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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 7 Length and normal of a curve

- Plane curve in the Euclidian plane

The first example is the line segment $[A, B]$ between the two points $A(\alpha, \beta)$ and $B(\gamma, \delta)$. We have a parameterization $[0, 1] \ni t \longmapsto M(t) = (X(t), Y(t)) = (1-t)A + tB$. In particular, $X(t) = (1-t)\,\alpha + t\,\gamma$ and $Y(t) = (1-t)\,\beta + t\,\delta$.

The second example is a circular arc. We introduce $R > 0$ and θ_1 and θ_2 such that $0 \leq \theta_1 < \theta_2 \leq 2\pi$ to fix the ideas. Then a point $M(\theta)$ of this curve satisfies the conditions $\theta_1 \leq \theta \leq \theta_2$ and $M(\theta) = R(\cos\theta, \sin\theta)$.

Functional curve (third exmple). For $a > b$ two given reals, we consider the mapping $[a, b] \ni t \longmapsto f(t) \in \mathbb{R}$ and the associate graph in the Euclidian plane: $X(t) = t$, $Y(t) = f(t)$.

In general, we have two regular functions X and Y from the interval $[a, b]$ and taking their values in $\mathbb{R}$. The curve Γ is composed by all the points $M(t) = (X(t), Y(t))$ for all $t \in [a, b]$.

- Velocity vector

When the mapping $t \longmapsto M(t)$ is derivable, we set $V(t) = \frac{\mathrm{d}M}{\mathrm{d}t}$. The components of the velocity vector are simply $\frac{\mathrm{d}M}{\mathrm{d}t} = \big(\frac{\mathrm{d}X}{\mathrm{d}t}, \frac{\mathrm{d}y}{\mathrm{d}t}\big)$.

For the previous examples, we have respectively $V(t) = -A + B = \overrightarrow{AB}$ for the first example, $V(\theta) = R(-\sin\theta, \cos\theta)$ in the second case and $V(t) = (1, f'(t))$ for a functional curve.

- Length of a regular curve

We introduce an integer $N \geq 1$ and we first define the approximated length L_N. With $h = \frac{b-a}{N}$, we consiter $a = t_0 < t_1 <\ldots< t_j = a + jh < t_{j+1} = t_j + h <\ldots\ < t_N = b$ and $M_j = M(t_j)$. We approach the length of the curvilear arc $\widehat{M_j M_{j+1}}$ by the length $||\overrightarrow{M_j M_{j+1}}||$ of the segment $[M_j, M_{j+1}]$. We have $||\overrightarrow{M_j M_{j+1}}|| = \sqrt{(X(t_{j+1}) - X(t_j))^2 + (Y(t_{j+1}) - Y(t_j))^2}$ and we set $L_N = \sum_{j=1}^{N} ||\overrightarrow{M_j M_{j+1}}||$ for the length of the polygoal approximation of the curve.

We have also the following expansions, if the functions X and Y are derivable: $X(t_j + h) = X(t_j) + h\,\frac{\mathrm{d}X}{\mathrm{d}t}(t_j) + h\,\varepsilon_j^X(h)$ and $Y(t_j + h) = Y(t_j) + h\,\frac{\mathrm{d}Y}{\mathrm{d}t}(t_j) + h\,\varepsilon_j^Y(h)$ with $\varepsilon_j^X(h)$ and $\varepsilon_j^Y(h)$ tending to zero as h tends to zero. Then $||\overrightarrow{M_j M_{j+1}}|| = h\,||\frac{\mathrm{d}M}{\mathrm{d}t}(t_j)|| + h\,\eta_j(h)$ and $\eta_j(h)$ tends to zero if h tends to zero. In consequence, we have the decomposition

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$L_N = \sum_{j=1}^{N} h \, || \frac{\mathrm{d}M}{\mathrm{d}t}(t_j) || + h \sum_{j=1}^{N} \eta_j(h)$. The second term tends to zero when h tends to zero and the first tends to the integral $\int_a^b || \frac{\mathrm{d}M}{\mathrm{d}t}(t) || \, \mathrm{d}t$ in the same conditions.

