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Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 11 Change of variable in a double integral

- Change of variable in a double integral: first steps

To fix the ideas, we give ourselves the unit square $K = [0, 1] \times [0, 1]$ and two strictly positive real numbers a and b. With the linear mapping F defined by $x = a\,\xi$, $y = b\,\eta$, the unit square is transformed into a rectangle $Q = [0, a] \times [0, b]$ (see the Figure 1). If we integrate the function $f \equiv 1$ in the rectangle Q, we find $|Q| = \int_Q \mathrm{d}x\,\mathrm{d}y = ab$ while we integrate this same function $f \equiv 1$ in the square K, we obtain $|K| = \int_K \mathrm{d}\xi\,\mathrm{d}\eta = 1$. We introduce the (constant) matrix J_F of the linear application F: $J_F = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$. Its determinant $\det J_F$ is equal to ab and we see that we have $\int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det J_F|\,\mathrm{d}\xi\,\mathrm{d}\eta$.

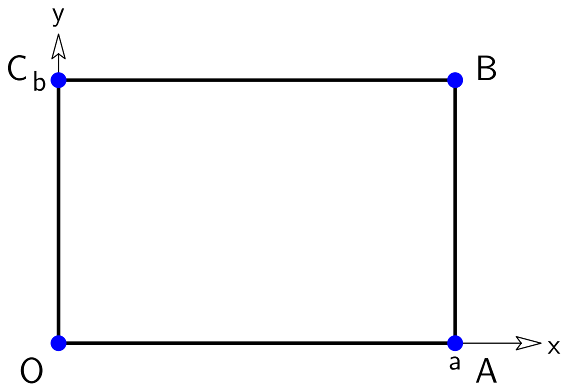

Figure 1. Rectangle with side parallel to the axes

- Change of variable in a double integral: a first parallelogram

We transform the unit square K with a linear transformation F defined now by $x = a\,\xi + c\,\eta$, $y = b\,\eta$. Then the unit square is transformed into a parallelogram Q whose can be given the coordinates of the four vertices: $\mathrm{O}(0, 0)$ $[\xi = \eta = 0]$, $\mathrm{A}(a, 0)$ $[\xi = 1, \eta = 0]$, $\mathrm{B}(a + c, b)$ $[\xi = \eta = 1]$ et $\mathrm{C}(c, b)$ $[\xi = 0, \eta = 1]$. The area of the parallelogram Q is equal to its base multiplied by the height, that is ab. Moreover, the matrix J_F of the linear application F is now $J_F = \begin{pmatrix} a & c \\ 0 & b \end{pmatrix}$. Its determinant $\det J_F$ is always ab and we still have $\int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det J_F|\,\mathrm{d}\xi\,\mathrm{d}\eta$.

François Dubois, december 2023.

