le cnam

Master Structural Mechanics and Coupled Systems

Applied Mathematics

Lecture 13 Surface integral

- Local parameterization

We give ourselves four real numbers $a < b$, $c < d$ and a space of parameters $(u, v) \in \mathbb{R}^2$ such that $a \le u \le b$ and $c \le v \le d$. We define the parametrized sheet Σ in space $\mathbb{R}^3$ by a continuously differentiable function Φ fom the rectangle $\widehat{\Sigma} \equiv [a, b] \times [c, d]$ and taking its values in $\mathbb{R}^3$. This application Φ is called the "local map". The parameterized sheet Σ is defined by the relation $\Sigma = \Phi(\widehat{\Sigma})$. A point $M(u, v)$ of the parametric sheet has coordinates x, y et z that are regular functions of the parameters u and v: $x = X(u, v)$, $y = Y(u, v)$ et $z = Z(u, v)$.

The fundamental example is a plane parallelogram. The equation of the plane is for example of the form $z = \alpha x + \beta y + \gamma$. For $a \le u \le b$ and $c \le v \le d$, we set $x = u$, $y = v$ and $z = \alpha u + \beta v + \gamma$.
A second example concerns a surface with equation $z = f(x, y)$. It is similar to the previous case except that the affine function $f(x, y) = \alpha x + \beta y + \gamma$ is replaced by a function f of two variables while remaining fairly regular.
The next example is the sphere centered at the origin O and of radius $R > 0$. We use the spherical coordinates of the three-dimensional space. We project the current point M of the sphere onto the xOy plane at a point m. We have: $z = R \cos\theta$ and O$m = R \sin\theta$. It comes then $x = \mathrm{O}m \cos\varphi = R \sin\theta \cos\varphi$ and $y = \mathrm{O}m \sin\varphi = R \sin\theta \sin\varphi$.

- Tangent plane

We suppose the function Φ différentiable at the point (u, v):
$\Phi(u + \delta u, v + \delta v) = \Phi(u, v) + \frac{\partial \Phi}{\partial u}(u, v)\, \delta u + \frac{\partial \Phi}{\partial v}(u, v)\, \delta v + \| (u, v) \| \, \varepsilon(\delta u, \delta v)$. The two tangent vectors $\frac{\partial \Phi}{\partial u}(u, v)$ and $\frac{\partial \Phi}{\partial v}(u, v)$ are two vectors of space $\mathbb{R}^3$. We suppose that the family $(\frac{\partial \Phi}{\partial u}, \frac{\partial \Phi}{\partial v})$ is non-degenerate and the point $M = \Phi(u, v)$ is a "regular point" of the surface. The tangent plane to the parametrized sheet Σ at the point $M = \Phi(u, v)$ is the affine plane that passes through the point $M(u, v)$ with an associated vector plane that has as its basis the family of two vectors $\frac{\partial \Phi}{\partial u}(u, v)$ and $\frac{\partial \Phi}{\partial v}(u, v)$.
In the case of a surface of the form $z = f(x, y)$, the two parameters are the abscissa x and the the ordinate y. We have $\frac{\partial \Phi}{\partial x} = (1, 0, \frac{\partial f}{\partial x})^{\mathrm{t}}$ and $\frac{\partial \Phi}{\partial y} = (0, 1, \frac{\partial f}{\partial y})^{\mathrm{t}}$. They always form a free family whatever the function f.

For a sphere of radius $R>0$ and centered at the origin, we introduce the spherical coordinates, so the two polar angles θ and φ such that $x=R\,\sin\theta\,\cos\varphi$, $y=R\,\sin\theta\,\sin\varphi$ and $z=R\,\cos\theta$. The moving reference frame $(e_r,\, e_\theta,\, e_\varphi)$ is defined by the relations
$e_r(\theta,\,\varphi)=\sin\theta\,(\cos\varphi\, e_1+\sin\varphi\, e_2)+\cos\theta\, e_3$, $e_\theta(\theta,\,\varphi)=\cos\theta\,(\cos\varphi\, e_1+\sin\varphi\, e_2)-\sin\theta\, e_3$ and $e_\varphi(\varphi)=-\sin\varphi\, e_1+\cos\varphi\, e_2$. In the case of the sphere of radius R, we have
$\mathrm{d}M=R\,(e_\theta\ \mathrm{d}\theta+e_\varphi\ \sin\theta\ \mathrm{d}\varphi)$; we deduce $\frac{\partial M}{\partial\theta}=R\,e_\theta$ and $\frac{\partial M}{\partial\varphi}=R\,\sin\theta\, e_\varphi$.