The length L of the curve Γ between the parameters a and b is given by the relation $L = \int_a^b || \frac{\mathrm{d}M}{\mathrm{d}t}(t) || \, \mathrm{d}t = \int_a^b || V(t) || \, \mathrm{d}t$.

For an arc segment, we recover the coherence $L = || \overrightarrow{AB} || = AB$. For an arc of circle, we have $|| \frac{\mathrm{d}M}{\mathrm{d}\theta} || = R$ and $L = R(\theta_2 - \theta_1)$. A functional curve satisfies $|| V(t) || = \sqrt{1 + (f'(t))^2}$ and $L = \int_a^b \sqrt{1 + (f'(t))^2} \, \mathrm{d}t$.

- Regular points

A regular point $M(t)$ of a curve Γ satisfies the condition $\frac{\mathrm{d}M}{\mathrm{d}t}(t) \neq 0$. All the previous examples are composed only with regular points.

- Curvilinear abscissa

With the notations used previously, we define the curvilinear abscissa by the relation $s(t) = \int_a^t || \frac{\mathrm{d}M}{\mathrm{d}t}(t) || \, \mathrm{d}t$. Then we have $s(a) = 0$, $s(b) = L$, the function $t \longmapsto s(t)$ is derivable and $\frac{\mathrm{d}s}{\mathrm{d}t} = || \frac{\mathrm{d}M}{\mathrm{d}t}(t) || > 0$ if all the points are regular. Then this function is continuous and strictly increasing. It realizes a bijection from the interval $[a, b]$ onto the interval $[0, L]$. Its reciprocal mapping $T : \; [0, L] \ni s \longmapsto T(s) \in [a, b]$ gives the value of the parameter t when the value of the curvilinear absissa is known. Moreover, this reciprocal function $s \longmapsto t = T(s)$ is derivable and we have the classical relation $\frac{\mathrm{d}T}{\mathrm{d}s} = \frac{1}{\frac{\mathrm{d}s}{\mathrm{d}t}} = \frac{1}{|| \frac{\mathrm{d}M}{\mathrm{d}t} ||}$.

- Tangent vector

We use the intrinsic parametrization of the curve Γ by the curvilinear abscissa. We consider the composed map $[0, L] \ni s \longmapsto P(s) = (M_\circ T)(s) = M(T(s))$. Then its derivate $\tau(s) = \frac{\mathrm{d}P}{\mathrm{d}s} = \frac{\mathrm{d}M}{\mathrm{d}t} \frac{\mathrm{d}T}{\mathrm{d}s} = \frac{1}{|| \frac{\mathrm{d}M}{\mathrm{d}t} ||} \frac{\mathrm{d}M}{\mathrm{d}t}$ is a unitary vector: $|| \tau(s) || = 1$. It is by definition the tangent vector to the curve Γ.

For the previous examples, we have $\tau(s) = \frac{1}{|| \overrightarrow{AB} ||} \overrightarrow{AB}$ for the line segment, $\tau(s) = (-\sin\theta, \cos\theta)$ for the arc of circle and $\tau(s) = \frac{1}{\sqrt{1 + (f'(t))^2}} (1, f'(t))$ for a functional curve.

- Normal vector

The normal vector $n(s)$ is defined in these lectures as the result of a rotation of angle $-\frac{\pi}{2}$ on the tangent vector $\tau(s)$. We have the relation $n(s) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \tau(s)$ and looking to the components: $n_x = \tau_y$, $n_y = -\tau_x$. Then the local basis $(n(s), \tau(s))$ is a direct orthonormal basis of the vector plane $\mathbb{R}^2$.