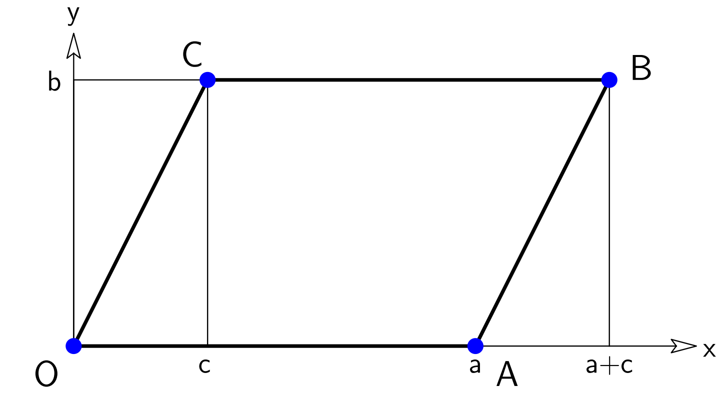

Figure 2. Parallelogram : first simple case

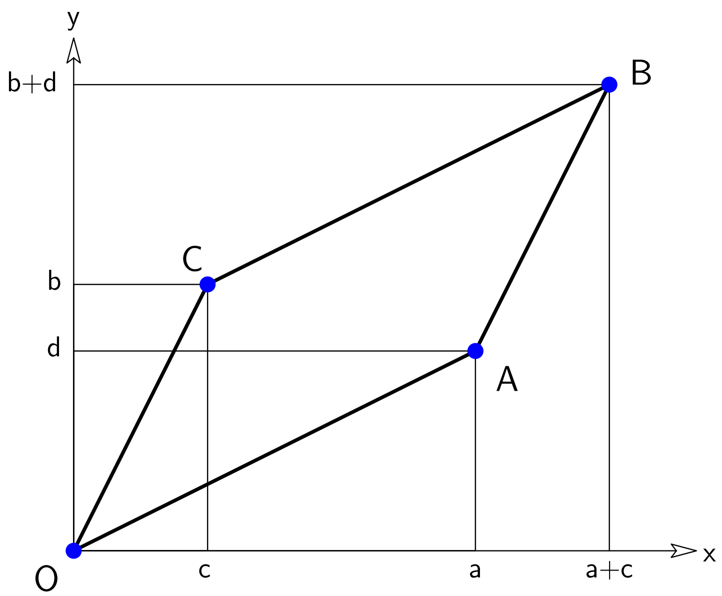

Figure 3. Parallelogram : second case

- Change of variable in a double integral: a second parallelogram

We now set the change of variables $(\xi, \eta) \longmapsto (x, y)$ *via* the linear application F defined by $x = a\xi + c\eta$, $y = d\xi + b\eta$, with a, b, c and d strictly positive to fix the ideas. Then the unit square K is transformed into another parallelogram Q. The coordinates of its four vertices are the following: O$(0, 0)$ $[\xi = \eta = 0]$, A(a, d) $[\xi = 1, \eta = 0]$, B$(a+c, b+d)$ $[\xi = \eta = 1]$ and C(c, b) $[\xi = 0, \eta = 1]$. If the quadrangle $OABC$ has a direct orientation (it turns counterclockwise) [we advise the reader to make a drawing!] then the area of the parallelogram Q can be calculated with a graphical approach [exercise!] and we have $|Q| = ab - dc$. If the quadrangle $OABC$ has a retrograde orientation [we advise the reader to make another drawing!], then we see that $|Q| = -ab + dc$. In all cases, $|Q| = |ab - dc|$. The matrix J_F of the linear application F is now equal to $J_F = \begin{pmatrix} a & c \\ d & b \end{pmatrix}$ and $\det J_F = ab - dc$. We notice that to calculate the area of this second parallelogram, it is enough to write $\int_Q \mathrm{d}x \, \mathrm{d}y = \int_K |\det J_F| \, \mathrm{d}\xi \, \mathrm{d}\eta$.

This result generalizes [exercise!] if we replace the unit square by any other square of side $\Delta x > 0$.

