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Master Structural Mechanics and Coupled Systems

## Applied Mathematics

# Lecture 11 Change of variable in a double integral

- Change of variable in a double integral: first steps

To fix the ideas, we give ourselves the unit square $K = [0, 1] \times [0, 1]$ and two strictly positive real numbers $a$ and $b$. With the linear mapping $F$ defined by $x = a\,\xi$, $y = b\,\eta$, the unit square is transformed into a rectangle $Q = [0, a] \times [0, b]$ (see the Figure 1). If we integrate the function $f \equiv 1$ in the rectangle $Q$, we find $|Q| = \int_Q \mathrm{d}x\,\mathrm{d}y = ab$ while we integrate this same function $f \equiv 1$ in the square $K$, we obtain $|K| = \int_K \mathrm{d}\xi\;\mathrm{d}\eta = 1$. We introduce the (constant) matrix $J_F$ of the linear application $F$: $J_F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Its determinant $\det J_F$ is equal to $ab$ and we see that we have $\int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det J_F|\,\mathrm{d}\xi\;\mathrm{d}\eta$.

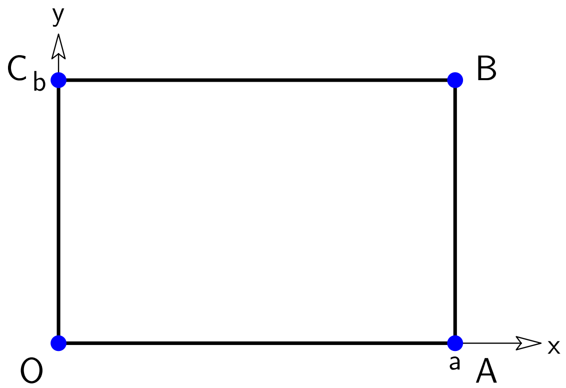


Figure 1. Rectangle with side parallel to the axes

- Change of variable in a double integral: a first parallelogram

We transform the unit square $K$ with a linear transformation $F$ defined now by $x = a\,\xi + c\,\eta$, $y = b\,\eta$. Then the unit square is transformed into a parallelogram $Q$ whose can be given the coordinates of the four vertices: $\mathrm{O}(0, 0)$ $[\xi = \eta = 0]$, $\mathrm{A}(a, 0)$ $[\xi = 1, \eta = 0]$, $\mathrm{B}(a + c, b)$ $[\xi = \eta = 1]$ et $\mathrm{C}(c, b)$ $[\xi = 0, \eta = 1]$. The area of the parallelogram $Q$ is equal to its base multiplied by the height, that is $ab$. Moreover, the matrix $J_F$ of the linear application $F$ is now $J_F = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$. Its determinant $\det J_F$ is always $ab$ and we still have $\int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det J_F|\,\mathrm{d}\xi\;\mathrm{d}\eta$.

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François Dubois, 31 january 2023.