- Vector product

Let u and v be two vectors in an euclidian space of dimension 3. The vector product $u\times v$ satisfies the following properties.
(i) The vector product $u\times v$ is orthogonal to the vectors u and v: $(u\times v,\, u)=(u\times v,\, v)=0$.
(ii) If the vectors u and v are collinear, the vector product $u\times v$ is zero.
(iii) If $P(u,\,v)$ is the parallelogram generated by the vectors u and v:
$P(u,\,v)=\{x\in\mathbb{R}^3,\,\exists\,\theta,\,\xi,\;0\leq\theta\leq 1,\,0\leq\xi\leq 1,\; x=\theta\,u+\xi\,v\}$, then the area of this parallelogram is equal to the norm of the vector product $u\times v$: $|P(u,\,v)|=\|u\times v\|$. Moreover, $\|u\times v\|\leq\|u\|\,\|v\|$.
(iv) We give ourselves a direct orthonormal basis $(e_1,\,e_2,\,e_3)$ and the components of the two vectors u and v: $u=\sum_{j=1}^{3}u_j\,e_j$ and $v=\sum_{k=1}^{3}v_k\,e_k$. The vector product $u\times v$ is expressed in the basis $(e_1,\,e_2,\,e_3)$ *via* the relation: $u\times v=\begin{vmatrix} u_1 & v_1\\ u_2 & v_2\end{vmatrix} e_3+\begin{vmatrix} u_2 & v_2\\ u_3 & v_3\end{vmatrix} e_1+\begin{vmatrix} u_3 & v_3\\ u_1 & v_1\end{vmatrix} e_2$.
(v) It is possible to prove that $w=u\times v$ is a biliear function of the two vectors u and v:
$(\alpha\,u+\beta\,u')\times v=\alpha\,(u\times v)+\beta\,(u'\times v)$ and $u\times(\alpha\,v+\beta\,v')=\alpha\,(u\times v)+\beta\,(u\times v')$, whatever the choice of vectors u, u', v, v' and whatever the choice of numbers α and β.
(vi) If $u\times v\neq 0$, the family $(u,\,v,\,u\times v)$ is a direct basis of $\mathbb{R}^3$: the change matrix of basis between a direct orthonormal basis $(e_1,\,e_2,\,e_3)$ and the family $(u,\,v,\,u\times v)$ is strictly positive.
(vii) Be careful, the vector product is not associative: in general, $u\times(v\times w)\neq(u\times v)\times w$.

- Normal vector

We assume that the point $M=\Phi(u,\,v)$ is a regular point of the surface, *i.e.* that the family $(\frac{\partial\Phi}{\partial u}(u,\,v),\,\frac{\partial\Phi}{\partial v}(u,\,v))$ is free. Then the vector product $\frac{\partial\Phi}{\partial u}(u,\,v)\times\frac{\partial\Phi}{\partial v}(u,\,v)$ is not zero. The norm $\|\,\frac{\partial\Phi}{\partial u}(u,\,v)\times\frac{\partial\Phi}{\partial v}(u,\,v)\,\|$ of this vector product is equal to the surface of the parallelogram constructed with the two tangent vectors $\frac{\partial\Phi}{\partial u}(u,\,v)$ and $\frac{\partial\Phi}{\partial v}(u,\,v)$. We define the normal vector $n(u,\,v)$ as the unit vector constructed from this vector product:
$n(u,\,v)=\frac{1}{\|\frac{\partial\Phi}{\partial u}(u,v)\times\frac{\partial\Phi}{\partial v}(u,v)\|}\,\frac{\partial\Phi}{\partial u}(u,\,v)\times\frac{\partial\Phi}{\partial v}(u,\,v)$.
It is a vector orthogonal to the tangent plane. Hence the name "normal" or "normal vector".
For a surface of the form $z=f(x,\,y)$, we have $\frac{\partial\Phi}{\partial x}\times\frac{\partial\Phi}{\partial y}=(-\frac{\partial f}{\partial x},\,-\frac{\partial f}{\partial y},\,1)^{\mathrm{t}}$ and
$n(x,\,y)=\frac{1}{\sqrt{1+(\frac{\partial\Phi}{\partial x})^2+(\frac{\partial f}{\partial y})^2}}\,(-\frac{\partial f}{\partial x},\,-\frac{\partial f}{\partial y},\,1)^{\mathrm{t}}$.
For the sphere of radius R, we have the relation $\frac{\partial M}{\partial\theta}\times\frac{\partial M}{\partial\varphi}=R^2\,\sin\theta\, e_r$. We deduce
$\|\,\frac{\partial M}{\partial\theta}\times\frac{\partial M}{\partial\varphi}\,\|=R^2\,\sin\theta$ and $n(\theta,\,\varphi)=e_r(\theta,\,\varphi)$.