- Change of variable in a double integral: curvilinear quadrangle

We transform the unit square $K = [0, 1] \times [0, 1]$ with a nonlinear application Φ which we assume to be assumed to be of class $\mathscr{C}^1$, bijective from K to $Q = \Phi(K)$. We assume the reciprocal application Φ^{-1} continuous from Q onto K. We cut the square K into $N \times N$ small squares $K_{i,j}$ of side $\Delta x = \frac{1}{N}$: $K_{i,j} = [\xi_i, \xi_{i+1}] \times [\eta_j, \eta_{j+1}]$, with $\xi_i = (i-1)\,\Delta x$ and $\eta_j = (j-1)\,\Delta x$. We introduce the points $M_{i,j} = \Phi(\xi_i, \eta_j)$ and the quadrangles $Q_{i,j} = \Phi(K_{i,j})$. Then we have $\int_Q \mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{Q_{i,j}} \mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{\Phi(K_{i,j})} \mathrm{d}x\,\mathrm{d}y$. We approach the application Φ in the square $K_{i,j}$ by a tangent affine application $F_{i,j}$ at the point (ξ_i, η_j):
$\Phi(\xi, \eta) \approx F_{i,j}(\xi, \eta) \equiv \Phi(\xi_i, \eta_j) + \mathrm{d}\Phi(\xi_i, \eta_j).(\xi - \xi_i, \eta - \eta_j)$. Then we can approximate the area of the curvilinear quadrangle $Q_{i,j}$ by that of the parallelogram $P_{i,j} = F_{i,j}(K_{i,j})$ obtained by replacing Φ by $F_{i,j}$: $\int_{\Phi(K_{i,j})} \mathrm{d}x\,\mathrm{d}y \approx \int_{P_{i,j}} \mathrm{d}x\,\mathrm{d}y$. But we have seen that for a parallelogram $P_{i,j}$, we have $\int_{P_{i,j}} \mathrm{d}x\,\mathrm{d}y = \int_{K_{i,j}} |\det J_{F_{i,j}}|\,\mathrm{d}\xi\,\mathrm{d}\eta$. In the present case, $J_{F_{i,j}} = \mathrm{d}\Phi(\xi_i, \eta_j)$ and we have $\int_Q \mathrm{d}x\,\mathrm{d}y \approx \sum_{1\leq i,\, j\leq N} \int_{K_{i,j}} |\det \mathrm{d}\Phi(\xi_i, \eta_j)|\,\mathrm{d}\xi\,\mathrm{d}\eta$.
If the integer N tends to infinity, the sum of the right-hand side of the last expression converges towards $\int_K |\det \mathrm{d}\Phi(\xi, \eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$ and we finally have
$|Q| = \int_Q \mathrm{d}x\,\mathrm{d}y = \int_K |\det \mathrm{d}\Phi(\xi, \eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$.

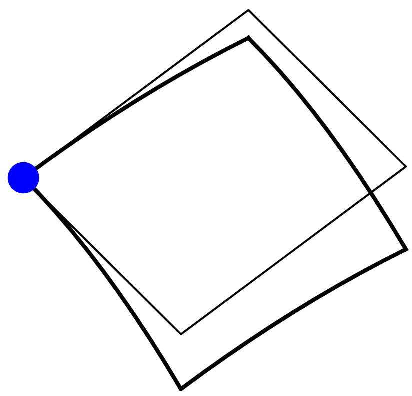

Figure 4. Around the point $M_{i,j} = \Phi(\xi_i, \eta_j)$ (big point in blue on the left), the curvilinear quadrangle $Q_{i,j}$ (in strong line) is well approximated by the parallelogram $P_{i,j}$ (in thin lines) associated with the tangent affine application $F_{i,j}$ if we have sufficiently cut out the initial square.

- Change of variable in a double integral: general case

As above, we transform the unit square $K = [0, 1] \times [0, 1]$ with a nonlinear function Φ of class $\mathscr{C}^1$, bijective from K onto $Q = \Phi(K)$ and the reciprocal application is assumed to be continuous from Q onto K. We now give ourselves a function f integrable in the sense of Riemann in Q and we try to write the integral $\int_Q f(x, y)\,\mathrm{d}x\,\mathrm{d}y$ with an integral in the square K. We use the notations from the previous paragraph and set $f_{i,j} = f(\Phi(\xi_i, \eta_j))$: this is an approximation of the function f in the (small) curvilinear quadrangle $Q_{i,j}$. We then have
$\int_Q f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{Q_{i,j}} f(x, y)\,\mathrm{d}x\,\mathrm{d}y = \sum_{1\leq i,\, j\leq N} \int_{\Phi(K_{i,j})} f(x, y)\,\mathrm{d}x\,\mathrm{d}y$.