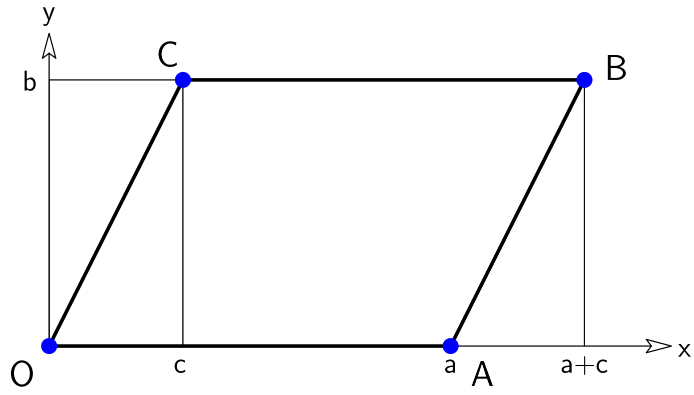


Figure 2. Parallelogram : first simple case

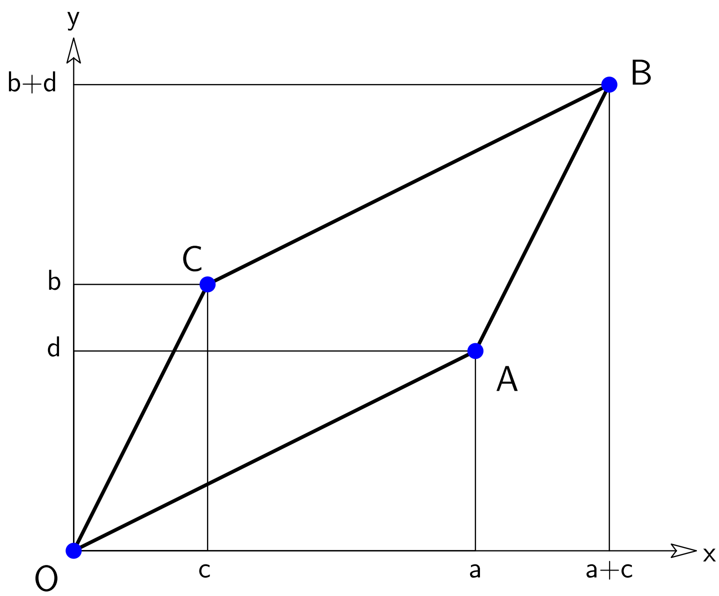


Figure 3. Parallelogram : second case

• Change of variable in a double integral: a second parallelogram

We now set the change of variables $(\xi,\eta) \longmapsto (x,y)$ *via* the linear application $F$ defined by $x = a\xi + c\eta$, $y = d\xi + b\eta$, with $a$, $b$, $c$ and $d$ strictly positive to fix the ideas. Then the unit square $K$ is transformed into another parallelogram $Q$. The coordinates of its four vertices are the following: O$(0,0)$ $[\xi = \eta = 0]$, A$(a,d)$ $[\xi = 1, \eta = 0]$, B$(a+c, b+d)$ $[\xi = \eta = 1]$ and C$(c,b)$ $[\xi = 0, \eta = 1]$. If the quadrangle $OABC$ has a direct orientation (it turns counterclockwise) [we advise the reader to make a drawing!] then the area of the parallelogram $Q$ can be calculated with a graphical approach [exercise!] and we have $|Q| = ab - dc$. If the quadrangle $OABC$ has a retrograde orientation [we advise the reader to make another drawing!], then we see that $|Q| = -ab + dc$. In all cases, $|Q| = |ab - dc|$. The matrix $J_F$ of the linear application $F$ is now equal to $J_F = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$ and $\det J_F = ab - dc$. We notice that to calculate the area of this second parallelogram, it is enough to write $\int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det J_F|\,\mathrm{d}\xi\,\mathrm{d}\eta$.

This result generalizes [exercise!] if we replace the unit square by any other square of side $\Delta x > 0$.

- Change of variable in a double integral: curvilinear quadrangle

We transform the unit square $K = [0, 1] \times [0, 1]$ with a nonlinear application $\Phi$ which we assume to be assumed to be of class $\mathscr{C}^1$, bijective from $K$ to $Q = \Phi(K)$. We assume the reciprocal application $\Phi^{-1}$ continuous from $Q$ onto $K$. We cut the square $K$ into $N \times N$ small squares $K_{i,j}$ of side $\Delta x = \frac{1}{N}$: $K_{i,j} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$, with $\xi_i = (i-1)\,\Delta x$ and $\eta_j = (j-1)\,\Delta x$. We introduce the points $M_{i,j} = \Phi(\xi_i, \eta_j)$ and the quadrangles $Q_{i,j} = \Phi(K_{i,j})$. Then we have $\int_Q \mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{Q_{i,j}} \mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{\Phi(K_{i,j})} \mathrm{d}x\,\mathrm{d}y$. We approach the application $\Phi$ in the square $K_{i,j}$ by a tangent affine application $F_{i,j}$ at the point $(\xi_i, \eta_j)$:
$\Phi(\xi, \eta) \approx F_{i,j}(\xi, \eta) \equiv \Phi(\xi_i, \eta_j) + \mathrm{d}\Phi(\xi_i, \eta_j).(\xi - \xi_i, \eta - \eta_j)$. Then we can approximate the area of the curvilinear quadrangle $Q_{i,j}$ by that of the parallelogram $P_{i,j} = F_{i,j}(K_{i,j})$ obtained by replacing $\Phi$ by $F_{i,j}$: $\int_{\Phi(K_{i,j})} \mathrm{d}x\,\mathrm{d}y \approx \int_{P_{i,j}} \mathrm{d}x\,\mathrm{d}y$. But we have seen that for a parallelogram $P_{i,j}$, we have $\int_{P_{i,j}} \mathrm{d}x\,\mathrm{d}y = \int_{K_{i,j}} |\det J_{F_{i,j}}|\,\mathrm{d}\xi\,\mathrm{d}\eta$. In the present case, $J_{F_{i,j}} = \mathrm{d}\Phi(\xi_i, \eta_j)$ and we have $\int_Q \mathrm{d}x\,\mathrm{d}y \approx \sum_{1\leq i,\, j\leq N} \int_{K_{i,j}} |\det \mathrm{d}\Phi(\xi_i, \eta_j)|\,\mathrm{d}\xi\,\mathrm{d}\eta$.
If the integer $N$ tends to infinity, the sum of the right-hand side of the last expression converges towards $\int_K |\det \mathrm{d}\Phi(\xi, \eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$ and we finally have
$|Q| = \int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det \mathrm{d}\Phi(\xi, \eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$.

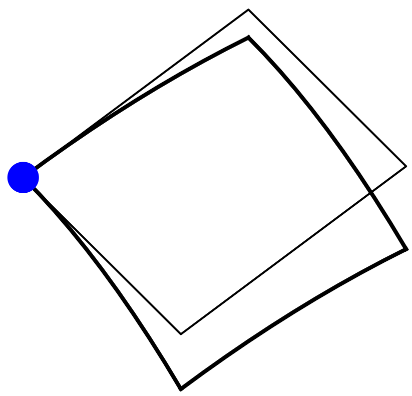

Figure 4. Around the point $M_{i,j} = \Phi(\xi_i, \eta_j)$ (in blue), the curvilinear quadrangle $Q_{i,j}$ (in strong line) is well approximated by the parallelogram $P_{i,j}$ (in thin lines) associated with the tangent affine application $F_{i,j}$ if we have sufficiently cut out the initial square.