- Scaled surface

To approximate a curve Γ, we give ourselves points on the curve and we approach the curve by the sequence of strings stretched between a point and its neighbor. Through two different points

there always passes one and only one line segment. Thus we obtain a continuous approximation of the curve Γ.
To approximate a surface, it is more delicate. Indeed, we give ourselves an integer $n \geq 1$ and we discretize first the rectangle $[a, b] \times [c, d]$ in the parameter space with small rectangles of the type $[a+ih, a+(i+1)h] \times [c+jk, c+(j+1)k]$ with $h = \frac{b-a}{n}$ and $k = \frac{b-a}{n}$. The image by the local map Φ is a curvilinear quadrangle whose four corners we denote by
$M_{ij} = \Phi(a+ih, c+jk)$, $M_{i+1,j} = \Phi(a+(i+1)h, c+jk)$,
$M_{i+1,j+1} = \Phi(a+(i+1)h, c+(j+1)k)$ and $M_{i,j+1} = \Phi(a+ih, c+(j+1)k)$. These four points are close enough if n is large enough belong to the surface Σ but are not coplanar in general. We propose to approximate the surface quadrilateral
$Q_{ij} = \Phi([a+ih, a+(i+1)h] \times [c+jk, c+(j+1)k])$ by a plane parallelogram $\widetilde{Q_{ij}}$ localized on the tangent plane to the surface Σ at the point M_{ij}. Precisely $\widetilde{Q_{ij}}$ is the parallelogram passing through the point M_{ij} and directed by two tangent vectors at the point M_{ij}, *i.e.* $h\frac{\partial\Phi}{\partial u}(a+ih, c+jk)$ and $k\frac{\partial\Phi}{\partial v}(a+ih, c+jk)$. We have
$\widetilde{Q_{ij}} = M_{ij} + \{\xi\, h\frac{\partial\Phi}{\partial u}(a+ih, c+jk) + \eta\, k\frac{\partial\Phi}{\partial v}(a+ih, c+jk), 0 \leq \xi, \eta \leq 1\}$. The area $|\widetilde{Q_{ij}}|$ of this parallelogram is given by the norm of the vector product of the two tangent vectors: $|\widetilde{Q_{ij}}| = hk\, \| (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(a+ih, c+jk) \|$.
Such a plane parallelogram is, for n large enough, *i.e.* h and k small enough, a good approximation of the surface parallelogram Q_{ij}. When we join all these parallelograms for $0 \leq i, j < n$, we obtain an approximation $\Sigma_n = \cup_{0 \leq i, j < n} \widetilde{Q_{ij}}$ fairly accurate surface Σ, but it has the defect of being discontinuous at the interfaces. Hence the expression "scaled surface".