For each curvilinear quadrangle $Q_{i,j}$, we have $\int_{\Phi(K_{i,j})} f(x,\,y)\,\mathrm{d}x\,\mathrm{d}y \approx f_{i,j}\int_{\Phi(K_{i,j})}\,\mathrm{d}x\,\mathrm{d}y$ and we saw in the previous paragraph that $\int_{\Phi(K_{i,j})}\,\mathrm{d}x\,\mathrm{d}y \approx \int_{P_{i,j}}\,\mathrm{d}x\,\mathrm{d}y = \int_{K_{i,j}} |\det \mathrm{d}\Phi(\xi_i,\,\eta_j)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. We deduce that $\int_{\Phi(K_{i,j})} f(x,\,y)\,\mathrm{d}x\,\mathrm{d}y \approx \sum_{1\leq i,\,j\leq N}\int_{K_{i,j}} f(\Phi(\xi_i,\,\eta_j))\,|\det \mathrm{d}\Phi(\xi_i,\,\eta_j)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. If the integer N tends to infinity, this last sum converges to the integral
$\int_K f(\Phi(\xi,\,\eta))\,|\det \mathrm{d}\Phi(\xi,\,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. We deduce the final form of the formula of change of variable of variable in a double integral :
$\int_Q f(x,\,y)\,\mathrm{d}x\,\mathrm{d}y = \int_K f(\Phi(\xi,\,\eta))\,|\det \mathrm{d}\Phi(\xi,\,\eta)|\,\mathrm{d}\xi\,\mathrm{d}\eta$. The trick is not to forget the jacobian $J(\xi,\,\eta) \equiv |\det \mathrm{d}\Phi(\xi,\,\eta)|$, absolute value of the determinant of the Jacobian matrix of partial derivatives partial derivatives $\mathrm{d}\Phi(\xi,\,\eta)$!
We admit that the previous result generalizes to the case of any open set K in $\mathbb{R}^n$ any integer $n \geq 1$ and a function f measurable on $Q = \Phi(K)$ and integrable on Q, that is, such that $\int_Q |f(x,\,y)|\,\mathrm{d}x\,\mathrm{d}y < \infty$.
As an exercise, the reader can try to find the "usual" formula of change of variable variable in the case of dimension one as a special case of the previous relation!

- Polar coordinates in the plane

The variables ξ and η are denoted r et θ and the application Φ of change of variable $(r,\,\theta) \longmapsto (x,\,y)$ is defined by $x = r\cos\theta$ et $y = r\sin\theta$. The Jacobian matrix of this transformation can be calculated without particular difficulty and we have, if we assume $r > 0$:
$J(r,\,\theta) = r$. Then we have $\int_Q f(x,\,y)\,\mathrm{d}x\,\mathrm{d}y = \int_K f(r\cos,\,r\sin\theta)\,r\,\mathrm{d}r\,\mathrm{d}\theta$ when $Q = \Phi(K)$.

- Revisiting the unidimensional case

Consider four reals numbers $a,\,b,\,\alpha$ and β with $a < b$ and $\alpha < \beta$. Introduce a derivable function $\varphi : [\alpha,\,\beta] \longmapsto [a,\,b]$ realizing a bijection from $[\alpha,\,\beta]$ onto $[a,\,b]$. Then we have in all cases $\int_a^b f(x)\,\mathrm{d}x = \int_\alpha^\beta f\big(\varphi(t)\big)\,|\varphi'(t)|\,\mathrm{d}t$.

Exercises

- Circular domain

a) We suppose given $R > 0$. Let D be the domain $D = \{(x,\,y) \in \mathbb{R}^2,\; x^2 + y^2 \leq R^2\}$. Compute the double integral $I = \iint_D x^3\,y^2\,\mathrm{d}x\mathrm{d}y$.
b) Same question with the analogous integral $I_+ = \iint_{D_+} x^3\,y^2\,\mathrm{d}x\mathrm{d}y$ in the domain
$D_+ = \{(x,\,y) \in \mathbb{R}^2, x \geq 0\,,\; x^2 + y^2 \leq R^2\}$. [$0,\ \frac{4}{105}R^7$]

- Elliptic domain

Let $a > 0$ and $b > 0$ be two fixed lengths. We introduce the domain D intersection of the interior of the ellipse satisfying the equation $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ with the first quadrant
$Q_+ = \{(x,\,y) \in \mathbb{R}^2, x \geq 0, y \geq 0\}$.
a) Draw the domain D.
b) With a not so conventional change of variables, transform the calculus of the double integral $I = \iint_D x\,y\,\mathrm{d}x\mathrm{d}y$.
c) Deduce from the previous question the surface $|D|$ of this quarter of elliptic domain.
d) Achieve the calculus of the double integral I. [$\frac{1}{4}\pi ab,\ \frac{1}{8}a^2\,b^2$]