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Three test cases for quasi-unidimensional nozzles

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• We consider the Euler equations of gas dynamics in one-dimensional geometries. Let $[0,\,1] \ni x \longmapsto A(x) \in]0,\,+\infty[$ be a regular function that modelizes the variations of the section of a given pipe. We introduce the density field $\rho(\bullet)$, the velocity $u(\bullet)$ and the total specific energy $E(\bullet)$ that can be splitted into internal energy $e(\bullet)$ and kinetic energy $\frac{1}{2}u(\bullet)^2$: $E \equiv e + \frac{1}{2}u^2$. We suppose that the fluid is a polytropic perfect gas with parameter $\gamma = \frac{7}{5}$; then the pressure $p(\bullet)$ is given by a simple state law : $p \equiv (\gamma - 1)\,\rho\, e$. The so-called "conservative variables" $W(\bullet)$ are defined as follows :

(0.1) $$W \;\equiv\; (\,\rho\, A\,,\; \rho\, u\, A\,,\; \rho\, E\, A\,)^{\rm t}$$

and the associated flux $f(\bullet)$ satisfies

(0.2) $$f(W) \;\equiv\; (\,\rho\, u\, A\,,\; (\rho\, u^2 + p)\, A\,,\; (\rho\, u\, E + p\, u)\, A\,)^{\rm t}\,.$$

The conservation laws for mass, momentum and energy take the following form

(0.3) $$\frac{\partial W}{\partial t} \;+\; \frac{\partial f(W)}{\partial x} \;-\; p\,\frac{{\rm d}A}{{\rm d}x} \;=\; 0$$

for these particular quasi-unidimensional geometries. We propose here to consider three test cases numbered from 1 to 3 and associated with three variations of the nozzle $A(\bullet)$, three different initial conditions $W^0(\bullet)$ and three different boundary sets $\mathcal{M}^{\rm left}$ and $\mathcal{M}^{\rm right}$ at the two extremities of the pipes.

• **Test case n° 1.**

The geometry of the nozzle for the first test case is a simple tube :

(1.1) $$A_1(x) \;\equiv\; 1\,,$$

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The initial conditions $W^0(\bullet)$ for these test cases are given in tems of initial density $\rho^0(\bullet)$, initial velocity $u^0(\bullet)$, and initial pressure $p^0(\bullet)$. For our first test case, the initial condition is the one of the Sod shock tube :

$$(1.2) \qquad \big(\rho_1^0(x)\,,\, u_1^0(x)\,,\, p_1^0(x)\big) \;=\; \begin{cases} (\,1\,,\,0\,,\,1)\,, & x < \dfrac{1}{2} \\ (\,0.125\,,\,0\,,\,0.1)\,, & x > \dfrac{1}{2} \end{cases} ,$$

The boundary conditions $\mathcal{M}^{\rm left}$ at $x = 0$ and $\mathcal{M}^{\rm right}$ at $x = 1$ are independent of the time for the three test cases. They consist of perfectly defined states for the Sod shock tube :

$$(1.3) \qquad \begin{cases} \mathcal{M}_1^{\rm left} \;=\; \{\, W_1^{\rm left}\,, \quad \rho_1^{\rm left} \;=\; 1\,,\; u_1^{\rm left} \;=\; 0\,,\; p_1^{\rm left} \;=\; 1\,\} \\ \mathcal{M}_1^{\rm right} \;=\; \{\, W_1^{\rm right}\,, \quad \rho_1^{\rm right} \;=\; 0.125\,,\; u_1^{\rm right} \;=\; 0\,,\; p_1^{\rm right} \;=\; 0.1\,\} \,. \end{cases}$$

- **Test case n° 2.**

We propose a divergent profile initially proposed by Shubin for the second test :

$$(2.1) \qquad A_2(x) \;=\; 1.398 + 0.347 \tanh{(8x-4)} \;, \qquad 0 \leq x \leq 1\,,$$

the initial condition for the second test case is uniform :

$$(2.2) \qquad \big(\rho_2^0(x)\,,\, u_2^0(x)\,,\, p_2^0(x)\big) \;\equiv\; (\,1\,,\,0\,,\,1\,)\,.$$

The left boundary for the second test case is a so-called "supersonic inflow" :

$$(2.3) \qquad \mathcal{M}_2^{\rm left} \;=\; \{\, W_2^{\rm left}\,,\; \rho_2^{\rm left} \;=\; 0.50179\,,\; u_2^{\rm left} \;=\; 1.299\,,\; p_2^{\rm left} \;=\; 0.38083\,\}$$

and we propose a so-called "supersonic outflow" for this second test case :

$$(2.4) \qquad \mathcal{M}_2^{\rm right} \;=\; \bigg\{ W_2^{\rm right}\,, \quad u_2^{\rm right} \;\geq\; \sqrt{\gamma\,\frac{p_2^{\rm right}}{\rho_2^{\rm right}}}\; \bigg\}\,.$$

- **Test case n° 3.**

We suggest a De Laval converging-diverging profile for the third experiment :

$$(3.1) \qquad A_3(x) \;=\; \begin{cases} 1 \,+\, 6\,\big(x - \dfrac{1}{2}\big)^2\,, & x < \dfrac{1}{2} \\ 1 \,+\, 2\,\big(x - \dfrac{1}{2}\big)^2\,, & x > \dfrac{1}{2} \end{cases} .$$

and the initial condition for the third test problem is uniform :

$$(3.2) \qquad \begin{pmatrix} \rho_3^0(x) \\ u_3^0(x) \\ p_3^0(x) \end{pmatrix} \;=\; \begin{pmatrix} \Big(\frac{3\,(\gamma-1)}{\gamma}\Big)^{\frac{1}{\gamma-1}} \\ 0 \\ \Big(\frac{3\,(\gamma-1)}{\gamma}\Big)^{\frac{\gamma}{\gamma-1}} \end{pmatrix} \;=\; \begin{pmatrix} 0.68019 \\ 0 \\ 0.58302 \end{pmatrix}.$$

For the left condition of the De Laval nozzle, we suppose to be given the total enthalpy $H \equiv \frac{\gamma}{\gamma-1} \frac{p}{\rho} + \frac{1}{2} u^2$ and the reduced entropy $\Sigma \equiv \frac{p}{\rho^\gamma}$; we set

$$\mathcal{M}_3^{\rm left} \,=\, \Big\{ W_3^{\rm left} \,, \quad \frac{\gamma}{\gamma - 1} \frac{p_3^{\rm left}}{\rho_3^{\rm left}} + \frac{1}{2} (u_3^{\rm left})^2 \,=\, 3 \,, \quad \frac{p_3^{\rm left}}{(\rho_3^{\rm left})^\gamma} \,=\, 1 \, \Big\} \,. \tag{3.3}$$

At the other extremity, the (static) pressure is imposed :

$$\mathcal{M}_3^{\rm right} \,=\, \Big\{ W_3^{\rm right} \,, \quad p_3^{\rm right} \,=\, 0.4 \, \Big\} \,. \tag{3.4}$$

• The difficulty of the first case is the treatment of nonlinear waves going outside the computational domain $]0, 1[$. For the second test, the initial condition contradicts the two boundary conditions. It is also the case for the pressure condition that concerns the third test case. We propose to consider a set of recursively refined meshes, with a number N of mesh cells satisfying the conditions

$$N \,=\, 25,\, 50,\, 100,\, 200,\, 400,\, 800,\, 1600$$

and to determine for each test case the discrete stationary state if it exists.