- Surface of a parametrized sheet

We first define the surface $|\Sigma_n|$ of the scaled surface Σ_n associated with the rectangle cutout $\widehat{\Sigma} = [a, b] \times [c, d]$ of the parameters into $n \times n$ small rectangles: $|\Sigma_n| = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} |\widetilde{Q_{ij}}|$. Given the surface of a piece of scale $\widetilde{Q_{ij}}$, we have $|\Sigma_n| = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} \| (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(a+ih, c+jk) \|\, hk$. We then make the integer n tend to infinity. The double sum converges to the double integral on the rectangle $\widehat{\Sigma}$ of the function $(u, v) \longmapsto \| (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(u, v) \|$. We deduce an expression for the surface of the parameterized sheet: $|\Sigma| = \int\!\!\int_{\widehat{\Sigma}} \| (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v})(u, v) \|\, \mathrm{d}u\, \mathrm{d}v$. We define the element of surface $\mathrm{d}\sigma$ by the relation $\mathrm{d}\sigma = \| \frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v} \|\, \mathrm{d}u\, \mathrm{d}v$ and then we write in a deceptively simple way: $|\Sigma| = \int\!\!\int_{\widehat{\Sigma}} \mathrm{d}\sigma$. The surface element $\mathrm{d}\sigma$ does not depend on the chosen parameterization and the relation $|\Sigma| = \int\!\!\int_{\widehat{\Sigma}} \mathrm{d}\sigma$ is intrinsic.
The metric term $\| (\frac{\partial\Phi}{\partial u} \times \frac{\partial\Phi}{\partial v}) \|$ is to be compared to the length when calculating the length of an curve Γ: $|\Gamma| = \int_a^b \| (\frac{\mathrm{d}M}{\mathrm{d}t}(t) \|\, \mathrm{d}t = \int_a^b \mathrm{d}s$. For a sphere Σ of radius R, we have seen that $n = e_r$ and $\frac{\partial M}{\partial\theta} \times \frac{\partial M}{\partial\varphi} = R^2 \sin\theta\, e_r$. We can therefore write the surface element
$\mathrm{d}\sigma = \| \frac{\partial M}{\partial\theta} \times \frac{\partial M}{\partial\varphi} \|\, \mathrm{d}\theta\, \mathrm{d}\varphi = R^2 \sin\theta\, \mathrm{d}\theta\, \mathrm{d}\varphi$. Then we have for the sphere Σ such that $0 \leq \theta \leq \pi$ and $0 \leq \varphi \leq 2\pi$, $|\Sigma| = \int_0^\pi \mathrm{d}\theta \int_0^{2\pi} \mathrm{d}\varphi\, R^2 \sin\theta = 4\pi R^2$.

- Surface integral

A function f defined on a parametric sheet Σ can also be written as a function of the parameters u and v: $\widehat{f}(u, v) = f(\Phi(u, v))$. For $n \geq 1$ we introduce the scaled surface Σ_n associated to a discretization $M_{ij} = \Phi(a+ih, c+jk)$ of $\widehat{\Sigma} = [a, b] \times [c, d]$. We can approximate the function

f on Σ_n by the stepped function equal to the constant $f(M_{ij})$ in each parallelogram $\widetilde{Q_{ij}}$. Given the value $\| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(a+ih,\, c+jk) \|$ of the surface of this parallelogram, we define the approximate integral I_n of the function f on the surface Σ by
$I_n = \sum_{i=0}^{n-1} \sum_{j=0}^{n-1} f(M_{ij}) \, \| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(a+ih,\, c+jk) \| \, hk$. If n tends to infinity and if the function function f is continuous to fix the ideas, the sequence I_n converges to the double integral $I = \iint_{\widehat{\Sigma}} f(\Phi(u,\, v)) \; \| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(u,\, v) \| \; \mathrm{d}u \, \mathrm{d}v$. We define the surface integral $\int_{\Sigma} f(M) \, \mathrm{d}\sigma$ by the relation $\int_{\Sigma} f(M) \, \mathrm{d}\sigma = \iint_{\widehat{\Sigma}} f(\Phi(u,\, v)) \; \| (\frac{\partial \Phi}{\partial u} \times \frac{\partial \Phi}{\partial v})(u,\, v) \| \; \mathrm{d}u \, \mathrm{d}v$. It does not depend on the parameterization.
In the case where $f(M) \equiv 1$, we do find the value $|\Sigma|$ for the area of the surface Σ: $\int_{\Sigma} \mathrm{d}\sigma = |\Sigma|$.