For the arc of circle, we have $M(\theta) = (R\cos\theta, R\sin\theta)$ and the normal proposed in this section is simply given by $n = (\cos\theta, \sin\theta)$. We observe that it is pointing outside the disc of radius R centered at the origin.

- Curvature

The curvature $\rho(s)$ is defined by the relation $\frac{\mathrm{d}\tau}{\mathrm{d}s} = -\rho(s) \, n(s)$ with the normal $n(s)$ introduced in the previous section.

For a line segment, the curvature is null and for an arc of circle, we obtain easily $\rho = \frac{1}{R}$.

Exercices

- Catenary curve

We recall some elements of hyperbolic trigonometry: $\cosh x = \frac{1}{2}\left(\exp(x)+\exp(-x)\right)$ and $\sinh x = \frac{1}{2}\left(\exp(x)-\exp(-x)\right)$.

a) Prove that for each real number x, we have $(\cosh x)^2-(\sinh x)^2=1$.

b) Prove the following rules for the derivatives of hyperbolic cosine and hyperbolic sinus: $\frac{\mathrm{d}}{\mathrm{d}x}\cosh x = \sinh x$ and $\frac{\mathrm{d}}{\mathrm{d}x}\sinh x = \cosh x$.

We suppose given $a>0$ and $X \geq 0$. A catenary curve has a cartesian equation given by the relation $y = a\,\cosh\left(\frac{x}{a}\right)$ in an orthonormal frame of reference.

c) Draw the catenary curve.

d) What is the length of the catenary curve between the points of abscissa $x=0$ and $x=X$? [$L = a\,\sinh\left(\frac{X}{a}\right)$]

- Length of an arch of parabola

We use hyperbolic cosine and hyperbolic sinus recalled in the previous exercice.

a) Show that the hyperbolic sinus map is continuous, strictly increasing, that $\sinh x$ approaches $+\infty$ [respectively $-\infty$] if x approaches $+\infty$ [respectively $-\infty$].

b) Deduce from the previous question that the hyperbolic sinus map is bijective from $\mathbb{R}$ to $\mathbb{R}$. We denote by argsh the inverse function: $x = \operatorname{argsh} y$ is equivalent to $y = \sinh x$.

c) What is the derivative of the function argsh?

d) Prove that we have $\operatorname{argsh} x = \log\left(x+\sqrt{1+x^2}\right)$.

We set $F(x) = \frac{1}{2}\left(\operatorname{argsh} x + x\sqrt{1+x^2}\right)$.

e) Show that the function F is derivable for $x \in \mathbb{R}$ and evaluate the derivative $\frac{\mathrm{d}F}{\mathrm{d}x}$.

We introduce $a>0$ and the parabola of equation $y = \frac{x^2}{2a}$ in an orthonormal frame of reference. We suppose also given an abscissa $X \geq 0$.

f) Compute the lenght of an arc of this parabola between the ponts with abscissa $x=0$ and $x=X$. We can explicit the result with the function F introduced previously. [$L = a\,F\left(\frac{X}{a}\right)$]

- Length of a cycloid

A cycloid associated with a circle of radius $R>0$ admits the following parametric representation $x(\theta) = R(\theta-\sin\theta)$, $y(\theta) = R(1-\cos\theta)$.

a) Draw this curve for $0 \leq \theta \leq 2\pi$.

b) Express the element of length $\mathrm{d}s$ in terms of the variable θ and the infinitesimal $\mathrm{d}\theta$.

c) What is the length of the arch of cycloid between the points A corresponding to $\theta=0$ and B associated with $\theta = 2\pi$? [$8R$]

- Curvature of a functional curve

We suppose that the function $t \longmapsto f(t)$ is two times derivable on the interval $[a, b]$.

a) Show that the curvature is given by the expression $\rho(t) = \frac{f''(t)}{\left(\sqrt{1+f'(t)^2}\right)^3}$.

b) Interpret in terms of curvature the classic condition for beeing an inflexion point.