- Change of variable in a double integral: general case

As above, we transform the unit square $K = [0, 1] \times [0, 1]$ with a nonlinear function $\Phi$ of class $\mathscr{C}^1$, bijective from $K$ onto $Q = \Phi(K)$ and the reciprocal application is assumed to be continuous from $Q$ onto $K$. We now give ourselves a function $f$ integrable in the sense of Riemann in $Q$ and we try to write the integral $\int_Q f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ with an integral in the square $K$. We use the notations from the previous paragraph and set $f_{i,j} = f(\Phi(\xi_i, \eta_j))$: this is an approximation of the function $f$ in the (small) curvilinear quadrangle $Q_{i,j}$. We then have
$\int_Q f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{Q_{i,j}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{\Phi(K_{i,j})} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

For each curvilinear quadrangle $Q_{i,j}$, we have $\int_{\Phi(K_{i,j})} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx f_{i,j} \int_{\Phi(K_{i,j})} \mathrm{d}x\,\mathrm{d}y$ and we saw in the previous paragraph that $\int_{\Phi(K_{i,j})} \mathrm{d}x\,\mathrm{d}y \approx \int_{P_{i,j}} \mathrm{d}x\,\mathrm{d}y = \int_{K_{i,j}} |\det \mathrm{d}\Phi(\xi_i,\,\eta_j)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. We deduce that $\int_{\Phi(K_{i,j})} f(x,y)\,\mathrm{d}x\,\mathrm{d}y \approx \sum_{1\leq i,\,j\leq N} \int_{K_{i,j}} f(\Phi(\xi_i,\,\eta_j))\,|\det \mathrm{d}\Phi(\xi_i,\,\eta_j)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. If the integer $N$ tends to infinity, this last sum converges to the integral
$\int_K f(\Phi(\xi,\,\eta))\,|\det \mathrm{d}\Phi(\xi,\,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. We deduce the final form of the formula of change of variable of variable in a double integral :
$\int_Q f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_K f(\Phi(\xi,\,\eta))\,|\det \mathrm{d}\Phi(\xi,\,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. The trick is not to forget the jacobian $J(\xi,\,\eta) \equiv |\det \mathrm{d}\Phi(\xi,\,\eta)|$, absolute value of the determinant of the Jacobian matrix of partial derivatives partial derivatives $\mathrm{d}\Phi(\xi,\,\eta)$ !
We admit that the previous result generalizes to the case of any open set $K$ in $\mathbb{R}^n$ any integer $n \geq 1$ and a function $f$ measurable on $Q = \Phi(K)$ and integrable on $Q$, that is, such that $\int_Q |f(x,y)|\,\mathrm{d}x\,\mathrm{d}y < \infty$.
As an exercise, the reader can try to find the "usual" formula of change of variable variable in the case of dimension one as a special case of the previous relation!

- Polar coordinates in the plane

The variables $\xi$ and $\eta$ are denoted $r$ et $\theta$ and the application $\Phi$ of change of variable $(r,\theta) \longmapsto (x,y)$ is defined by $x = r\cos\theta$ et $y = r\sin\theta$. The Jacobian matrix of this transformation can be calculated without particular difficulty and we have, if we assume $r > 0$ :
$J(r,\theta) = r$. Then we have $\int_Q f(x,y)\,\mathrm{d}x\,\mathrm{d}y = \int_K f(r\cos, r\sin\theta)\, r\,\mathrm{d}r\,\mathrm{d}\theta$ when $Q = \Phi(K)$.

## Exercises

- Circular domain

a) We suppose given $R > 0$. Let $D$ be the domain $D = \{(x,y) \in \mathbb{R}^2,\ x^2+y^2 \leq R^2\}$. Compute the double integral $I = \iint_D x^3\,y^2\,\mathrm{d}x\,\mathrm{d}y$.
b) Same question with the analogous integral $I_+ = \iint_{D_+} x^3\,y^2\,\mathrm{d}x\,\mathrm{d}y$ in the domain $D_+ = \{(x,y) \in \mathbb{R}^2, x \geq 0,\ x^2+y^2 \leq R^2\}$. [0, $\frac{4}{105}R^7$]

- Elliptic domain

Let $a > 0$ and $b > 0$ be two fixed lengths. We introduce the domain $D$ intersection of the interior of the ellipse satisfying the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the first quandrant $Q_+ = \{(x,y) \in \mathbb{R}^2, x \geq 0, y \geq 0\}$.
a) Draw the domain $D$.
b) With a not so conventional change of variables, transform the calculus of the double integral $I = \iint_D x\,y\,\mathrm{d}x\,\mathrm{d}y$.
c) Deduce from the previous question the surface $|D|$ of this quarter of elliptic domain.
d) Achieve the calculus of the double integral $I$. [$\frac{1}{4}\pi ab$, $\frac{1}{8}a^2\,b^2$]