- Flow of a vector field

We give ourselves a vector field $\varphi:\ \mathbb{R}^3 \longmapsto \mathbb{R}^3$ continuous to fix the ideas. If $f(M) = (\Phi,\, n)$, scalar product of the field φ against the normal vector of the surface Σ, the corresponding surface integral defines the flux Φ of the vector field φ on the surface Σ: $\Phi = \int_{\Sigma} (\Phi,\, n) \, \mathrm{d}\sigma$.

- Integration by parts in three dimensions

Finally, we consider a domain Ω included in $\mathbb{R}^3$ and its boundary $\Sigma = \partial\Omega$. We observe that Σ is a closed surface without any boundary. We suppose that the normal vector $n(M)$ along the surface Σ is pointing in the direction outside of the domain Ω. Let f be a regular application defined on the adherence $\overline{\Omega}$: $f:\ \overline{\Omega} \longrightarrow \mathbb{R}$. The theorem of integration by parts expresses that the triple integral of a derivative of the function f in the domain Ω reduces to a surface integral on the boundary $\Sigma = \partial\Omega$: $\iiint_{\Omega} \frac{\partial f}{\partial x_j} \, \mathrm{d}x \, \mathrm{d}y \, \mathrm{d}z = \iint_{\partial\Omega} f \, n_j \, \mathrm{d}\sigma$.

Exercises

- On half-spheres

We denote by Σ the half-sphere centered at the origin, of radius $R > 0$ and defined also by the inequality $z \geq 0$.
a) In troduce a parameterization of this half-sphere with the spherical coordinate system r, θ and φ such that $x = r \sin\theta \cos\varphi$, $y = r \sin\theta \sin\varphi$ and $z = r \cos\theta$.
b) In what intervals vary the angles θ and φ ?
c) Propose an expression for the surface element $\mathrm{d}\sigma$ as a function of the variables of the problem.
d) Compute the integral $I = \int_{\Sigma} z \, \mathrm{d}\sigma$. $[\pi R^3]$
e) Go back to the questions b), c) and d) of this exercice raplacing on one hand the half-sphere Σ by the half-sphere $\widetilde{\Sigma}$ of radius $R > 0$, centered at the origin and defined by the inequality $x \geq 0$ and on the other hand the integral I by $J = \int_{\widetilde{\Sigma}} x \, \mathrm{d}\sigma$.
f) Why the questions d) and e) are related in a simple manner ?

- Surface of a truncated cone

We consider a truncated cone with a circular basis, a radius $R > 0$ and a height equal to $h > 0$.

a) Show that the half-angle θ at the summit satisfies to the relation $\tan\theta = \frac{R}{h}$.

b) Show that truncated cone can be parameterized with the relations $x = R\cos\varphi\,(1-\frac{z}{h})$, $y = R\sin\varphi\,(1-\frac{z}{h})$ and $z = z$, with $0 \leq \varphi \leq 2\pi$ and $0 \leq z \leq h$.

c) Compute the cartesian components of the vectors $\frac{\partial M}{\partial\varphi}$ and $\frac{\partial M}{\partial z}$.

d) Express the element of surface $\mathrm{d}\sigma$ as a function of the geometrical parameters R and h, of the coordinates φ and z along the truncated cone and of the product $\mathrm{d}\varphi\ \mathrm{d}z$.

e) Show that the surface S of this truncated cone is equal to $\pi R\sqrt{R^2+h^2}$.

- Computation of a flux

We consider the half-sphere Σ with radius $R > 0$ centered at the origin and defined by the inequality $z \geq 0$. We denote by n the normal vector field pointing in a direction such that $n_z \geq 0$. We consider also the vector field $\psi(x, y, z) = (x, y, 0)$.

a) Compute the scalar product (ψ, n) on the half-sphere Σ. $[R\sin^2\theta]$

b) Computer the flux $\Phi = \int_\Sigma (\psi.n)\,\mathrm{d}\sigma$ of the vector field ψ on the half-sphere Σ. $\left[\frac{4}{3}\pi R^3\right]$