Change of basis

Let E be the vector space $< e_1, e_2, \ldots, e_n >$ of dimension n. Then the family $(e_1, e_2, \ldots, e_n)$ is a basis of E. Each vector $x \in E$ can be decomposed as a linear combination of the vectors of this basis: $x = \sum_{j=1}^{n} x_j e_j$ and the coordinates x_j are uniquely defined. We introduce a new family of vectors $\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n$ defined by their decomposition in the previous basis: $\widetilde{e}_k = \sum_{j=1}^{n} P_{jk} e_j$. The coefficients P_{jk} for $1 \leq j, k \leq n$ compose a square matrix P with n lines and n columns, called the transfer matrix. The components of the new vector $\widetilde{e}_k$ define the kth column of the transfer matrix. We have the following result.

Theorem. The family of vectors $(\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n)$ is a basis of the space E if and only if the transfer matrix P is invertible.

If we wish to write the new coordinates $\widetilde{x}_k$ of the previous vector $x \in E$, we have the relation $P\widetilde{X} = X$ between the column vector X of the old coordinates x_j and the column vector $\widetilde{X}$ of the new coordinates $\widetilde{x}_k$: $x = \sum_{j=1}^{n} x_j e_j = \sum_{k=1}^{n} \widetilde{x}_k \widetilde{e}_k$. To explicit the coordinates in the new basis, it is necessary to solve a linear system associated with the transfer matrix.

- Change of matrix of a linear map when changing the basis of the vector space

With the standard hypothesis of a finite dimensional vector space E of dimension $n \in \mathbb{N}$, we consider a linear map $u \in \mathscr{L}(E)$ and the associated matrix M_u relatively a given basis $(e_1, e_2, \ldots, e_n)$. When we change the basis of E for a new basis $(\widetilde{e}_1, \widetilde{e}_2, \ldots, \widetilde{e}_n)$ of the same space, we introduce an invertible transfer matrix P. Then the matrix $\widetilde{M}_u$ of the linear map u in the new basis is related to the previous data according to the relation $\widetilde{M}_u = P^{-1} M_u P$.

Exercices

- A change of basis in the space of affine functions

We denote by P_1 the space of affine functions. The basis functions f_0 and f_1 are defined by the relations $f_0(t) = 1$ and $f_1(t) = t$ for any arbitrary $t \in \mathbb{R}$. We consider the two new functions φ_0 and φ_1 defined by the relations $\varphi_0(t) = 1 + t$ and $\varphi_1(t) = 1 - t$ for an arbitrary $t \in \mathbb{R}$.

a) Express the two vectors φ_0 and φ_1 as linear combinations of f_0 and f_1.

b) What is the transfer matrix P between the family (f_0, f_1) and the new family (φ_0, φ_1) ?

c) Prove that the family (φ_0, φ_1) is a basis of the space P_1.

d) What are the coordinates of the affine function f defined by $f(t) = at + b$ (for an arbitrary real number $t \in \mathbb{R}$) in the basis (φ_0, φ_1) ?

- Changing the basis of a linear map

We still denote by P_1 the space of affine functions and by (f_0, f_1) and (φ_0, φ_1) the bases defined previously. The operator w (or the linear map w) is defined by the relation
$w(b\,f_0 + a\,f_1) = (2a + 3b)\,f_1$.

a) Recall the value of the matrix M_w of the linear map w relatively to the basis (f_0, f_1).

b) With a relation introduced in this chapter, precise the value of the matrix $\widetilde{M}_w$ in the new basis (φ_0, φ_1).

c) Express the vectors $w(\varphi_0)$ and $w(\varphi_1)$ in the basis (φ_0, φ_1) and recover the result of the previous question.

- Changing the basis for an other linear map

We still denote by P_1 the space of affine functions and by (f_0, f_1) and (φ_0, φ_1) the bases introduced during the first exercice. The operator θ is defined by the relation
$\theta(b\,f_0 + a\,f_1) = a\,f_0 + b\,f_1$.

a) Recall the value of the matrix M_θ of the linear map w relatively to the basis (f_0, f_1).

b) Prove that the map θ is a bijection from P_1 on the space P_1.

c) With an algebraic relation introduced in this chapter, precise the value of the matrix $\widetilde{M}_\theta$ in the new basis (φ_0, φ_1).

d) Express the vectors $\theta(\varphi_0)$ and $\theta(\varphi_1)$ in the basis (φ_0, φ_1) and recover the result of the previous question.

- Determinant of a linear map.

Let E be of dimension n, $u \in \mathscr{L}(E)$ a linear map from E to E, M_u the matrix of this map u relatively to a given basis and P the transfer matrix from the given basis and a new basis of E. We denote by $\widetilde{M}_u$ the matrix of u relatively the new basis.

a) Propose an algebraic relation between the matrices P, M_u and $\widetilde{M}_u$.

b) Prove that the determinant does not depend on the choice of the basis: $\det \widetilde{M}_u = \det M_u$.